After the integrations have been performed, one must perform a Lorentz transformation parallel to the momentum of the pion, $\mathbf{P}$, in order to return to the barycentric frame of the initial electron-positron system. The parameter for this Lorentz transformation is

$$\beta = P/(2Q-E). \qquad (A.7)$$

Eliminating the remaining delta function by integrating over the meson energy variable E results in the spectral-differential cross section, Eq. (17).

It was found that not all of the integrations over meson angles in Eq. (17) were possible in terms of tabulated functions. However, when the terms were collected according to whether they would contribute as $[\ln(2Q/m)]^2$, $\ln(2Q/m)$, or unity, to the total cross section, it was possible to do the integrations over meson angles for the terms proportional to $[\ln(2Q/m)]^2$ and $\ln(2Q/m)$. Since $\ln(2Q/m)\approx 6$, these two sets of terms are also the most important. In order to verify this, the nonlogarithmic contributions to the pion spectrum were evaluated numerically for $Q=1.0\mu$ and $P=0.1\mu$, 0.4μ, and 0.74μ. The values obtained for the nonlogarithmic corrections relative to the logarithmic contributions were $+0.2\%$, -1.3%, and $+5.4\%$, respectively.

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Optimal Measuring Apparatus

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An upper limit for the accuracy of the measurement of a simple quantity which does not commute with a conserved quantity is obtained in terms of the "size" of the apparatus. The "size" of the apparatus is defined as the mean square value $\hbar^2M^2$ of the conserved quantity for the apparatus which is, in the example chosen, the z component of the angular momentum. The measured quantity is the projection of a spin in a perpendicular direction. It is found that the probability of an unsuccessful measurement is at least $1/8M^2$.

1. INTRODUCTION

It was shown recently that a quantum mechanical operator which does not commute with the operator of a conserved quantity can be measured only approximately. There is a finite probability that the measurement is unsuccessful, but this probability can be very small if the measuring apparatus contains a large amount of the conserved quantity.[1] It was shown, in particular, that if the product of the probability of an unsuccessful measurement and of the maximum value of the conserved quantity which is present in the measuring apparatus exceeds a certain value, no contradiction with the conservation law occurs. The objective of the present article is to find the "best" measuring apparatus for a given "size." The conserved quantity will not have an upper limit in the initial state of this; rather, we specify the mean-square value of the conserved quantity and ask for the minimum probability for an unsuccessful measurement, consistent with the prescribed mean square of the conserved quantity and, of course, the validity of the conservation law.

We require that the operator of the conserved quantity for the apparatus commute with the operator, by which the final state of the apparatus is measured. This condition is necessary because otherwise—as a consequence of the result of our previous work—we cannot ascertain the result of the measurement exactly.

The condition to be obtained will be only a necessary one. In other words, for the given mean square of the conserved quantity, the probability of a malfunctioning of the apparatus cannot be smaller than the value to be calculated. Whether an apparatus with the specified properties is actually possible will not be decided. All that can be claimed is that the existence of such an apparatus is not in conflict with the conservation law considered.

The quantity to be measured and the conservation law to be considered will be the same as in the first publication on this subject: The quantity is the component of the spin of a particle in a given direction; the conserved quantity, the angular momentum about a direction perpendicular to the aforementioned direction. It will be shown in Sec. 2 that the minimum probability for the malfunctioning of the apparatus is inversely proportional to the mean square of the conserved quantity, and the proportionality constant will be determined. Section 3 will contain a discussion of the results.

2. MINIMIZATION OF THE MALFUNCTIONING PROBABILITY

We measure the x component of the spin of a particle with spin $\frac{1}{2}$; the z component of the spin of this particle is the additive conserved quantity. To make the compu-

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[1] E. P. Wigner, Z. Physik **131**, 101 (1952); H. Araki and M. M. Yanase, Phys. Rev. **120**, 622 (1960).

tation simpler, we choose the eigenvalues of the z component as 0 and 1 and denote the corresponding eigenstates ψ_0 and ψ_1, respectively. Then eigenstates of the x component are $2^{-\frac{1}{2}}(\psi_0+\psi_1)$ and $2^{-\frac{1}{2}}(\psi_0-\psi_1)$. The state of the measuring apparatus before the measurement will be denoted by ξ. The measurement will result in a unitary transformation U in the Hilbert space of the combined system of the object and the measuring apparatus:

$$U[2^{-\frac{1}{2}}(\psi_0+\psi_1)\xi]=2^{-\frac{1}{2}}(\psi_0+\psi_1)X+2^{-\frac{1}{2}}(\psi_0-\psi_1)\eta, \quad (2.1)$$

$$U[2^{-\frac{1}{2}}(\psi_0-\psi_1)\xi]=2^{-\frac{1}{2}}(\psi_0-\psi_1)X'+2^{-\frac{1}{2}}(\psi_0+\psi_1)\eta', \quad (2.2)$$

where X, X', η, η' are the states of the measuring apparatus after the measurement.

From (2.1) and (2.2) we have

$$U(\psi_0\xi)=\psi_0u+\psi_1v, \quad (2.3)$$

$$U(\psi_1\xi)=\psi_0w+\psi_1z, \quad (2.4)$$

where

$$\begin{aligned} u&=\tfrac{1}{2}(X+X'+\eta+\eta'),\\ v&=\tfrac{1}{2}(X-X'-\eta-\eta'),\\ w&=\tfrac{1}{2}(X-X'+\eta-\eta'),\\ z&=\tfrac{1}{2}(X+X'-\eta-\eta'). \end{aligned} \quad (2.4a)$$

It follows from the conservation law for the z component of the angular momentum that

$$U(\psi_0\xi_\mu)=\psi_0u_\mu+\psi_1v_{\mu-1}, \quad (2.5)$$

$$U(\psi_1\xi_{\mu-1})=\psi_0w_\mu+\psi_1z_{\mu-1}, \quad (2.6)$$

where we decomposed ξ, u, v, w, z, into $\xi=\sum\xi_\mu$, $u=\sum u_\mu$, $v=\sum v_\mu$, $w=\sum w_\mu$, $z=\sum z_\mu$, the index μ being the eigenvalues of the z component of the angular momentum of the apparatus, and the ξ_μ, u_μ, etc., corresponding eigenstates. We normalize

$$(\xi,\xi)=1. \quad (2.7)$$

After the measurement, the apparatus is separated from the object, and subject to the second measurement to distinguish the states corresponding to the states of the measured object. For this second measurement we require the following conditions. First, the two states X and X' should be the two orthogonal eigenstates of the operator N of the second measurement, i.e.,

$$(X,X')=0. \quad (2.8)$$

Secondly, N should commute with the operator L_z of the conserved quantity for the apparatus, otherwise we cannot measure N exactly,[1] i.e.,

$$[N,L_z]=0. \quad (2.9)$$

This leads to the relation

$$(X_\mu,X_\mu')=0, \quad (2.10)$$

the X_μ being eigenfunctions of both N and L_z. Because of (2.4a) this can be written also as

$$(u_\mu+z_\mu,\, u_\mu+z_\mu)=(v_\mu+w_\mu,\, v_\mu+w_\mu). \quad (2.10a)$$

In an ideal measurement, both η and η' would be zero. It is not possible to accomplish this, but we shall try to find the minimum ϵ of the probability that the measurement was unsuccessful, i.e., the minimum of

$$\begin{aligned} \epsilon&=(\eta,\eta)+(\eta',\eta')\\ &=\tfrac{1}{2}[(u-z)^2+(v-w)^2]\\ &=\tfrac{1}{2}[(\textstyle\sum u_\mu-\sum z_\mu)^2+(\sum v_\mu-\sum w_\mu)^2], \end{aligned} \quad (2.11)$$

consistent with a definite "size" of the measuring apparatus, to be defined below.

The conservation law, together with (2.5), (2.6), (2.7), and the unitary nature of U gives

$$\begin{aligned} (\xi_\mu,\xi_\mu)&=(u_\mu,u_\mu)+(v_{\mu-1},v_{\mu-1})\\ &=(w_{\mu+1},w_{\mu+1})+(z_\mu,z_\mu); \end{aligned} \quad (2.12)$$

$$\begin{aligned} 1&=(u,u)+(v,v)=\textstyle\sum(u_\mu,u_\mu)+\sum(v_\mu,v_\mu)\\ &=(w,w)+(z,z)=\textstyle\sum(w_\mu,w_\mu)+\sum(z_\mu,z_\mu); \end{aligned} \quad (2.13)$$

and

$$(u_\mu,w_\mu)+(v_{\mu-1},z_{\mu-1})=0. \quad (2.14)$$

We define the "size" of the measuring apparatus as the mean square of the additive conserved quantity which we denote by M^2

$$(\xi,L_z^2\xi)=\sum\mu^2\xi_\mu^2=M^2. \quad (2.15)$$

The problem is, therefore, to obtain the smallest value of ϵ consistent with Eqs. (2.12) to (2.15), the ξ_μ, u_μ, v_μ, w_μ, z_μ, being otherwise arbitrary eigenstates of L_z to the eigenvalue μ. The ξ_μ can be eliminated from (2.13) by means of (2.10) and one obtains

$$\sum\mu^2(u_\mu^2+v_{\mu-1}^2)=\sum\mu^2(w_{\mu-1}^2+z_\mu^2)=M^2. \quad (2.16)$$

We note that (2.14) can be always satisfied because no equation depends on the angles between the u_μ and w_μ, or the v_μ and z_μ, except (2.14). It is easy to see that, in order to obtain the smallest possible ϵ at given $\|u_\mu\|$, $\|v_\mu\|$, $\|w_\mu\|$, $\|z_\mu\|$, the Hilbert vectors u_μ and z_μ must be parallel for all μ and the same applies for the vectors v_μ and w_μ.

To solve the preceding equations, we note that, because of (2.10a) and (2.12),

$$u_\mu+z_\mu=v_\mu+w_\mu, \quad (2.17)$$

$$u_\mu-z_\mu=(w_{\mu+1}^2-v_{\mu-1}^2)/(v_\mu+w_\mu), \quad (2.18)$$

where the u_μ, z_μ are now the *lengths* of the corresponding vectors (which are parallel), and the same applies to v_μ and z_μ.

We introduce new variables s_μ, t_μ such that

$$w_\mu+v_\mu=s_\mu, \quad (2.19)$$

$$w_\mu-v_\mu=t_\mu. \quad (2.20)$$

If ϵ is small, M is large and the s_μ, t_μ can be considered to be continuous functions of μ. It can be verified from

the solutions to be obtained that

$$s_\mu = O(M^{-\frac{1}{2}}), \quad \partial s_\mu/\partial\mu = \dot{s}_\mu = O(M^{-\frac{3}{2}}), \quad \text{etc.}; \qquad (2.21)$$

$$t_\mu = O(M^{-\frac{3}{2}}), \quad \partial t_\mu/\partial\mu = \dot{t}_\mu = O(M^{-\frac{5}{2}}), \quad \text{etc.} \qquad (2.22)$$

Using these estimates, we obtain from (2.18), (2.19), (2.20)

$$u_\mu - z_\mu = \dot{s}_\mu + t_\mu + O(M^{-7/2}). \qquad (2.23)$$

whereas (2.11) becomes

$$\begin{aligned}\epsilon &= \tfrac{1}{2}\{\textstyle\sum [\dot{s}_\mu + t_\mu + O(M^{-7/2})]^2 + \sum t_\mu{}^2\} \\ &= \tfrac{1}{2}\textstyle\sum [\dot{s}_\mu{}^2 + 2t_\mu{}^2 + 2\dot{s}_\mu t_\mu + O(M^{-5})]. \qquad (2.24)\end{aligned}$$

The s and t are still subject to the conditions (2.13) and (2.14) which now read

$$\tfrac{1}{2}\textstyle\sum [s_\mu{}^2 + O(M^{-3})] = 1, \qquad (2.25)$$

$$\tfrac{1}{2}\textstyle\sum \mu^2[s_\mu{}^2 + O(M^{-3})] = M^2. \qquad (2.26)$$

Since the t do not occur in these equations, the derivative of (2.24) with respect to t_μ will be zero at the minimum for ϵ:

$$2t_\mu = -\dot{s}_\mu + O(M^{-7/2}). \qquad (2.27)$$

Hence ϵ becomes

$$\epsilon = \textstyle\sum [\tfrac{1}{4}\dot{s}_\mu{}^2 + O(M^{-5})]. \qquad (2.28)$$

Using $-\lambda$ and $2m^2$ as the Lagrange multipliers for (2.25) and (2.26), we obtain the Euler equation for s_μ

$$-\ddot{s}_\mu + (4m^2\mu^2 - 2\lambda)s_\mu = 0, \qquad (2.29)$$

where terms of lower order of magnitude have been omitted. The solution of (2.29) is

$$s_\mu = c\exp(-m\mu^2); \quad c^2 = 2(m/\pi)^{\frac{1}{2}}. \qquad (2.30)$$

This already satisfies the normalization condition (2.25). The condition (2.26) gives

$$M^2 = 1/4m, \qquad (2.31)$$

and this gives, finally, for the probability (2.28) of an unsuccessful measurement

$$\epsilon = 1/8M^2, \qquad (2.32)$$

for a given M. Tracing through the omitted low-order terms, one finds that (2.32) is accurate up to terms of the order $1/M^3$.

3. CONCLUSIONS AND DISCUSSIONS

(1) The relation (2.32) tells us that the probability of an unsuccessful measurement is $1/8M^2$, if one uses the "best possible" apparatus with given M^2. Therefore, neglecting terms of the order $1/M^3$, the inequality

$$\epsilon > 1/8M^2 \qquad (3.1)$$

holds. In other words, with given M^2, we cannot make the probability of unsuccessful measurement smaller than $\hbar^2/8M^2$ (in dimensionless units). This limitation of the measurement is not the consequence of an uncertainty relation for the simultaneous measurement of *two* noncommuting operators, but the consequence of the existence of the additive conserved quantity whose operator does not commute with the *single* operator to be measured.

(2) We required the condition $(X_\mu, X_\mu') = 0$, (2.10), so that the conservation law does not interfere with the possibility of distinguishing X and X'. However, if we loosen this condition and require only $(X, X') = 0$, we obtain the same value for ϵ except for terms of order $1/M^3$ or lower. However, in this case the second measurement (which distinguishes X and X') can be carried out only approximately, and the total probability for an unsuccessful measurement becomes larger.

(3) The theory of measurement, as described by von Neumann,[2] contains no limitation for the size of the measuring apparatus. As we have seen, there always are such limitations unless the operator to be measured commutes with all the operators of additive conserved quantities.[3] Recently the macroscopic character of the measuring apparatus has been studied in detail by several authors,[4] but no quantitative conditions have been established before.

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[2] J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[3] See reference 1. We will discuss this problem, "exactly measurable operators," in another article.

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