Domains of Definition for Feynman Integrals over Real Feynman Parameters

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Given a Feynman diagram, the corresponding integral over real Feynman parameters is meaningful and analytic in a certain domain in the space of the Lorentz invariants formed from the external momenta, each of which is on the mass shell. In the case where all the masses are equal, the intersection of these domains for all proper convergent diagrams is studied. For the cases of four and five external lines, the real intersections are explicitly found; for the case of six, seven, and eight external lines, procedures for finding the real intersections are given. A knowledge of the real intersection makes it possible to construct geometrically a subset of the complex intersection. Generalization to unequal external masses is briefly considered.

I. INTRODUCTION

In order to study the analytic property of scattering and production amplitudes in perturbation theory, the first step is to express the contribution F_0 from a given Feynman diagram as the limiting value of an integral over Feynman parameters. Here the external particles are on their respective mass shells, and hence F_0 is a function of the remaining independent Lorentz invariants formed from the external momenta. In the case of the two-particle scattering amplitude, there are two such invariants. Furthermore, the integral is over non-negative values of the Feynman parameters. One can then define a function F of complex variables by formally the same integral over Feynman parameters but allowing the independent invariants to take on complex values. In the definition of F, the integral is still over non-negative values of the Feynman parameters. Thus it is a well-defined problem to find the conditions under which the integral is meaningful. It is the purpose of the present paper to study the region in the complex space of the invariants where the integral is meaningful and analytic for *every* proper convergent Feynman diagram. Of course complex values of the Feynman parameters are needed for the purpose of analytically continuing F as defined above, but throughout the present paper these parameters remain real and non-negative.

When there are selection rules, the Feynman diagrams must be constructed in accordance with these rules. For example, it is not permissible to have a three-nucleon vertex. Even in the simplest case where each of the diagrams has four external lines, the presence of the selection rules complicates the problem enormously. In the present paper, it is assumed throughout that there is no selection rule at all, and that three-point vertices are allowed. Furthermore, for simplicity it is also assumed that the particles involved all have no internal degrees of freedom, i.e., all propagators are of the form $(p^2+m^2-i\epsilon)^{-1}$, and that all the masses, both internal and external, are equal to 1. This last assumption can be relaxed, as discussed in Sec. 8. It should be emphasized that all questions concerning divergence and renormalization are to be ignored, and hence basically only convergent Feynman diagrams can be considered. It seems to be a very interesting but difficult problem to find the analytic property of various amplitudes after renormalization.

The method of studying the region mentioned above makes use of the close relation between a Feynman diagram and the corresponding electric circuit. Since the problem is algebraic in nature and it is certainly possible to work with the algebraic expressions directly, there is no necessity of introducing electric circuits. However, this analog proves to be very convenient. Of course, any linear system, for example a system of elastic rods, can serve equally well. The choice of electric circuits is made only for the sake of definiteness.

Consider the case $n\geq 4$, where n is the number of external lines, or the number of four-momenta from which invariants can be formed. In this case, the number of independent invariants is $3n-10$. The case $n=4$ is by far the simplest and probably the most familiar one; this case is treated in Sec. 2. In Sec. 3, some properties of electric circuits are written down as a preparation for the general case treated in Secs. 4–7. In Sec. 4, the relevance of the electric circuit to the Feynman diagram is established. In Sec. 5, the problem of the majorization of Feynman diagrams is discussed. Section 6 is devoted to the method of calculation for n between 5 and 8, and in Sec. 7, the calculation is explicitly carried through for the case $n=5$. It may be noted that there is no necessity of ever referring to the so-called "Euclidean region."

2. THE CASE $n=4$

Consider a Feynman diagram with four external lines. Let p_1, p_2, p_3, and p_4 be the four-momenta associated with these external lines, all pointing inward. Then

$$p_1+p_2+p_3+p_4=0, \tag{2.1}$$

and

$$p_1^2=p_2^2=p_3^2=p_4^2=1, \tag{2.2}$$

where the metric used is $(1, -1, -1, -1)$. Let s, t, and

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u be the Mandelstam variables[1]

$$s=(p_1+p_2)^2,\quad t=(p_1+p_3)^2,\quad \text{and } u=(p_1+p_4)^2, \tag{2.3}$$

then

$$s+t+u=4. \tag{2.4}$$

Let the r internal lines of the diagram be labeled by $i=1, 2, \cdots r$, and $\tilde{p}_i$ be the four-momentum associated with the internal line i. As shown by Symanzik[2] and Nambu,[3] a convergent Feynman diagram gives the following contribution to the scattering amplitude, except for a multiplicative constant,

$$F_0(s,t,u)$$

$$=\lim_{\epsilon\to 0+}\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2}$$

$$\times[Q(s,t,u;\alpha_i)-(1-i\epsilon)\textstyle\sum_i\alpha_i]^{-2N+r+2}, \tag{2.5}$$

where N is the number of vertices, $d(\alpha_i)$ is a non-negative function of α_i and is of no concern here, and Q is the extreme value of $\sum_i\alpha_i\tilde{p}_i^2$ under the constraint of the conservation of momentum at each vertex. Equation (2.5) holds in each of the three physical regions, i.e., the regions in the (s,t,u) space where the scattering processes $1+2\rightleftharpoons 3+4$, $1+3\rightleftharpoons 2+4$, and $1+4\rightleftharpoons 2+3$ are respectively possible kinematically. For example, the s physical region, corresponding to the first process, is given by

$$s\geq 4,\ t\leq 0,\quad \text{and } u\leq 0. \tag{2.6}$$

Since $\sum_i\alpha_i\tilde{p}_i^2$ is a quadratic form, Q must be linear in the variables s, t, and u. In view of (2.4), Q can be expressed uniquely in the following way

$$Q(s,t,u;\alpha_i)=\tfrac{1}{4}[sP_s(\alpha_i)+tP_t(\alpha_i)+uP_u(\alpha_i)]. \tag{2.7}$$

If (2.7) is used to define $Q(s,t,u;\alpha_i)$ for complex values of s, t, and u that satisfy (2.4), then it is possible to define F analogous to (2.5) as a function of two complex variables:

$$F(s,t,u)$$

$$=\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2}$$

$$\times[Q(s,t,u;\alpha_i)-\textstyle\sum_i\alpha_i]^{-2N+r+2}. \tag{2.8}$$

The problem is to study $\mathfrak{D}_4$, defined as the set of complex $\{s,t,u\}$ satisfying (2.4) such that for all proper convergent Feynman diagrams

$$Q(s,t,u;\alpha_i)\neq\textstyle\sum_i\alpha_i \tag{2.9}$$

for all non-negative α_i. Since Q is by definition linear in α_i, i.e., for positive Z,

$$Q(s,t,u;Z\alpha_i)=ZQ(s,t,u;\alpha_i), \tag{2.10}$$

the condition (2.9) may be expressed as

$$Q(s,t,u;\alpha_i)\neq 1 \tag{2.11}$$

for all non-negative α_i for which $\sum_i\alpha_i=1$. Let D_4 be subset of $\mathfrak{D}_4$ where s, t, and u are all real.

It follows from (2.7) that

$$P_s(\alpha_i)=Q(4,0,0;\alpha_i). \tag{2.12}$$

In particular, $P_s(\alpha_i)$ is the extremal value of $\sum_i\alpha_i\tilde{p}_i^2$ when $p_1=p_2=-p_3=-p_4=(1,0,0,0)$. This extremal value is the same as the extremal value of $\sum_i\alpha_i\tilde{p}_{i0}^2$ under the same circumstances, where $\tilde{p}_{i0}$ is the zeroth component of $\tilde{p}_i$. If an electric circuit is constructed from the Feynman diagram by replacing each internal line i by a resistor with resistance α_i, and replacing the four-momenta p_1, p_2, p_3, and p_4 by currents $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$, and $\hat{p}_4$ which are scalars that satisfy

$$\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{p}_4=0, \tag{2.13}$$

then $P_s(\alpha_i)$ is the power dissipated in the circuit when $\hat{p}_1=\hat{p}_2=-\hat{p}_3=-\hat{p}_4=1$. Thus

$$P_s(\alpha_i)\geq 0. \tag{2.14}$$

Furthermore, since the Feynman diagram is proper, for any $j\leq r$

$$P_s(\delta_{ij})=0. \tag{2.15}$$

In the language of circuit theory, when only one resistance is positive, all the vertices are short-circuited together. Similarly $P_t(\alpha_i)$ and $P_u(\alpha_i)$ are the amounts of power dissipated in the same circuit when $\hat{p}_1=-\hat{p}_2=\hat{p}_3=-\hat{p}_4=1$ and $\hat{p}_1=-\hat{p}_2=-\hat{p}_3=\hat{p}_4=1$, respectively. Thus it follows from (2.15) that for any $j\leq r$

$$Q(s,t,u;\delta_{ij})=0. \tag{2.16}$$

Since Q is a continuous function of α_i for non-negative α_i, (2.16) immediately gives the following condition for D_4 as a consequence of (2.11). A set of real $\{s,t,u\}\in D_4$ if and only if

$$Q(s,t,u;\alpha_i)<1 \tag{2.17}$$

for all non-negative α_i such that $\sum_i\alpha_i=1$. If $s\geq 4$, then consider the self-energy diagram shown in Fig. 1(a). In that case $P_t(\alpha_i)=P_u(\alpha_i)=0$ and thus

$$Q(s,t,u;\alpha_i)=s\alpha_1\alpha_2, \tag{2.18}$$

when $\alpha_1+\alpha_2=1$. This is incompatible with (2.17). Accordingly, D_4 is contained in the triangle[4]

$$s<4,\ t<4,\quad \text{and } u<4. \tag{2.19}$$

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] K. Symanzik, Progr. Theoret. Phys. (Kyoto) **20**, 690 (1958).
[3] Y. Nambu, Nuovo cimento **6**, 1064 (1957).

[4] Strictly speaking, the derivation of (2.19) from (2.17) should be modified as follows. A necessary condition for a Feynman integral to converge is that $2N\geq r+3$. This is not satisfied by the diagram of Fig. 1(a). To satisfy this condition, consider instead the diagram of Fig. 1(b). When $\alpha_3=0$, (2.18) holds and a contradiction with (2.17) is obtained.

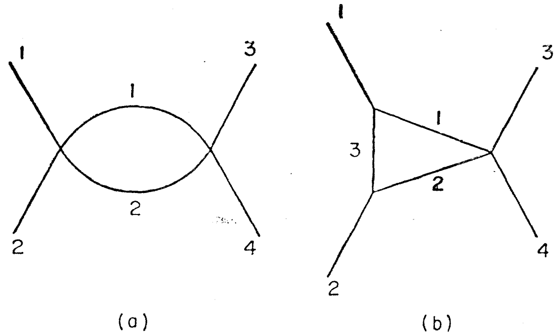


FIG. 1. Two simple Feynman diagrams.

Let Q_{13} be the power dissipated in the circuit when $\hat{p}_1=-\hat{p}_3=1$ and $\hat{p}_2=\hat{p}_4=0$, then Q_{13} is the resistance between the two points where the two external lines are attached. Under the constraint $\sum_i \alpha_i=1$, it is intuitively obvious that Q_{13} is largest among all proper Feynman diagrams for the diagram of Fig. 1 with $\alpha_1=\alpha_2=\frac{1}{2}$. Thus,

$$Q_{13}\leq\tfrac{1}{4} \tag{2.20}$$

for $\sum_i \alpha_i=1$. A formal proof in a more general case is given in Sec. 3. Similarly Q_{12}, Q_{32}, etc. may be defined. Let

$$Q_s=2(Q_{12}+Q_{34}),\quad Q_t=2(Q_{13}+Q_{24}),\quad \text{and } Q_u=2(Q_{14}+Q_{23})\,; \tag{2.21}$$

then it follows from (2.20) that

$$Q_s\leq 1,\ Q_t\leq 1,\quad \text{and } Q_u\leq 1, \tag{2.22}$$

and furthermore, it follows from the linearity of the circuit that

$$P_s=\tfrac{1}{2}(-Q_s+Q_t+Q_u), \tag{2.23}$$

and hence by (2.14)

$$Q_s\leq Q_t+Q_u,\ Q_t\leq Q_u+Q_s,\quad \text{and } Q_u\leq Q_s+Q_t. \tag{2.24}$$

With (2.22) and (2.24) it can immediately be verified that (2.19) implies (2.17). Hence D_4 is given by the triangle (2.19). This result is not new, and has indeed been used by Eden[5] and Landshoff, Polkinghorne, and Taylor[6] in their proofs of the Mandelstam representation in perturbation theory. However, the present proof seems to be more direct and conceptually simpler.

Consider next a set of complex numbers $\{s,t,u\}$ that satisfy (2.4). Let their real and imaginary parts be represented by s', t', u', s'', t'', and u''. If $\{s',t',u'\}\in D_4$, then clearly $\{s,t,u\}\in\mathfrak{D}_4$. If $\{s,t,u\}\notin\mathfrak{D}_4$, then there exists a Feynman diagram and non-negative numbers α_i such that $Q(s,t,u;\alpha_i)=1$. By (2.7) this implies that

$$\tfrac{1}{4}[s'P_s(\alpha_i)+t'P_t(\alpha_i)+u'P_u(\alpha_i)]=1$$

and

$$\tfrac{1}{4}[s''P_s(\alpha_i)+t''P_t(\alpha_i)+u''P_u(\alpha_i)]=0.$$

[5] R. J. Eden, Phys. Rev. **121**, 1567 (1961).

[6] P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor (to be published). See also reference 9.

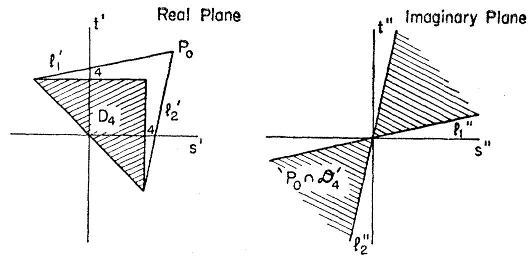


FIG. 2. The construction of $\mathfrak{D}_4'$.

Hence, for any real λ, $Q(s'+\lambda s'', t'+\lambda t'', u'+\lambda u'';\alpha_i)=1$, which means that the real point $\{s'+\lambda s'',\ t'+\lambda t'',\ u'+\lambda u''\}\notin D_4$. Take $\{s',t',u'\}\in D_4$, carry out the geometric construction as shown in Fig. 2, where the lines l_1'' and l_2'' are parallel to the lines l_1' and l_2', respectively. If $\{s'', t'', u''\}$ are in the shaded region in the imaginary plane as shown, then by the above argument $\{s,t,u\}\in\mathfrak{D}_4$. This result may be stated alternatively as follows: define a regime $\mathfrak{D}_4'$ in the complex $\{s,t,u\}$ space by the requirement that $\{s,t,u\}\in\mathfrak{D}_4'$ if and only if there exists a real number λ such that $\{s'+\lambda s'',\ t'+\lambda t'',\ u'+\lambda u''\}\in D_4$, then $\mathfrak{D}_4$ contains $\mathfrak{D}_4'$. Algebraically, $\mathfrak{D}_4'$ may be characterized as follows. Let $\mathfrak{D}_4{}^0$ be the set defined by the inequalities

$$s''\geq 0,\ t''\geq 0,\ u''<0$$

$$\operatorname{Im}[(u-4)/(s-4)]>0,$$

and

$$\operatorname{Im}[(u-4)/(t-4)]>0, \tag{2.25}$$

then $\mathfrak{D}_4'$ is the union of D_4 with the six sets obtained from $\mathfrak{D}_4{}^0$ by permutations of s, t, and u and complex conjugations. An alternative method of getting (2.25) is given in the Appendix, where it is shown that $\mathfrak{D}_4-\mathfrak{D}_4'$ is not empty. It may be of some interest to note that the envelope of holomorphy of $\mathfrak{D}_4'$ is $\mathfrak{D}_4'$ itself.

From the knowledge of $\mathfrak{D}_4'$, a connection between F and F_0 can be found. Given $\{s,t,u\}$ so that (2.6) is satisfied, choose real $\{\bar{s},\bar{t},\bar{u}\}$ so that

$$\{s+\lambda\bar{s},\ t+\lambda\bar{t},\ u+\lambda\bar{u}\}\in D_4 \tag{2.26}$$

for some $\lambda<0$. Given a Feynman diagram and a set of non-negative numbers $\{\alpha_i\}$ with $\sum_i \alpha_i=1$ such that

$$Q(s,t,u;\alpha_i)\geq 1,$$

then it follows from (2.26) that for $\epsilon>0$,

$$\begin{aligned}\operatorname{Im}Q(s+i\epsilon\bar{s},\ t+i\epsilon\bar{t},\ u+i\epsilon\bar{u};\alpha_i) &= (\epsilon/\lambda)[Q(s+\lambda\bar{s},\ t+\lambda\bar{t},\ u+\lambda\bar{u};\alpha_i)\\ &\quad -Q(s,t,u;\alpha_i)]>0.\end{aligned} \tag{2.27}$$

Thus, a comparison of (2.5) and (2.8) gives that

$$F_0(s,t,u)=\lim_{\epsilon\to 0} F(s+i\epsilon\bar{s},\ t+i\epsilon\bar{t},\ u+i\epsilon\bar{u}). \tag{2.28}$$

In other words, F_0 is a boundary value of F restricted to $\mathfrak{D}_4'$.

3. PROPERTIES OF AN ELECTRIC CIRCUIT

It has been seen in the last section how electric quantities can be used conveniently in obtaining properties of the scattering amplitude. In this section, relevent properties of an electric circuit are written down and a generalization of (2.20) is proved.

Consider a direct current circuit of N nodes and r resistances denoted by α's. Let the indices i, j, etc. be used to indicate the resistances, and the indices a, b, etc. to indicate the nodes. If the resistance i is connected between the nodes a and b, then there is a correspondence $i \leftrightarrow (a,b,\omega)$ where the index ω distinguishes the possibility of more than one resistance between the two nodes.

Let the current I_a be injected at node a, where I_a is real. To be consistent, it is necessary that

$$\sum_a I_a = 0. \tag{3.1}$$

Under (3.1), let V_b be the voltage at node b. Note that V_b is not defined to an additive constant independent of b. If $i \leftrightarrow (a,b,\omega)$, let

$$I_{ab\omega} = \alpha_i^{-1}(V_a - V_b), \tag{3.2}$$

and

$$I_i^2 = I_{ab\omega}^2. \tag{3.3}$$

Here $I_{ab\omega}$ is the current in the resistance α_i from node a to node b. Also note that $\alpha_i \geq 0$ and that the sign of I_i is not defined. The equation of continuity for current is

$$\sum_{b,\omega} I_{ab\omega} = I_a. \tag{3.4}$$

Equations (3.2) and (3.4) can be used to find V_b in terms of I_a. Define

$$P = \sum_a I_a V_a \tag{3.5}$$

as the power dissipated in the circuit. It follows from (3.2) that $I_{ab\omega}$ has the symmetry

$$I_{ab\omega} = -I_{ba\omega}, \tag{3.6}$$

and thus the substitution of (3.4) into (3.5) gives

$$P = \sum_i \alpha_i I_i^2. \tag{3.7}$$

Under the constraint (3.4), (3.2) is the condition such that P is stationary as given by (3.7), provided that the Lagrangian multipliers are identified with $-2V_a$ In other words, if

$$P' = P - 2\sum_a V_a\Big[\sum_{b,\omega} I_{ab\omega} - I_a\Big], \tag{3.8}$$

then (3.2) is identical with

$$\partial P'/\partial I_{ab\omega} = 0. \tag{3.9}$$

Now consider α_i for a fixed i and all $I_{ab\omega}$ as variables. Then it follows from (3.9) that

$$dP'/d\alpha_i = \partial P'/\partial \alpha_i = I_i^2, \tag{3.10}$$

provided that all I_a are fixed. Therefore, when P is considered to be a function of I_a and α_i,

$$\partial P/\partial \alpha_i = I_i^2. \tag{3.11}$$

Let a and b be two fixed nodes; consider the case $I_a = -I_b = 1$ with $I_c = 0$ for all $c \neq a$, b. Define V_{cab} to be the value of $V_c - V_b$, P_{ab} the value of P, and I_{iab} the value of I_i in this case. Then

$$P_{ab} = P_{ba} = V_{aab}. \tag{3.12}$$

The reciprocity theorem for electric circuits states that

$$V_{cab} = V_{acb}, \tag{3.13}$$

and the linearity of the circuit implies that in general

$$V_a - V_b = \sum_c I_c V_{acb}, \tag{3.14}$$

and hence,

$$P = \sum_{a,c} I_a I_c V_{acb}. \tag{3.15}$$

In particular,

$$P_{ac} = P_{ab} + P_{bc} - 2V_{cab}. \tag{3.16}$$

The substitution of (3.16) into (3.15) gives

$$P = \tfrac{1}{2}\sum_{a,c} I_a I_c [P_{ab} + P_{bc} - P_{ac}].$$

But it is a consequence of (3.1) that the first two terms here are zero. Thus,

$$P = -\tfrac{1}{2}\sum_{a,c} I_a I_c P_{ac} = -\sum_{a<b} I_a I_b P_{ab}. \tag{3.17}$$

Since

$$V_{aab} \geq V_{cab} \geq V_{bab} = 0, \tag{3.18}$$

it follows from (3.16) that

$$P_{ac} \leq P_{ab} + P_{ac}, \tag{3.19}$$

and in particular the three quantities $P_{ab}+P_{cd}$, $P_{ac}+P_{bd}$, and $P_{ad}+P_{bc}$ satisfy triangular inequalities. Note also that, if $V_{aab} = V_{cab}$ for $a \neq c$, then c and b are disconnected when a is removed, and that

$$|I_{iab}| \leq 1. \tag{3.20}$$

Next the effect of changing one α on P is to be considered. First define

$$\bar{\alpha}_i = (\alpha_i - P_{ab})^{-1}\alpha_i^2 \tag{3.21}$$

for $i \leftrightarrow (a,b,\omega)$. Electrically, $\bar{\alpha}_i$ is the resistance seen by a generator inserted in the resistance branch i. Suppose the resistance α_i is increased to $\alpha_i + \Delta\alpha_i$; then this change may be compensated by adding a voltage generator with voltage $I_i\Delta\alpha_i$ in this branch. Suppose ΔV_c is the voltage at node a due to this generator; then the change in P due to $\Delta\alpha_i$ is given by

$$\Delta P = -\sum_c I_c \Delta V_c. \tag{3.22}$$

Furthermore, the change in I_i is

$$\Delta I_{ab\omega} = -(\bar{\alpha}_i + \Delta\alpha_i)^{-1}\Delta\alpha_i I_{ab\omega}. \tag{3.23}$$

The ΔV_c can also be produced by the following external currents

$$\Delta I_a = -\Delta I_b = \frac{-\bar{\alpha}_i \Delta\alpha_i}{\alpha_i(\bar{\alpha}_i + \Delta\alpha_i)} I_{ab\omega}. \tag{3.24}$$

Thus

$$\Delta V_c = \frac{-\bar{\alpha}_i \Delta\alpha_i}{\alpha_i(\bar{\alpha}_i + \Delta\alpha_i)} I_{ab\omega} V_{cab}, \tag{3.25}$$

and it follows from (3.13), (3.14), and (3.22) that

$$\Delta P = I_i^2 \frac{\bar{\alpha}_i \Delta\alpha_i}{\bar{\alpha}_i + \Delta\alpha_i}. \tag{3.26}$$

This is the required result. Note that (3.26) is consistant with (3.23) and that, when $\Delta\alpha_i \to \infty$,

$$\Delta P \to I_i^2 \bar{\alpha}_i. \tag{3.27}$$

This gives the increase in power dissipation when a resistance is increased to infinity, or "removed."

Finally, the following question is considered. Let I_a be given; what is the maximum value of P under a single constraint of the form

$$\sum_i K_i \alpha_i = 1, \tag{3.28}$$

where $K_i > 0$ for all i? Actually, only the case where all $K_i = 1$ is needed in the present paper. The procedure is to find the extreme value of P first. Call it P_1; then P_1 is the extremal value of $\sum_i \alpha_i I_i^2$ under the constraints (3.28), (3.2), and (3.4) with the prescribed I_a. By (3.9), the constraint (3.2) can be removed. But the constraint (3.4) does not involve α_i; therefore, for each i either

$$(\partial/\partial\alpha_i)(\sum_i \alpha_i I_i^2 - \lambda \sum_i K_i \alpha_i) = 0 \quad \text{or } \alpha_i = 0,$$

i.e.,

$$I_i^2 = \lambda K_i \quad \text{or } \alpha_i = 0. \tag{3.29}$$

Since only rational functions are involved, P_1 can be correctly obtained if all $(K_i)^{\frac{1}{2}}$ are approximated by rational numbers and I_a approximated by numbers mutually irrational and finally a limit is taken. But with these approximations, (3.29) cannot be satisfied unless the situation is as follows. The set $\mathfrak{A}$ of all nodes a is split into disjoint sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that all nodes in $\mathfrak{A}_i$, $i=1$ or 2, are short circuited to each other. This means that if the nodes a and b both belong to $\mathfrak{A}_1$ or both belong to $\mathfrak{A}_2$, $i \leftrightarrow (a,b,\omega)$, then $\alpha_i = 0$. If $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2$, and $i \leftrightarrow (a,b,\omega)$, then there exists a number C independent of i such that

$$\alpha_i I_{ab\omega} = C. \tag{3.30}$$

Since all $I_{ab\omega}$ are of the same sign, it follows from (3.29) and (3.30) that

$$\alpha_i = C\lambda^{-\frac{1}{2}} K_i^{-\frac{1}{2}}. \tag{3.31}$$

The substitution of (3.31) into (3.28) gives

$$C = \lambda^{\frac{1}{2}} [\sum_i{}' (K_i)^{\frac{1}{2}}]^{-1}, \tag{3.32}$$

where $\sum_i{}'$ means summation over those i that satisfies $i \leftrightarrow (a,b,\omega)$ with $a \in \mathfrak{A}_1$ and $b \in \mathfrak{A}_2$. Let

$$I = \sum_{a \in \mathfrak{A}_1} I_a, \tag{3.33}$$

then

$$I = \lambda^{\frac{1}{2}} [\sum_i{}' (K_i)^{\frac{1}{2}}]. \tag{3.34}$$

Therefore, it follows from (3.29), (3.31), and (3.32) that

$$P_1 = I^2 [\sum_i{}' (K_i)^{\frac{1}{2}}]^{-2}. \tag{3.35}$$

The maximum value of P is thus to be found as follows. 1. Split the circuit into two disjoint connected pieces by removing a number of resistances. 2. Define I by (3.33) and let $K^{\frac{1}{2}}$ be the sum of $(K_i)^{\frac{1}{2}}$ over the resistances removed. 3. The maximum value of P is the maximum of I^2/K over all possible ways of carrying out step 1.

In the special case where all $K_i = 1$ with $I_a = -I_b = 1$ and $I_c = 0$ for all $c \neq a$, b, the result is that

$$\max P_{ab} = r_{ab}^{-2}, \tag{3.36}$$

where r_{ab} is the minimum number of resistances to be removed to disconnect a and b.

Let the currents I_a be injected from n "external lines," which are numbered by α, β, etc. The number of these external lines attached to a node may be anywhere between zero and n. Let $a = A(\alpha)$ denote the node to which the αth external line is attached; then if I_α is the current carried in the line α,

$$I_b = \sum_{\alpha=1}^{n} I_\alpha \delta_{b,A(\alpha)}. \tag{3.37}$$

Let

$$Q_{\alpha\beta} = P_{A(\alpha),A(\beta)}, \tag{3.38}$$

and

$$\bar{r}_{\alpha\beta} = r_{A(\alpha),A(\beta)}; \tag{3.39}$$

then (3.36) gives

$$Q_{\alpha\beta} \leq \bar{r}_{\alpha\beta}^{-2}, \tag{3.40}$$

which is a generalization of (2.20).

4. CIRCUIT ANALOG OF A FEYNMAN DIAGRAM

Some of the considerations in Sec. 2 are now to be generalized to the case $n > 4$. Consider a Feynman diagram with n external lines and the associated four momenta p_α, $\alpha = 1, 2, \cdots n$, that satisfy

$$\sum_\alpha p_\alpha = 0, \tag{4.1}$$

and

$$p_\alpha^2 = 1 \tag{4.2}$$

for each α. It is found convenient to use the invariants

$$t_{\alpha\beta} = t_{\beta\alpha} = p_\alpha p_\beta \tag{4.3}$$

for $\alpha \neq \beta$. It follows from (4.1) and (4.2) that these invariants satisfy

$$\sum_\beta t_{\alpha\beta} = -1 \tag{4.4}$$

for each α. In addition, when $n > 5$, there are nonlinear

relations between the t's

$$T_\xi(t_{ab})=0, \tag{4.5}$$

because there are only four space-time dimensions. The number of such relations is $\frac{1}{2}(n-4)(n-5)$. As stated in the Introduction, the number of independent t's is $3n-10$; but, if an arbitrarily large number of space-time dimensions is allowed so that (4.5) can be disregarded, the number of independent t's becomes $\frac{1}{2}n(n-3)$.

Associated with a Feynman diagram, there is a corresponding electric circuit obtained by calling the vertices nodes and by assigning a resistance α_i to each internal line. The labeling for the electric circuit used in the last section can then be taken over for the Feynman diagram also. Then, also shown by Symanzik[2] and Nambu,[3] (2.5) can be generalized to

$$F_0(t_{\alpha\beta})$$
$$=\lim_{\epsilon\to0+}\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2}$$
$$\times[Q(t_{\alpha\beta},\alpha_i)-(1-i\epsilon)\textstyle\sum_i\alpha_i]^{-2N+r+2}. \tag{4.6}$$

Explicitly, $d(\alpha_i)$ may be written as an $(N-1)\times(N-1)$ determinant,

$$d(\alpha_i)=\textstyle\prod_i\alpha_i\,\det B_{ab}, \tag{4.7}$$

where

$$B_{ab}=\begin{cases}-\sum_\omega(\alpha_{ab\omega})^{-1} & \text{for } a\neq b,\\ -\sum_{c\neq a}B_{ac} & \text{for } a=b,\end{cases} \tag{4.8}$$

with a, $b\leq N-1$ and $\alpha_{ab\omega}=\alpha_i$ if $i\leftrightarrow(a,b,\omega)$. In (4.6), Q is again the extremal value of $\sum_i\alpha_i\tilde{p}_i^2$ under the constraint of the conservation of momentum at each vertex. Equation (4.6) holds in the many physical regions, which may be characterized as follows. Let the n external momenta p_α be segregated into two disjoint sets $\tilde{A}_1$ and $\tilde{A}_2$ and consider the reaction written symbolically as $\tilde{A}_1\rightleftharpoons\tilde{A}_2$; then the $(\tilde{A}_1,\tilde{A}_2)$ physical region is the set in the real t_{ab} space such that there exist p_α satisfying (4.3) for which this reaction is kinematically possible. Because of momentum conservation, the physical region is empty unless each of the sets $\tilde{A}_1$ and $\tilde{A}_2$ consists of at least two elements. The $(\tilde{A}_1,\tilde{A}_2)$ physical region is contained in the set of $\{t_{\alpha\beta}\}$ satisfying

$$t_{\alpha\beta}\geq1, \tag{4.9}$$

if both α and β are in $\tilde{A}_1$ or both are in $\tilde{A}_2$, and

$$t_{\alpha\beta}\leq0, \tag{4.10}$$

if $\alpha\in\tilde{A}_1$, $\beta\in\tilde{A}_2$. Therefore, if $\tilde{A}_1\neq\tilde{A}_1'$, then the $(\tilde{A}_1,\tilde{A}_2)$ and the $(\tilde{A}_1',\tilde{A}_2')$ physical regions are disjoint.

The next step is to define F. Let

$$p_b'=\sum_{\alpha=1}^n p_\alpha\delta_{b,A(\alpha)} \tag{4.11}$$

analogous to (3.37). If p_a' were a scalar equal to I_a, then as shown in the last section it would be possible to define P in terms of p_a' and α_i. But P is a quadratic form in I_a as seen from (3.17); thus, formally P can still be defined even when each p_a' is a four-vector. The result is precisely the Q of (4.6). Therefore, by (3.17) and (3.38),

$$Q=-\sum_{a<b}p_a'p_b'P_{ab}. \tag{4.12}$$

By (4.11) and (4.3), (4.12) gives

$$Q(t_{\alpha\beta},\alpha_i)=-\sum_{\alpha<\beta}t_{\alpha\beta}Q_{\alpha\beta}(\alpha_i). \tag{4.13}$$

If (4.13) is used to define $Q(t_{\alpha\beta},\alpha_i)$ for complex values of $t_{\alpha\beta}$ which may or may not satisfy (4.5), then as a generalization of (2.8), F may be defined as a function of $\frac{1}{2}n(n-3)$ complex variables

$$F(t_{\alpha\beta})=\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2}$$
$$\times[Q(t_{\alpha\beta},\alpha_i)-\textstyle\sum_i\alpha_i]^{-2N+r+2}. \tag{4.14}$$

As a generalization of $\mathfrak{D}_4$, let $\mathfrak{D}_n$ be the region in the complex $\{t_{\alpha\beta}\}$ space where, for every proper convergent Feynman diagram with n external lines,

$$Q(t_{\alpha\beta},\alpha_i)\neq1 \tag{4.15}$$

for all non-negative α_i satisfying $\sum_i\alpha_i=1$. Strictly speaking, the region $\mathfrak{D}_n$ with $n=4$ is not $\mathfrak{D}_4$ because the variables used are different, but this point should not cause undue confusion. Since $Q_{\alpha\beta}$ is a well-defined quantity within the framework of electric circuits, $\mathfrak{D}_n$ can be studied in this framework. Let $\mathcal{R}_n$ be the set where all t's are real, and $\mathcal{S}_n$ be the set where (4.5) is satisfied. Define

$$D_n=\mathfrak{D}_n\cap\mathcal{R}_n, \tag{4.16}$$

$$\bar{\mathfrak{D}}_n=\mathfrak{D}_n\cap\mathcal{S}_n, \tag{4.17}$$

and

$$\bar{D}_n=\mathfrak{D}_n\cap\mathcal{R}_n\cap\mathcal{S}_n; \tag{4.18}$$

then the problem posed in the Introduction is to study $\bar{\mathfrak{D}}_n$.

Let $t_{\alpha\beta}'$ and $t_{\alpha\beta}''$ be the real and imaginary parts of $t_{\alpha\beta}$, respectively. As a generalization for $\mathfrak{D}_4'$, define a region $\mathfrak{D}_n'$ in the complex $\{t_{\alpha\beta}\}$ space by the requirement that $\{t_{\alpha\beta}\}\in\mathfrak{D}_n'$ if and only if there exists a real number λ uch that $\{t_{\alpha\beta}'+\lambda t_{\alpha\beta}''\}\in D_n$. Since all $Q_{\alpha\beta}$ are real, it follows from the identity

$$Q(t_{\alpha\beta},\alpha_i)=Q(t_{\alpha\beta}'+\lambda t_{\alpha\beta}'',\alpha_i)-(\lambda-i)\,\mathrm{Im}Q(t_{\alpha\beta},\alpha_i), \tag{4.19}$$

that, if $\{t_{\alpha\beta}\}\in\mathfrak{D}_n'$, (4.15) is satisfied. Accordingly,

$$\mathfrak{D}_n'\subset\mathfrak{D}_n. \tag{4.20}$$

In the present paper, only the subset $\mathfrak{D}_n'$ of $\mathfrak{D}_n$ is to be considered. This gives a subset of $\bar{\mathfrak{D}}_n$. For the purpose of studying $\mathfrak{D}_n'$, it is only necessary to find D_n, and

thus all $t_{\alpha\beta}$ may be considered to be real. The reasoning in Sec. 2 can be immediately generalized to give that for $j\leq r$

$$Q_{\alpha\beta}(\delta_{ij})=0, \quad Q(t_{\alpha\beta},\delta_{ij})=0, \tag{4.21}$$

and hence $\{t_{\alpha\beta}\}\in D_n$ if and only if $t_{\alpha\beta}$ are all real and for all proper convergent Feynman diagrams with n external lines

$$Q(t_{\alpha\beta},\alpha_i)<1 \tag{4.22}$$

for all non-negative α_i satisfying $\sum_i \alpha_i=1$.

It is also an immediate generalization from the case $n=4$ that if $\{t_{\alpha\beta}\}\in D_n$, then for any $\tilde{A}_1$

$$\sum t_{\alpha\beta}<2-\tfrac{1}{2}n(\tilde{A}_1), \tag{4.23}$$

where the sum is over all $\alpha<\beta$ such that α and β are in $\tilde{A}_1$, and $n(\tilde{A}_1)$ is the number of elements in $\tilde{A}_1$. If $\tilde{A}_1$ consists of two elements α and β, then (4.23) gives

$$t_{\alpha\beta}<1 \tag{4.24}$$

for all α and β. Thus, D_n does not intersect any physical region. Given $\{t_{\alpha\beta}\}$ in any physical region, then choose $\{\bar{t}_{\alpha\beta}\}$ so that $\{t_{\alpha\beta}+\lambda\bar{t}_{\alpha\beta}\}\in D_n$ for some $\lambda<0$. This is always possible if D_n is not empty. Note that $\{\bar{t}_{\alpha\beta}\}$ has to satisfy the constraints $\sum_\beta \bar{t}_{\alpha\beta}=0$. Then

$$F_0(t_{\alpha\beta})=\lim_{\epsilon\to 0+} F(t_{\alpha\beta}+i\epsilon\bar{t}_{\alpha\beta}). \tag{4.25}$$

This means that F_0 is a boundary value of F restricted to $\mathfrak{D}_n'$.

5. MAJORIZATION OF FEYNMAN DIAGRAMS

In the case $n=4$ considered above, all proper Feynman diagrams are studied simultaneously. This does not seem feasible when n is larger. Instead, in this section a result of the following type is to be obtained: if $t_{\alpha\beta}$ are all real, then (3.24) is satisfied for all proper Feynman diagrams if and only if (3.24) is satisfied for a particular subset $\mathfrak{F}$ of proper Feynman diagrams. This procedure of eliminating from consideration all Feynman diagrams except a small number of them has been called majorization. Problems of this type have been studied by Nambu,[3] Symanzik,[2] and more recently by Chernikov, Logunov, and Todorov.[7] The present treatment is somewhat different. It may be worth reemphasizing that no reference to the "Euclidean region" need ever be made here.

Given a set of real numbers $t_{\alpha\beta}$ that satisfy (4.3) but may or may not satisfy (4.5), there exist real vectors q_α such that

$$\textstyle\sum_\alpha q_\alpha=0, \tag{5.1}$$

and

$$q_\alpha q_\beta=t_{\alpha\beta} \tag{5.2}$$

provided that the number of components of each vector is allowed to be sufficiently large. The product $q_\alpha q_\beta$ is defined with an indefinite metric, i.e., if $q_{\alpha\mu}$ are the components of q_α, then

$$q_\alpha q_\beta=\textstyle\sum_\mu s_\mu q_{\alpha\mu}q_{\beta\mu}, \tag{5.3}$$

where s_μ equals $+1$ or -1 for each μ. The choice of s_μ depends on the prescribed numbers $t_{\alpha\beta}$. For example, suppose that there is a correspondence $\mu\leftrightarrow(\alpha,\beta)$ with $\alpha<\beta$, then the quantities q_γ defined by

$$q_{\gamma\mu}=|t_{\alpha\beta}|^{\frac{1}{2}}(\delta_{\alpha\gamma}-\delta_{\beta\gamma}) \tag{5.4}$$

satisfy (5.1) and (5.2) when s_μ are chosen to be

$$s_\mu=-t_{\alpha\beta}/|t_{\alpha\beta}|. \tag{5.5}$$

Other possible representations of q_α may be obtained by applying a transformation that preserves the indefinite metric with s_μ.

With q_α, define q_a' analogous to (4.11) by

$$q_b'=\sum_{\alpha=1}^{n} q_\alpha\delta_{b,A(\alpha)}, \tag{5.6}$$

and define the vectors J_i and $J_{ab\omega}$ such that $J_{i\mu}$ and $J_{ab\omega\mu}$ are the values of I_i and $I_{ab\omega}$ when $I_a=q_{a\mu}'$. It then follows from (4.13) and (5.2) that

$$Q(t_{\alpha\beta},\alpha_i)=-\sum_{\alpha<\beta} q_\alpha q_\beta Q_{\alpha\beta}(\alpha_i)=\textstyle\sum_i \alpha_i J_i^2, \tag{5.7}$$

where

$$J_i^2=\textstyle\sum_\mu s_\mu J_{i\mu}^2. \tag{5.8}$$

For a given Feynman diagram G and real $t_{\alpha\beta}$, let $\tilde{Q}(t_{\alpha\beta},G)$ be the maximum value of $Q(t_{\alpha\beta},\alpha_i)$ for non-negative α_i with $\sum_i \alpha_i=1$. It follows from (4.21) that

$$\tilde{Q}(t_{\alpha\beta},G)\geq 0 \tag{5.9}$$

for all G and all $t_{\alpha\beta}$. If G is proper, let G^i be the diagram obtained from G by removing the internal line i, and G_i be the diagram obtained from G^i by identifying a and b if $i\leftrightarrow(a,b,\omega)$.

Lemma 1. For all $t_{\alpha\beta}$ and all proper Feynman diagram G,

$$\tilde{Q}(t_{\alpha\beta},G)\leq\max[\tilde{Q}(t_{\alpha\beta},G^i),\tilde{Q}(t_{\alpha\beta},G_i)], \tag{5.10}$$

where i is any prescribed integer less than $r+1$.

Proof. Since the set of allowed values of α_j is compact, there exist allowed α_j^0 such that

$$\tilde{Q}(t_{\alpha\beta},G)=Q(t_{\alpha\beta},\alpha_j^0). \tag{5.11}$$

If $\alpha_i^0=0$, then (5.10) follows trivially. If $\alpha_i^0>0$ then, with the Lagrangian multiplier τ, it follows from (3.11) and

$$(\partial/\partial\alpha_i)[Q(t_{\alpha\beta},\alpha_j)-\tau\textstyle\sum_j\alpha_j]=0 \tag{5.12}$$

at $\alpha_j=\alpha_j^0$ that

$$J_i^2=\tau. \tag{5.13}$$

The value of τ is easily found to be

$$\tau=\tilde{Q}(t_{\alpha\beta},G). \tag{5.14}$$

[7] N. A. Chernikov, A. A. Logunov, and T. Todorov (to be published).

Thus, by (5.9),

$$J_i^2 \geq 0. \tag{5.15}$$

Equation (3.27) then implies that

$$\lim_{\alpha_i\to\infty} Q(t_{\alpha\beta};\alpha_1^0,\alpha_2^0,\cdots\alpha_{i-1}^0,\alpha_i,\alpha_{i+1}^0\cdots,\alpha_r^0) -Q(t_{\alpha\beta},\alpha_j^0)=J_i^2\bar{\alpha}_i^0\geq 0. \tag{5.16}$$

Equation (5.10) then follows from (5.9) and (5.16).

The following result is a direct consequence of (5.7).

Lemma 2. Suppose that G becomes disconnected when the vertex a is removed. Let $\mathfrak{A}_\nu(\nu=1,2,\cdots v)$ be the sets of vertices connected to each other after the removal of a, $\mathfrak{I}_\nu$ be the corresponding sets of internal lines, and G_ν be the Feynman diagram obtained from G by removing all α_i except those with $i\in\mathfrak{I}_\nu$ and identifying all vertices in $\mathfrak{A}_{\nu'}$ with a for $\nu'\neq\nu$. Then

$$\tilde{Q}(t_{\alpha\beta},G)=\max_\nu \tilde{Q}(t_{\alpha\beta},G_\nu). \tag{5.17}$$

Lemma 3. Suppose that $a=A(\gamma)$ and that $A(\alpha)\neq a$ for $\alpha\neq\gamma$. Also suppose that $i\leftrightarrow(a,b)$ and $j\leftrightarrow(a,c)$ with no ω needed, and that if $k\leftrightarrow(a,d,\omega)$, then $k=i$ or j. If G is proper, and if

$$\tilde{Q}(t_{\alpha\beta},G_i{}^j)<1,\ \tilde{Q}(t_{\alpha\beta},G_j{}^i)<1, \tag{5.18}$$

and

$$\tilde{Q}(t_{\alpha\beta},G_{ij})\leq 1, \tag{5.19}$$

where $G_i{}^j=(G_i)^j$ etc., then

$$\tilde{Q}(t_{\alpha\beta},G)<1. \tag{5.20}$$

Proof. Under the circumstances, by (5.11), if either $\alpha_i^0=0$ or $\alpha_j^0=0$, then this lemma is a direct consequence of lemma 1. If $\alpha_i^0>0$ and $\alpha_j^0>0$, then by (5.11)

$$\tilde{Q}(t_{\alpha\beta},G)\leq \lim_{\alpha_i\to\infty} Q(t_{\alpha\beta};\alpha_1^0,\cdots\alpha_{i-1}^0,\alpha_i,\alpha_{i+1}^0,\cdots\alpha_r^0) \leq\alpha_j^0+(1-\alpha_j^0)\tilde{Q}(t_{\alpha\beta},G_j{}^i). \tag{5.21}$$

In (5.21) the first equality sign holds only for $\tilde{Q}(t_{\alpha\beta},G)=0$. Equation (5.20) then follows from (5.21) with (5.18).

Lemma 4. Suppose that $b=A(\gamma)$, $c=A(\delta)$, and that $A(\alpha)\neq b$ or c for $\alpha\neq\gamma$ or δ. Also suppose that $i\leftrightarrow(a,b)$, $j\leftrightarrow(b,c)$, and $k\leftrightarrow(c,d)$ with no ω needed, and that if $l\leftrightarrow(b,e,\omega)$ or $l\leftrightarrow(c,e,\omega)$, then $l=i$, j, or k. If G is proper, let G_{bc} be the Feynman diagram obtained from G by removing i, j, and k, and identifying the nodes a and c and the nodes b and d. If, with $G_{ij}{}^k=(G_{ij})^k$ etc.,

$$\begin{aligned}&\tilde{Q}(t_{\alpha\beta},G_{ijk})<1,\quad \tilde{Q}(t_{\alpha\beta},G_{ij}{}^k)<1,\\ &\tilde{Q}(t_{\alpha\beta},G_{jk}{}^i)<1,\quad \tilde{Q}(t_{\alpha\beta},G_{ki}{}^j)<1,\end{aligned} \tag{5.22}$$

and

$$\tilde{Q}(t_{\alpha\beta},G_{bc})<1, \tag{5.23}$$

then

$$\tilde{Q}(t_{\alpha\beta},G)<1, \tag{5.24}$$

provided that

$$(q_\gamma+q_\delta)^2<4. \tag{5.25}$$

Proof. If $\alpha_j^0>0$, then the proof of lemma 3 applies here. Thus, it is sufficient to consider the special case $b=c$. In this case, it follows from

$$q_\gamma{}^2=q_\delta{}^2=1, \tag{5.26}$$

$$(J_i^0)^2=(J_k^0)^2=\tilde{Q}(t_{\alpha\beta};G), \tag{5.27}$$

and

$$q_\gamma+q_\delta=J_{ba}+J_{cd}, \tag{5.28}$$

that

$$(J_{ba}-q_\gamma)^2+(J_{cd}-q_\gamma)^2=\tilde{Q}(t_{\alpha\beta},G)-t_{\gamma\delta}. \tag{5.29}$$

If $\tilde{Q}(t_{\alpha\beta},G)\geq 1$, then either

$$(J_{ba}-q_\gamma)^2>0 \tag{5.30a}$$

or

$$(J_{cd}-q_\gamma)^2>0. \tag{5.30b}$$

In the case of (5.30b), it follows from (2.27) that either

$$\tilde{Q}(t_{\alpha\beta},G)=\tilde{Q}(t_{\alpha\beta},G_{ijk}), \tag{5.31}$$

or

$$\tilde{Q}(t_{\alpha\beta},G)<\alpha_i+\alpha_k+(1-\alpha_i-\alpha_k)\tilde{Q}(t_{\alpha\beta},G_{bc}). \tag{5.32}$$

This is a contradiction. Since the same argument applies in the case of (5.30a), lemma 4 is proved.

From the inequality

$$\tilde{Q}(t_{\alpha\beta},G)\geq\tilde{Q}(t_{\alpha\beta},G_i) \tag{5.33}$$

and the above four lemmas, together with (4.24), it is seen that in $\mathfrak{F}$ it is sufficient to include only proper diagrams G that satisfy the following conditions:

(i) For all i, G^i is not proper.

(ii) For any vertex a, the removal of a does not make the diagram disconnected.

(iii) In the situation prescribed in lemma 3, $G_i{}^j$ is not proper.

(iv) In the situation prescribed in lemma 4, $G_{ij}{}^k$ is not proper.

(v) There does not exist $\bar{G}\in\mathfrak{F}$ such that $G=\bar{G}_i$.

Define a loop diagram L_n as a proper Feynman diagram with n external lines satisfying the conditions that $A(\alpha)\neq A(\beta)$ for $\alpha\neq\beta$, and that for each i there exists $a=B(i)$ such that $i\leftrightarrow(a,a+1)$ with the interpretation $(N,N+1)=(N,1)$. For L_n,

$$n=N=r. \tag{5.34}$$

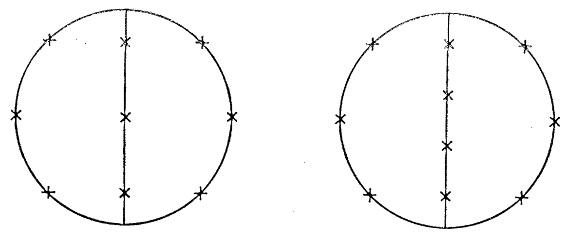
Only the cases $n\geq 3$ are to be considered in order to avoid divergence. The above consideration yields in particular the following theorem.

Theorem 1. $\{t_{\alpha\beta}\}\in D_n$ if and only if

$$\tilde{Q}(t_{\alpha\beta},G)<1 \tag{5.35}$$

for all $G\in\mathfrak{F}_n$, where $\mathfrak{F}_n$ consists of loop diagrams only for $4\leq n\leq 8$, $\mathfrak{F}_9$ contains the diagram shown in Fig. 3(a) besides the loop diagrams, and $\mathfrak{F}_{10}$ contains the one in Fig. 3(b) besides the loop diagrams, where the crosses denote the places where the external lines are attached.

This theorem is supplemented by the following.



(a) (b)

FIG. 3. Feynman diagrams for $\mathfrak{F}_9$ and $\mathfrak{F}_{10}$.

Theorem 2.[8] If $n\geq 11$, D_n is empty.

Proof. It follows from (4.13) and (4.22) that D_n is convex. Also D_n is symmetrical under permutations of the index α. Thus if D_n is not empty, then

$$\{t_{\alpha\beta}=-(n-1)^{-1}\}\in D_n. \tag{5.36}$$

Thus,

$$(n-1)^{-1}\sum_{\alpha<\beta} Q_{\alpha\beta}(\alpha_i)<1. \tag{5.37}$$

For the case of L_n with all $\alpha_i=n^{-1}$, it is found that

$$(n-1)^{-1}\sum_{\alpha<\beta} Q_{\alpha\beta}=(n+1)/12. \tag{5.38}$$

Therefore, when $n\geq 11$, (5.37) and (5.38) are not consistent. This proves the theorem.

6. LOOP DIAGRAMS

In view of theorem 1, the loop diagrams L_n are to be studied in some detail here. Because of (5.34), the function $A(\alpha)$ is a permutation and the function $B(i)$ can be chosen to be the identity, i.e., $i=a\leftrightarrow(a,a+1)$. Define a region D_n' by the requirement that $\{t_{\alpha\beta}\}\in D_n'$ if and only if

$$\tilde{Q}(t_{\alpha\beta},L_n)<1 \tag{6.1}$$

for all permutations A. Also define a region D_n'' in the following way. Let M_A be an $n\times n$ matrix given by

$$M_{ii}=1,$$
$$M_{i,i+1}=M_{i+1,i}=M_{1n}=M_{n1}=\tfrac{1}{2},$$

and otherwise,

$$M_{ij}=M_{ji}=1-\tfrac{1}{2}(\textstyle\sum q_\alpha)^2, \tag{6.2}$$

where the sum is over those α satisfying $\min(i,j)<A(\alpha)\leq\max(i,j)$. Note that M_A is completely determined by $t_{\alpha\beta}$. Let $M_A{}^\nu(\nu=1,2,\cdots 2^n-1)$ be a principal minor of M_A. Define a set $\mathcal{S}_A{}^\nu$ in the $t_{\alpha\beta}$ space by the requirement that on $\mathcal{S}_A{}^\nu$ $\det M_A{}^\nu=0$ and all cofactors of $M_A{}^\nu$ are of the same sign, i.e., either all positive or all negative. If $\mathcal{S}$ is the union of $\mathcal{S}_A{}^\nu$ for all ν and all A, and $\mathcal{S}'$ the complement of $\mathcal{S}$, then D_n'' is defined to be the connected component of $\mathcal{S}'$ that contains the point $\{t_{\alpha\beta}=-(n-1)^{-1}\}$.

[8] This result is due to Professor C. N. Yang (private communication).

Theorem 3. $D_n'=D_n''$.

Proof. Consider a point $\{t_{\alpha\beta}\}$ on the boundary $\partial D_n'$ of D_n'. For this $\{t_{\alpha\beta}\}$, there exists an A such that

$$\tilde{Q}(t_{\alpha\beta},L_n)=1. \tag{6.3}$$

For this L_n, it is possible to choose $\alpha_i{}^0$ such that

$$Q(t_{\alpha\beta},\alpha_i{}^0)=1. \tag{6.4}$$

Let $J_i{}^0$ be the value of J_i for this choice of $\alpha_i{}^0$; then

$$\textstyle\sum_i{}'\,\alpha_i{}^0J_i{}^0=0, \tag{6.5}$$

where the sum is taken over those i such that $\alpha_i{}^0\neq 0$, and the convention is used that $J_i=J_{a,a+1}$ for $i\leftrightarrow(a,a+1)$. For each j, either $\alpha_j{}^0=0$ or

$$(J_i{}^0)^2=1. \tag{6.6}$$

If there exist numbers $\alpha_i{}^1$ not proportional to $\alpha_i{}^0$ that satisfies

$$\textstyle\sum_i{}'\,\alpha_i{}^1J_i{}^0=0, \tag{6.7}$$

then

$$\textstyle\sum_i{}'\,\alpha_i{}^{0\prime}J_i{}^0=0, \tag{6.8}$$

where

$$\alpha_i{}^{0\prime}=K_1[\alpha_i{}^0-K_2\alpha_1{}^1], \tag{6.9}$$

with K_1 is determined by the requirement $\sum_i{}'\,\alpha_i{}^{0\prime}=1$ and K_2 is the minimum value of $\alpha_i{}^0/\alpha_i{}^1$ taken over the set of i where $\alpha_i{}^0\neq 0$. Then the set $\{\alpha_i{}^{0\prime}\}$ can be chosen instead of $\{\alpha_i{}^0\}$. Therefore, without loss of generality, it may be assumed that the set $\{\alpha_i{}^1\}$ does not exist, i.e., if

$$\textstyle\sum_i{}'\,\alpha_iJ_i{}^0=0, \tag{6.10}$$

then α_i is proportional to $\alpha_i{}^0$.

Choose ν so that $M_A{}^\nu$ is obtained from M_A by deleting those rows and columns of M_A that correspond to $\alpha_i{}^0=0$. Let $M_A{}^\nu$ be an $n_\nu\times n_\nu$ matrix, let the elements of $M_A{}^\nu$ be $(M_A{}^\nu)_{xy}$, and the cofactors be $(M_A{}^\nu)^{xy}$, where $x,y=1,\cdots n_\nu$. Let $\bar{J}_x$ be the same as $J_i{}^0$ for $\alpha_i{}^0\neq 0$ and α_x the same as $\alpha_i{}^0\neq 0$ except a renumbering of the index. Choose a representation such that

$$\bar{J}_{x\mu}=0, \tag{6.11}$$

for $\mu\geq n_\nu$. Let $\tilde{J}$ be the $n_\nu\times n_\nu$ matrix formed from $\bar{J}_{x\mu}$ with $\mu\leq n_\nu$; then by (6.5) $\tilde{J}$ is singular. On the other hand, by (6.2) and (6.6),

$$(M_A{}^\nu)_{xy}=\bar{J}_x\bar{J}_y. \tag{6.12}$$

If

$$\bar{J}_x\bar{J}_y=\textstyle\sum_\mu s_\mu{}'\bar{J}_{x\mu}\bar{J}_{y\mu}, \tag{6.13}$$

and S be the matrix with elements

$$S_{xy}=s_x{}'\delta_{xy}, \tag{6.14}$$

then

$$M_A{}^\nu=\tilde{J}S\tilde{J}^\dagger. \tag{6.15}$$

Accordingly,

$$\det M_A{}^\nu=0. \tag{6.16}$$

It then follows from determinent expansion and sym-

metry that

$$\sum_y (M_A{}^\nu)_{xy}(M_A{}^\nu)^{x'y}=0, \tag{6.17}$$

which leads to, by (6.12),

$$\sum_y \bar{J}_y(M_A{}^\nu)^{x'y}=0, \tag{6.18}$$

and hence to the existence of a real number $C_0\neq 0$ so that

$$(M_A{}^\nu)^{xy}=C_0\alpha_x\alpha_y. \tag{6.19}$$

Therefore,

$$\{t_{\alpha\beta}\}\in \mathcal{S}_A{}^\nu \tag{6.20}$$

with this particular choice of A and ν, or

$$\partial D_n'\subset \mathcal{S}. \tag{6.21}$$

Conversely, by the same type of argument, it can be shown that given $\{t_{\alpha\beta}\}\in\mathcal{S}_A{}^\nu$, it is possible to choose $\alpha_i{}^0$ such that

$$Q(t_{\alpha\beta},\alpha_i{}^0)=1 \tag{6.22}$$

for the graph L_n with the permutation A. Accordingly,

$$D_n'\subset \mathcal{S}'. \tag{6.23}$$

Theorem 3 then follows from (6.21), (6.23) and the convexity of D_n'.

Corollary 1. Let D_n''' be constructed in the same way as D_n'' except that only those $M_A{}^\nu$ with $n_\nu\leq 5$ are considered; then

$$D_n'''\cap\mathcal{S}_n=D_n'\cap\mathcal{S}_n. \tag{6.24}$$

Corollary 2. For $3<n\leq 8$,

$$D_n=D_n'=D_n''. \tag{6.25}$$

At least in the four cases $5\leq n\leq 8$, this gives in principle an explicit prescription to construct D_n. The following result gives the geometry of $\mathcal{S}$.

Theorem 4. At every point on the boundary $\partial\mathcal{S}_A{}^\nu$, at least one principal cofactor of $M_A{}^\nu$ is zero. Furthermore, $\partial\mathcal{S}_A{}^\nu\subset\mathcal{S}$.

Proof. If (6.16) is satisfied, then it is always possible to find C_0 and α_x so that (6.19) is satisfied and $\sum_x\alpha_x=1$. Let $\{t_{\alpha\beta}{}^0\}\in\partial\mathcal{S}_A{}^\nu$, then as $\{t_{\alpha\beta}\}\to\{t_{\alpha\beta}{}^0\}$ in $\mathcal{S}_A{}^\nu$, at least one α_x approaches zero. Suppose $\alpha_1\to 0$, and all other α's remain positive. Let $M_A{}^{\nu\prime}$ be the matrix obtained from $M_A{}^\nu$ by deleting the first row and first column, then

$$\det M_A{}^{\nu\prime}=0, \tag{6.26}$$

and

$$(M_A{}^{\nu\prime})^{xy}=(M_A{}^\nu)^{xy} \tag{6.27}$$

for x, $y>1$. Thus, $\{t_{\alpha\beta}{}^0\}\in\mathcal{S}_A{}^{\nu\prime}$. The case where more than one α approach zero simultaneously can be similarly treated.

7. THE CASE $n=5$

As an application of the results of the last section, D_5 [defined to be the real region in the space of the invariants where the Feynman integral is defined in terms of real Feynman parameters, see (4.16) and the paragraph preceding it] is to be found explicitly here. Consider the situation where $A(\alpha)=\alpha$, and write M_A as

$$M_A=\begin{bmatrix} 1 & \frac{1}{2} & -t_{23} & -t_{51} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & -t_{34} & -t_{12} \\ -t_{23} & \frac{1}{2} & 1 & \frac{1}{2} & -t_{45} \\ -t_{51} & -t_{34} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -t_{12} & -t_{45} & \frac{1}{2} & 1 \end{bmatrix}. \tag{7.1}$$

Let the rows and columns kept be used as the index for $M_A{}^\nu$; for example,

$$M_A{}^{245}=\begin{bmatrix} 1 & -t_{34} & -t_{12} \\ -t_{34} & 1 & \frac{1}{2} \\ -t_{12} & \frac{1}{2} & 1 \end{bmatrix}. \tag{7.2}$$

The case where $n_\nu=2$ gives the following planes: $\mathcal{S}_A{}^{35}$: $t_{45}=1$; $\mathcal{S}_A{}^{25}$: $t_{12}=1$; $\mathcal{S}_A{}^{24}$: $t_{34}=1$; $\mathcal{S}_A{}^{14}$: $t_{51}=1$; and $\mathcal{S}_A{}^{13}$: $t_{23}=1$. By permutation, $\mathcal{S}$ contains the following planes:

$$t_{\alpha\beta}=1 \tag{7.3}$$

for all α and β.

Consider next the case $n_\nu=3$. For example, (7.2) gives for $\mathcal{S}_A{}^{245}$

$$t_{12}{}^2+t_{34}{}^2-t_{12}t_{34}-\tfrac{3}{4}=0, \tag{7.4}$$

provided that

$$t_{34}-\tfrac{1}{2}t_{12}>0,$$

and

$$t_{12}-\tfrac{1}{2}t_{34}>0. \tag{7.5}$$

Thus $\mathcal{S}_A{}^{245}$ is the segment of the ellipse (7.4) with $t_{12}+t_{34}>\frac{3}{2}$. By permutation, for α, β, γ, and δ all unequal, $\mathcal{S}$ contains the segment of the ellipse

$$t_{\alpha\beta}{}^2+t_{\gamma\delta}{}^2-t_{\alpha\beta}t_{\gamma\delta}-\tfrac{3}{4}=0, \tag{7.6}$$

where

$$t_{\alpha\beta}+t_{\gamma\delta}>\tfrac{3}{2}. \tag{7.7}$$

Theorem 5. Let $D_5{}^0$ be the region in the t space where, for α, β, γ, and δ unequal,

$$t_{\alpha\beta}<1, \tag{7.8}$$

and furthermore,

$$t_{\alpha\beta}{}^2-t_{\gamma\delta}{}^2-t_{\alpha\beta}t_{\gamma\delta}-\tfrac{3}{4}<0, \tag{7.9}$$

provided that (7.7) is satisfied; then

$$D_5=D_5{}^0. \tag{7.10}$$

Proof. It is clear that $D_5\subset D_5{}^0$. In $D_5{}^0$, it follows from (7.8) that

$$\sum_i (M_A)_{ij}>0. \tag{7.11}$$

Therefore, $\mathcal{S}_A{}^{12345}$ does not intersect $D_5{}^0$. Similar considerations give the result that on $\mathcal{S}_A{}^{2345}$

$$t_{12}>\tfrac{1}{2}, \tag{7.12}$$

$$t_{12}+t_{34}>\tfrac{1}{2},\quad t_{12}+t_{45}>\tfrac{1}{2}, \tag{7.13}$$

and either

$$t_{34}>\tfrac{1}{2}\quad\text{or } t_{45}>\tfrac{1}{2}. \tag{7.14}$$

But $\det M_A{}^{234}$ and $\det M_A{}^{345}$ must have the same sign; thus (7.12) and (7.14) may be replaced by

$$t_{12}>\tfrac{1}{2},\ t_{34}>\tfrac{1}{2},\quad\text{and } t_{45}>\tfrac{1}{2}. \tag{7.15}$$

By theorem 4, on $\partial\mathcal{S}_A{}^{2345}$, either

$$\det M_A{}^{245}=0 \quad \text{or} \det M_A{}^{235}=0. \tag{7.16}$$

In the former case, (7.4) holds and

$$t_{12}(1+4t_{45})-2t_{34}(1+t_{45})=\tfrac{3}{2}. \tag{7.17}$$

Given t_{45}, (7.4) and (7.17) cannot have a double root in the region where (7.15) is satisfied. When $t_{12}=\frac{1}{2}$, (7.4) and (7.17) give $t_{34}=-\frac{1}{2}$ and $t_{45}=0$; when $t_{34}=t_{45}$, (7.4) and (7.17) give either $t_{12}=-1$, $t_{34}=-\frac{1}{2}$ or $t_{12}=-\frac{1}{2}$, $t_{34}=-1$ or $t_{12}=1$, $t_{34}=\frac{1}{2}$. Thus $\partial\mathcal{S}_A{}^{2345}$ cannot intersect $\partial D_5{}^0$, since D_5 is convex. Theorem 5 then follows.

In conventional terminology, (7.3) gives the normal thresholds, while (7.6) and (7.7) give the anomalous thresholds. The boundary of D_5 consists of thresholds only. Finally the geometrical construction of $\mathfrak{D}_5'$ from D_5 is entirely analogous to the case $n=4$.

8. DISCUSSIONS

The basic difference between the cases $n=4$ and $n=5$ is the presence of the condition (7.10) for D_5. Since the boundary of D_5 contains anomalous thresholds, the appearance of complex singularities for the production amplitude in perturbation theory cannot be avoided. This point has been discussed by Eden, Landshoff, Polkinghorne, and Taylor.[9] Furthermore, since the proof of theorem 2 requires the consideration of the eleven-point loop diagram with all α's nonzero, it may be expected that the boundary of D_n for $n>5$ does not consist of thresholds only.

It remains to discuss the possibility of different masses, still under the assumption of the absence of selection rules. It is further assumed that each one of the finite number of admissible masses is positive; then it is possible to choose units so that the smallest admissible mass is 1. Consider a Feynman diagram G where a mass m_i is associated with the internal line i; then (4.14) is replaced by

$$F(t_{\alpha\beta})=\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2} \times[Q(t_{\alpha\beta},\alpha_i)-\textstyle\sum_i m_i^2\alpha_i]^{-2N+r+2}. \tag{8.1}$$

On the one hand, it follows from $m_i\geq 1$ and (4.22) that the right-hand side (8.1) is analytic in D_n. On the other hand, in the absence of selection rule it is necessary to consider the Feynman diagram which is identical to G except that a mass 1 is instead associated with the internal line i. Accordingly, so far as the problems treated in the present paper are concerned, there is no loss of generality in assuming all the internal masses to be 1.

A number of the results obtained so far are also valid even when the external masses m_α are not equal. The internal masses are still assumed to be 1.

Theorem 6. If

$$m_\alpha\leq\sqrt{2} \tag{8.2}$$

for all α, then lemmas 1–4 and theorem 1 are still valid. Also D_4 is still given by (2.19).

Proof. The problem is to show that lemmas 3 and 4 hold under the condition (8.2). Equation (3.27) needs to be used. For lemma 3, it is only necessary to replace (5.21) by the following argument. Since

$$\tilde{Q}(t_{\alpha\beta},G)=\lim_{\alpha_i\to\infty}Q(t_{\alpha\beta};\alpha_1{}^0,\cdots,\alpha_{i-1}{}^0,\alpha_i,\alpha_{i+1}{}^0,\cdots\alpha_r{}^0) -\tilde{Q}(t_{\alpha\beta},G)\bar{\alpha}_i{}^0, \tag{8.3}$$

and

$$\lim_{\alpha_i\to\infty}Q(t_{\alpha\beta};\alpha_1{}^0,\cdots,\alpha_{i-1}{}^0,\alpha_i,\alpha_{i+1}{}^0,\cdots\alpha_r{}^0) \leq\alpha_j{}^0m_\gamma{}^2+(1-\alpha_j{}^0)\tilde{Q}(t_{\alpha\beta},G_j{}^i), \tag{8.4}$$

(5.10) follows from (8.2) and the inequality

$$\bar{\alpha}_i{}^0>\alpha_j{}^0. \tag{8.5}$$

Although somewhat more complicated, the modification of the proof of lemma 4 is of the same nature.

The validity of theorem 2 requires an inequality on m_α in the opposite direction. To generalize theorem 3, a suitable modification in the definition of D_n'' is needed. On the other hand, if (8.2) is violated for more than two α's, the situation may become much more complicated.

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APPENDIX

Equation (2.25) may be obtained alternatively as follows. Let

$$s=s'+is'',\quad t=t'+it'',\quad \text{and } u=u'+iu''. \tag{A1}$$

Since $\mathfrak{D}_4$ is invariant under complex conjugation and permutations of s, t, and u, it is sufficient to study the case

$$s''\geq 0,\ t''\geq 0,\quad \text{and } u''<0. \tag{A2}$$

Consider such a fixed set of values satisfying

$$s''+t''+u''=0 \tag{A3}$$

and

$$s''Q_s+t''Q+u''Q_u=0. \tag{A4}$$

Equation (A4) defines a plane in the space of Q_s, Q_t, and Q_u. Call the plane $\mathcal{P}$. Define also the following

[9] R. J. Eden, P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), pp. 227, 234.

planes:

$$\begin{cases}\mathcal{P}_1: Q_s=Q_t+Q_u, \\ \mathcal{P}_2: Q_t=Q_u+Q_s, \\ \mathcal{P}_3: Q_u=Q_s+Q_t,\end{cases} \quad \begin{cases}\mathcal{P}_4: Q_s=1, \\ \mathcal{P}_5: Q_t=1, \\ \mathcal{P}_6: Q_u=1.\end{cases} \tag{A5}$$

Furthermore, define $\mathcal{Q}$ to be the set where Q_s, Q_t, and Q_u satisfy the three triangular inequalities. If $\mathcal{Q}$ and the planes are projected along the line $Q_s=Q_t=Q_u$, the resulting situation is shown in Fig. 4. If (A2) is satisfied and $\partial\mathcal{Q}$ is the boundary of $\mathcal{Q}$, then the intersections $\mathcal{P}\cap\mathcal{P}_3\cap\partial\mathcal{Q}$ and $\mathcal{P}\cap\mathcal{P}_6\cap\partial\mathcal{Q}$ each consist of the two points (0,0,0) and (1,1,1) only. With this knowledge, it is straightforward to compute the four vertices of $\mathcal{P}\cap\mathcal{Q}$:

$$\begin{aligned}
&1.\ \mathcal{P}\cap\mathcal{P}_1\cap\mathcal{P}_2: && Q_s=Q_t=Q_u=0,\\
&2.\ \mathcal{P}\cap\mathcal{P}_4\cap\mathcal{P}_5: && Q_s=Q_t=Q_u=1,\\
&3.\ \mathcal{P}\cap\mathcal{P}_1\cap\mathcal{P}_4: && Q_s=1,\, Q_t=-t''/(u''-t''),\\
& && \text{and } Q_u=u''/(u''-t''),\\
&4.\ \mathcal{P}\cap\mathcal{P}_2\cap\mathcal{P}_5: && Q_s=-s''/(u''-s''),\, Q_t=1,\\
& && \text{and } Q_u=u''/(u''-s'').
\end{aligned} \tag{A6}$$

$$Q=\begin{cases}0 & \text{at vertex 1,}\\ \frac{1}{2} & \text{at vertex 2,}\\ \frac{1}{4}(u't''-t'u'')/(t''-u'') & \text{at vertex 3,}\\ \frac{1}{4}(u's''-s'u'')/(s''-u'') & \text{at vertex 4.}\end{cases} \tag{A7}$$

This immediately gives (2.25).

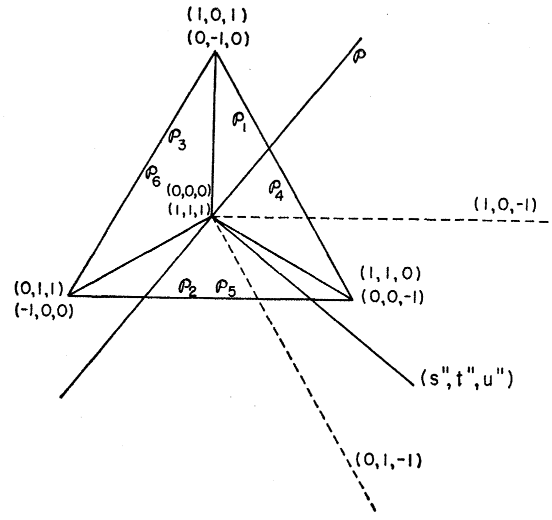

FIG. 4. Geometry for $\mathfrak{D}_4'$.

The important point here is that, contrary to the case of D_4, the vertices 3 and 4 are in general not on the boundary of $\mathcal{Q}'$, where $\mathcal{Q}'$ is the set of $\{Q_s,Q_t,Q_u\}$ that can be realized with non-negative α_i satisfying $\sum_i\alpha_i=1$. Therefore, the requirement that the four values in (A7) are each less than 1 is sufficient but not necessary for $\mathfrak{D}_4$, i.e., $\mathfrak{D}_4-\mathfrak{D}_4'$ is not empty.

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Properties of Normal Thresholds in Perturbation Theory

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Making use of the relation between a Feynman diagram and the corresponding electric circuit, several properties of the normal thresholds are established.

IN the proof of the Mandelstam representation for scattering amplitude in perturbation theory by either Eden[1] or Landshoff, Polkinghorne, and Taylor,[2] the following statements are needed: (1) In the physical region on the boundary of the first sheet, the only singularities of the scattering amplitude are the normal thresholds. (2) A normal threshold and another Landau curve cannot have any finite "effective intersection." It is the purpose of this note to give, for these statements, an alternative proof which is entirely algebraic. Only the case of equal masses and no internal degree of freedom is considered.

The notations of reference 3 are to be used. The scattering amplitude F for a given proper Feynman diagram G_0 with four external lines is expressed as a function of the Mandelstam variables s, t, and u:

$$F(s,t,u)=\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2} \times[Q(s,t,u;\alpha_i)-\textstyle\sum_i\alpha_i]^{-2N+r+2}, \tag{1}$$

where

$$Q(s,t,u;\alpha_i)=\tfrac{1}{4}[sP_s(\alpha_i)+tP_t(\alpha_i)+uP_u(\alpha_i)]. \tag{2}$$

The symbol $G\leftarrow G_0$ shall be used to denote that G is a reduced graph of G_0, i.e., there exist a set $\mathcal{I}(G)$ of indices i, j, $\cdots$ such that $G=(G_0)_{ij}\cdots$. In the following, G is to be studied in detail; for simplicity of nota-

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[1] R. J. Eden, Phys. Rev. **119**, 1763 (1960); Phys. Rev. Letters **5**, 213 (1960); Phys. Rev. **121**, 1567 (1961).

[2] P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor (to be published).

[3] T. T. Wu, preceding paper [Phys. Rev. **123**, 678 (1961)].