planes:

$$\begin{cases}\mathcal{P}_1: Q_s=Q_t+Q_u, \\ \mathcal{P}_2: Q_t=Q_u+Q_s, \\ \mathcal{P}_3: Q_u=Q_s+Q_t,\end{cases} \quad \begin{cases}\mathcal{P}_4: Q_s=1, \\ \mathcal{P}_5: Q_t=1, \\ \mathcal{P}_6: Q_u=1.\end{cases} \tag{A5}$$

Furthermore, define $\mathfrak{Q}$ to be the set where Q_s, Q_t, and Q_u satisfy the three triangular inequalities. If $\mathfrak{Q}$ and the planes are projected along the line $Q_s=Q_t=Q_u$, the resulting situation is shown in Fig. 4. If (A2) is satisfied and $\partial\mathfrak{Q}$ is the boundary of $\mathfrak{Q}$, then the intersections $\mathcal{P}\cap\mathcal{P}_3\cap\partial\mathfrak{Q}$ and $\mathcal{P}\cap\mathcal{P}_6\cap\partial\mathfrak{Q}$ each consist of the two points (0,0,0) and (1,1,1) only. With this knowledge, it is straightforward to compute the four vertices of $\mathcal{P}\cap\mathfrak{Q}$:

$$\begin{aligned}
&1.\ \mathcal{P}\cap\mathcal{P}_1\cap\mathcal{P}_2: && Q_s=Q_t=Q_u=0,\\
&2.\ \mathcal{P}\cap\mathcal{P}_4\cap\mathcal{P}_5: && Q_s=Q_t=Q_u=1,\\
&3.\ \mathcal{P}\cap\mathcal{P}_1\cap\mathcal{P}_4: && Q_s=1,\ Q_t=-t''/(u''-t''),\\
& && \qquad \text{and } Q_u=u''/(u''-t''),\\
&4.\ \mathcal{P}\cap\mathcal{P}_2\cap\mathcal{P}_5: && Q_s=-s''/(u''-s''),\ Q_t=1,\\
& && \qquad \text{and } Q_u=u''/(u''-s'').
\end{aligned} \tag{A6}$$

$$Q=\begin{cases}0 & \text{at vertex 1,}\\ \frac{1}{2} & \text{at vertex 2,}\\ \frac{1}{4}(u't''-t'u'')/(t''-u'') & \text{at vertex 3,}\\ \frac{1}{4}(u's''-s'u'')/(s''-u'') & \text{at vertex 4.}\end{cases} \tag{A7}$$

This immediately gives (2.25).

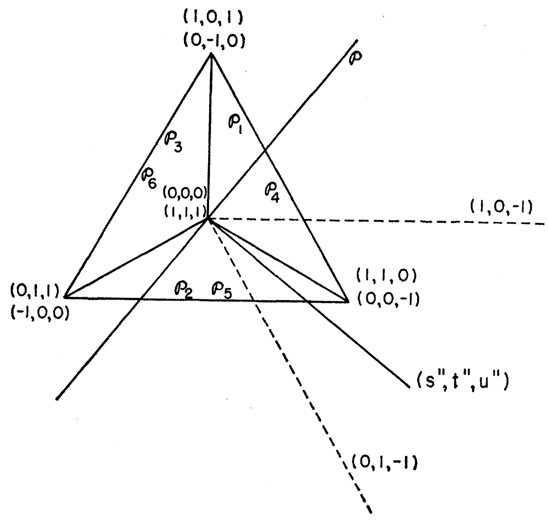

FIG. 4. Geometry for $\mathfrak{D}_4'$.

The important point here is that, contrary to the case of D_4, the vertices 3 and 4 are in general not on the boundary of $\mathfrak{Q}'$, where $\mathfrak{Q}'$ is the set of $\{Q_s,Q_t,Q_u\}$ that can be realized with non-negative α_i satisfying $\sum_i \alpha_i=1$. Therefore, the requirement that the four values in (A7) are each less than 1 is sufficient but not necessary for $\mathfrak{D}_4$, i.e., $\mathfrak{D}_4-\mathfrak{D}_4'$ is not empty.

PHYSICAL REVIEW VOLUME 123, NUMBER 2 JULY 15, 1961

Properties of Normal Thresholds in Perturbation Theory

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(Received March 1, 1961)

Making use of the relation between a Feynman diagram and the corresponding electric circuit, several properties of the normal thresholds are established.

IN the proof of the Mandelstam representation for scattering amplitude in perturbation theory by either Eden[1] or Landshoff, Polkinghorne, and Taylor,[2] the following statements are needed: (1) In the physical region on the boundary of the first sheet, the only singularities of the scattering amplitude are the normal thresholds. (2) A normal threshold and another Landau curve cannot have any finite "effective intersection." It is the purpose of this note to give, for these statements, an alternative proof which is entirely algebraic. Only the case of equal masses and no internal degree of freedom is considered.

The notations of reference 3 are to be used. The scattering amplitude F for a given proper Feynman diagram G_0 with four external lines is expressed as a function of the Mandelstam variables s, t, and u:

$$F(s,t,u)=\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_r\,\delta(1-\textstyle\sum_i\alpha_i)[d(\alpha_i)]^{-2} \times[Q(s,t,u;\alpha_i)-\textstyle\sum_i\alpha_i]^{-2N+r+2}, \tag{1}$$

where

$$Q(s,t,u;\alpha_i)=\tfrac{1}{4}[sP_s(\alpha_i)+tP_t(\alpha_i)+uP_u(\alpha_i)]. \tag{2}$$

The symbol $G\leftarrow G_0$ shall be used to denote that G is a reduced graph of G_0, i.e., there exist a set $\mathcal{I}(G)$ of indices i, j, $\cdots$ such that $G=(G_0)_{ij}\cdots$. In the following, G is to be studied in detail; for simplicity of nota-

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[1] R. J. Eden, Phys. Rev. **119**, 1763 (1960); Phys. Rev. Letters **5**, 213 (1960); Phys. Rev. **121**, 1567 (1961).
[2] P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor (to be published).
[3] T. T. Wu, preceding paper [Phys. Rev. **123**, 678 (1961)].

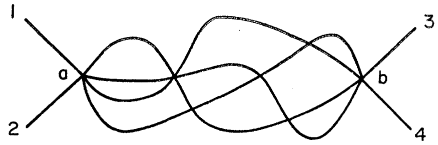

FIG. 1. An example of a Feynman diagram G with nonempty $\mathfrak{N}_s(G)$.

tion, the conventions are used that if $i\in\mathcal{I}(G)$ then $\alpha_i=0$, and that the indices $i, j, \cdots\notin\mathcal{I}(G)$ unless otherwise noted. With this convention, the values of P_s, P_t, P_u, and Q are the same for G and G_0. Strictly speaking, the numbering of nodes are different for G and for G_0; in the following the numbering always refers to G. Given G, let $\mathfrak{N}_s(G)$ be the set of all $\{s,t,u\}$ for which there exist positive α_i such that $\sum_i \alpha_i=1$ and for each i

$$(\partial/\partial\alpha_i)Q(s,\bar{t},\bar{u};\alpha_i)=1 \tag{3}$$

for all $\bar{t}$ and $\bar{u}$ that satisfy

$$s+\bar{t}+\bar{u}=s+t+u=4. \tag{4}$$

The normal thresholds in the s variable are then the connected components of

$$\mathfrak{N}_s=\bigcup_{G\leftarrow G_0}\mathfrak{N}_s(G). \tag{5}$$

Similarly, $\mathfrak{N}_t(G)$, $\mathfrak{N}_u(G)$, $\mathfrak{N}_t$, and $\mathfrak{N}_u$ may be defined; and $\mathfrak{N}=\mathfrak{N}_s\cup\mathfrak{N}_t\cup\mathfrak{N}_u$.

Since the physical regions are on the boundary of where the right-hand side of (1) is defined, it is sufficient to consider only non-negative values of α_i. Given G, let $\mathfrak{L}(G)$ be the set of all $\{s,t,u\}$ for which there exist positive α_i such that $\sum_i\alpha_i=1$ and for each i

$$(\partial/\partial\alpha_i)Q(s,t,u;\alpha_i)=1. \tag{6}$$

The real Landau signularities[4] $\mathfrak{L}$ are then defined as

$$\mathfrak{L}=\bigcup_{G\leftarrow G_0}\mathfrak{L}(G). \tag{7}$$

By definition

$$\mathfrak{N}_s\subset\mathfrak{L} \quad\text{and}\quad \mathfrak{N}_s(G)\subset\mathfrak{L}(G). \tag{8}$$

Note that $\mathfrak{L}$ and $\mathfrak{N}$ are both closed sets. Define $\mathfrak{L}'$ to be the closure of $\mathfrak{L}-\mathfrak{N}$.

Consider an infinitesimal increment ds, dt, du, $d\alpha_i$ so that $\{s,t,u\}\in\mathfrak{L}$ with α_i and $\{s, +ds, t+dt, u+du\}\in\mathfrak{L}$ with $\alpha_i+d\alpha_i$, then it follows from

$$Q(s,t,u,\alpha_i)=1 \tag{9}$$

on $\mathfrak{L}$ that

$$\frac{ds}{dt}=-\frac{P_s(\alpha_i)-P_u(\alpha_i)}{P_t(\alpha_i)-P_u(\alpha_i)}. \tag{10}$$

Thus the slope is determined unless

$$P_s(\alpha_i)=P_t(\alpha_i)=P_u(\alpha_i)=1. \tag{11}$$

[4] L. D. Landau, Nuclear Phys. **13**, 181 (1959).

But (11) is not possible. Thus every point on $\mathfrak{L}$ satisfies a polynomial equation $R(s,t)=0$, where R is not identically zero.

The following convention is also convenient. Given G, label the nodes in a definite way. If $i\leftrightarrow(a,b,\omega)$ with $a<b$, then defined I_i to be the $I_{ab\omega}$. Let this convention be applied to $I_{i\alpha\beta}$, defined as $I_{i,A(\alpha),A(\beta)}$, then relabeling the nodes can change the sign of $I_{i\alpha\beta}$, but not that of $I_{i\alpha\beta}I_{i\gamma\delta}$. In the following, I_{i12}, for example, always means $I_{i\alpha\beta}$ with $\alpha=1$ and $\beta=2$. Let I_{is} be the value of I_i when $I_\alpha=1$ for $\alpha=1$, 2 and $=-1$ for $\alpha=3$, 4. Then, with the above convention

$$I_{is}=I_{i13}+I_{i24}=I_{i14}+I_{i23}. \tag{12}$$

Similarly,

$$I_{it}=I_{i12}+I_{i34}=I_{i14}-I_{i23}, \tag{13}$$

and

$$I_{iu}=I_{i12}-I_{i34}=I_{i13}-I_{i24}. \tag{14}$$

Note that

$$I_{i\alpha\beta}=-I_{i\beta\alpha}, \tag{15}$$

and

$$I_{i\alpha\beta}+I_{i\beta\gamma}=I_{i\alpha\gamma}. \tag{16}$$

Theorem 1. If $\{s,t,u\}\in\mathfrak{N}_s(G)$, then for G $A(1)=A(2)=a$, $A(3)=A(4)=b$, and the internal lines consist of r_{ab} distinct but possibly intersecting continuous curves each joining $A(1)$ and $A(3)$. Furthermore,

$$s=r_{ab}^2. \tag{17}$$

Proof. An example of G is given in Fig. 1. If $\{s,t,u\}\in\mathfrak{N}_s(G)$, then

$$sP_s(\alpha_i)+\bar{t}P_t(\alpha_i)+\bar{u}P_u(\alpha_i)=4 \tag{18}$$

for all $\bar{t}$ and $\bar{u}$ satisfying (4). Thus,

$$P_s(\alpha_i)=4/s, \quad\text{and}\quad P_t(\alpha_i)=P_u(\alpha_i)=0, \tag{19}$$

which implies that $A(1)=A(2)$ and $A(3)=A(4)$. Equation (3) then gives that for each i

$$I_{is}^2=4/s, \tag{20}$$

and it follows from (19) and (12) that

$$I_{is}=2I_{13}. \tag{21}$$

If $i\leftrightarrow(a,b,\omega)$, label each internal line with an arrow from a to b if $I_{is}>0$ and from b to a if $I_{is}<0$. Starting from $A(1)$, trace a line along the direction of the arrows until the arrival at $A(3)$. Repeating this process without tracing any internal line twice exhausts all the internal lines. This proves the first part of the theorem, and (17) follows directly from the conservation of I at $A(1)$.

Cor. Given G, if $\mathfrak{N}_s(G)$ is not empty, then $\mathfrak{N}_t(G)$ and $\mathfrak{N}_u(G)$ are empty and $\mathfrak{L}(G)=\mathfrak{N}_s(G)$.

Theorem 2. If $s\geq4$, $t\leq0$, $u\leq0$, and $\{s,t,u\}\in\mathfrak{L}$, then $\{s,t,u\}\in\mathfrak{N}_s$.

Proof. Since $\{s,t,u\}\in\mathcal{L}$, there exist $G\leftarrow G_0$ such that $\{s,t,u\}\in\mathcal{L}(G)$. It is a consequence of (6) that for each i

$$sI_{is}^2+tI_{it}^2+uI_{iu}^2=4. \tag{22}$$

By using (12)–(14), (22) can be rewritten as

$$tI_{i14}I_{i23}+uI_{i13}I_{i24}=I_{is}^2-1. \tag{23}$$

Note that

$$|I_{i\alpha\beta}|\leq 1. \tag{24}$$

If $|I_{is}|>1$, then it follows from (12) and (24) that $I_{i14}I_{i23}>0$ and $I_{i13}I_{i24}>0$. Since $t\leq 0$ and $u\leq 0$, this violates (23). Therefore for each i

$$|I_{is}|\leq 1. \tag{25}$$

With the help of (16), rewrite (23) in the form

$$(tI_{i14}+uI_{i13})(tI_{i23}+uI_{i24}) = (s-4)(1-I_{is}^2)+tuI_{i34}^2. \tag{26}$$

The right-hand side is non-negative. Therefore $tI_{i14}+uI_{i13}$ and $tI_{i23}+uI_{i24}$ must have the same sign or one of them is zero. Let $A(1)=1$ and $\mathcal{A}$ be the set of i with the property that $i\leftrightarrow(a,b,\omega)$ with $a=1$. Then, if $A(1)\neq A(2)$,

$$\sum_{i\in\mathcal{A}} I_{i23}\leq 0, \quad \sum_{i\in\mathcal{A}} I_{i24}\leq 0, \tag{27}$$

and

$$\sum_{i\in\mathcal{A}} I_{i13}\geq 0, \quad \sum_{i\in\mathcal{A}} I_{i14}\geq 0. \tag{28}$$

Thus,

$$tI_{i14}+uI_{i13}\leq 0, \tag{29}$$

and hence

$$tI_{i23}+uI_{i24}\leq 0, \tag{30}$$

for $i\in\mathcal{A}$. A comparison of (27) and (30) yields that for $i\in\mathcal{A}$

$$I_{i23}=I_{i24}=0. \tag{31}$$

Since this is not possible, the conclusion is reached that $A(1)=A(2)$. Similarly, $A(3)=A(4)$. Therefore $P_t(\alpha_i)=P_u(\alpha_i)=0$, $I_{it}=I_{iu}=0$, and (6) implies (3). This proves theorem 2.

Theorem 3. If $\{s,t,u\}\in\mathfrak{N}$ with the set of numbers $\{\alpha_i\}$, and $\{s,t,u\}\in\mathcal{L}'$ with $\{\alpha_i'\}$, then for at least one $i,\alpha_i\neq\alpha_i'$.

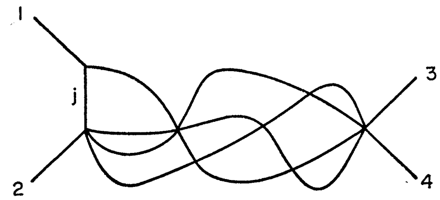

FIG. 2. An example of a Feynman diagram G that appears in connection with theorem 3.

Proof. Assume the contrary, i.e., $\alpha_i=\alpha_i'$ for each i. Without loss of generality, let $G'\leftarrow G_0$ such that $\{s,t,u\}\in\mathfrak{N}_s(G')$ with $\{\alpha_i\}$. Since every point of $\mathcal{L}$ satisfies a polynomial equation, $\mathcal{L}'$ consists of a finite number of branches of algebraic curves. There is a Feynman diagram $G''\leftarrow G_0$ such that G'' gives a branch which contains the point $\{s,t,u\}$ with $\{\alpha_i\}$. Then $G'\leftarrow G''$. If $A''(1)=A''(2)$ and $A''(3)=A''(4)$ for G'', then $P_t=P_u=I_{it}=I_{iu}=0$, and thus $\mathcal{L}(G'')=\mathfrak{N}_s(G'')$. Thus for G'', either $A''(1)\neq A''(2)$ or $A''(3)\neq A''(4)$. Assume that $A''(1)\neq A''(2)$. Then there exists G such that $A(1)\neq A(2)$, $G\leftarrow G''$, and $G'=G_j$ with $j\leftrightarrow(A(1), A(2))$. An example of G is shown in Fig. 2. If $A(1)=1$ and $A(2)=2$, then

$$I_{is}=2/s^{\frac{1}{2}}, \quad I_{it}=I_{iu}=0 \tag{32}$$

for either $i\leftrightarrow(1,b,\omega)$ or $i\leftrightarrow(2,b,\omega)$ with $b>2$. Since G_0 is proper, the conservation of I gives

$$|I_{js}|<1, \quad \text{and } I_{jt}=I_{ju}=1. \tag{33}$$

But $\{s,t,u\}\in\mathcal{L}'$; therefore it follows from (6) that

$$sI_{js}^2+tI_{jt}^2+uI_{ju}^2=4. \tag{34}$$

Since (33) and (34) are not consistent, this proves the theorem.

ACKNOWLEDGMENTS

I am greatly indebted to Professor C. N. Yang, without whose continuous guidance and encouragement this paper could not possibly have been written. I also would like to thank Professor F. J. Dyson for simplifying the proof of theorem 2, and Professor J. R. Oppenheimer for his hospitality at the Institute.