Note on Rearrangement Collisions*

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The conventional Born approximation formula for rearrangement collisions is used extensively in both atomic and nuclear physics. This formula contains a direct contribution from the heavy-particle or "core" interaction. A straightforward demonstration shows that for the usual case of a massive core this contribution does not appear, so that the only effect of this interaction is to distort the incident and outgoing waves. Such problems as the "post-prior" discrepancy are clarified.

1. INTRODUCTION

THE conventional Born approximation formula for rearrangement collisions is used extensively in both atomic and nuclear physics.[1] This approximation contains a direct contribution from the heavy-particle or "core" interaction about which there has been much controversy.[2] We shall show, by a simple application of the two-potential approach of Gell-Mann and Goldberger,[3] that the offending term vanishes identically when the core (usually the nucleus in atomic problems or the closed shell nucleons in nuclear problems) is infinitely heavy. The "post-prior" discrepancy, which arises when this term is evaluated using approximate wave functions, is thereby completely eliminated. Previous work on this problem has not sufficiently emphasized the vanishing of the core contribution, so that Born approximation calculations which include it continue to appear.[4]

The exact result for the transition amplitude, given by Eq. (11) of the following section, lends itself directly to numerical calculations in which the distorting effects of the core interactions in both the initial and final states may be included. These calculations may be readily performed using modern computing techniques, so that great improvements in the older calculations are now possible.

2. THE TRANSITION AMPLITUDE

As a simple example of the distorted-wave approach, let us consider an exchange or knock-out process in which a projectile a is incident upon a target composed of a particle b bound to a core c which we suppose to be infinitely heavy. In the final state we suppose that a is bound to c while b is free. Then we have the reaction

$$a+(b,c)\rightarrow b+(a,c). \qquad \text{(R)}$$

The following discussion will emphasize this process, but pickup or stripping reactions may be discussed in a quite similar way.

The transition amplitude for process R may be written in either the "post" or the "prior" form[3,5]:

$$M_R=\langle\eta_a\phi_b|\,V_f\,|\psi_i^{(+)}\rangle \qquad \text{(1a)}$$

$$=\langle\psi_f^{(-)}|\,V_i\,|\phi_a\eta_b\rangle. \qquad \text{(1b)}$$

V_f is the sum of the interaction between b and the core, denoted by U_b, and the interaction between a and b, denoted by V_{ab}; V_i is similarly defined. Thus,

$$V_f=U_b+V_{ab} \qquad \text{(2a)}$$

$$V_i=U_a+V_{ab}. \qquad \text{(2b)}$$

ϕ_a and ϕ_b are plane-wave functions describing the motion of a and b while η_b and η_a are the bound-state wave functions for the initial and final states. They are solutions of

$$(-\epsilon_a-K_a-U_a)\eta_a=0, \qquad \text{(3a)}$$

$$(-\epsilon_b-K_b-U_b)\eta_b=0, \qquad \text{(3b)}$$

where, for instance, K_a is the kinetic energy operator for particle a and ϵ_b is the binding energy of the target. Finally $\psi_i^{(+)}$ is the outgoing-wave solution for the total

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[1] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd edition, Chap. IX; N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), Chap. VIII; H. S. W. Massey, Revs. Modern Phys. **28**, 199 (1956) and references therein; J. D. Jackson and H. Schiff, Phys. Rev. **89**, 359 (1953); D. R. Bates and A. Dalgarno, Proc. Phys. Soc. (London) **A65**, 919 (1952) and **A66**, 972 (1953); M. R. C. McDowell and G. Peach, Proc. Phys. Soc. (London) **A74**, 463 (1959); G. E. Owen and L. Madansky, Phys. Rev. **105**, 1766 (1957).

[2] D. R. Bates, A. Fundaminsky, J. W. Leech, and H. S. W. Massey, Trans. Roy. Soc. (London) **A243**, 117 (1960); S. Altschuler, Phys. Rev. **92**, 1157 (1953); B. A. Lippman, *ibid.* **102**, 264 (1956); R. H. Bassel and E. Gerjuoy, *ibid.* **117**, 749 (1960).

[3] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[4] A recent example is that of R. A. Mann and M. E. Rose, Phys. Rev. **121**, 293 (1961). These authors consider the depolarization of μ mesons during their capture by atoms (carbon) and the subsequent de-excitation of the mesic atoms. Thus, the first part of the paper is devoted to a calculation of the relative capture probabilities of the meson into various atomic states with the ejection of an atomic electron. For this, the usual Born approximation to Eq. (10) is used; consequently, incorrect results are obtained. In particular, the results expressed in their Fig. 2 (in which it appears that the muon is preferentially captured into a low orbit, $n\sim 8$) are in error. The results of Fig. 2 arise from the tendency of the core term to ignore the requirement of maximum overlap of initial free meson and bound electron wave functions demanded by the muon-electron interaction term. It is essentially this overlap which forces the meson into the electron's orbit (i.e., $n\sim 15$), as has been demonstrated for pion capture in hydrogen by direct calculation using Eq. (13) [G. A. Baker, Jr., Phys. Rev. **117**, 1130 (1960)].

[5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950); E. Gerjuoy, Ann. Phys. **5**, 58 (1958).

Hamiltonian and is given by

$$\psi_i^{(+)}=\phi_a\eta_b+\frac{1}{E-K_a-K_b-U_b+i\epsilon}(U_a+V_{ab})\psi_i^{(+)}. \quad (4)$$

$\psi_f^{(-)}$ is defined in an analogous manner.

We shall use the "post" form Eq. (1a) although exactly the same results are obtained if the "prior" form is used [note the symmetry of Eq. (11)]. If scattering by the core represents a solvable problem, it is advantageous to use the two-potential formula of Gell-Mann and Goldberger to express Eq. (1a) in terms of the solutions of this problem. The outgoing wave solution for the scattering of a by the core is

$$\chi_a^{(+)}=\phi_a+\frac{1}{E_a-K_a-U_a+i\epsilon}U_a\phi_a, \quad (5)$$

where E_a is the kinetic energy of a: $E_a=E+\epsilon_b$. Similarly, the ingoing wave solution describing scattering of b in the final state is given by

$$\chi_b^{(-)}=\phi_b+\frac{1}{E_b-K_b-U_b-i\epsilon}U_b\phi_b. \quad (6)$$

The total wave function $\psi_i^{(+)}$ may be expressed in terms of $\chi_a^{(+)}$ if one introduces into Eq. (4) the identity (with $K=K_a+K_b$)

$$\frac{1}{E-K-U_b+i\epsilon}=\frac{1}{E-K-U_a-U_b+i\epsilon}-\frac{1}{E-K-U_a-U_b+i\epsilon}U_a\frac{1}{E-K-U_b+i\epsilon}. \quad (7)$$

Using Eqs. (3b) and (5), one finds

$$\psi_i^{(+)}=\chi_a^{(+)}\eta_b+\frac{1}{E-K_a-K_b-U_a-U_b+i\epsilon}V_{ab}\psi_i^{(+)} \quad (8a)$$

$$=\chi_a^{(+)}\eta_b+\frac{1}{E-K_a-K_b-U_a-U_b-V_{ab}+i\epsilon}\times V_{ab}\chi_a^{(+)}\eta_b. \quad (8b)$$

If Eq. (6) is inserted into M_R, we obtain

$$M_R=\langle\eta_a\phi_b|U_b+V_{ab}|\psi_i^{(+)}\rangle$$
$$=\langle\eta_a\phi_b|U_b|\psi_i^{(+)}\rangle+\langle\eta_a\chi_b^{(-)}|V_{ab}|\psi_i^{(+)}\rangle-\left\langle\eta_a\phi_b\left|U_b\frac{1}{E_b-K_b-U_b+i\epsilon}V_{ab}\right|\psi_i^{(+)}\right\rangle. \quad (9)$$

Since U_b commutes with K_a+U_a, we can use Eq. (3a) to replace the Green's function in the last term by $(E-K_a-K_b-U_a-U_b+i\epsilon)^{-1}$. The resulting term may then be combined with the first term of Eq. (9) using Eq. (8a), so that we obtain

$$M_R=\langle\eta_a\phi_b|U_b|\chi_a^{(+)}\eta_b\rangle+\langle\eta_a\chi_b^{(-)}|V_{ab}|\psi_i^{(+)}\rangle. \quad (10)$$

Since η_a and $\chi_a^{(+)}$ are eigenfunctions of K_a+U_a having different eigenvalues, they are orthogonal, and the first term vanishes identically. It is reasonable on physical grounds that the ejection of particle b cannot proceed unless its interaction with the incident projectile is included: This interaction is present only in the second term of Eq. (10). Thus, our final result is[6,7]

$$M_R=\langle\eta_a\chi_b^{(-)}|V_{ab}|\psi_i^{(+)}\rangle. \quad (11)$$

The symmetry of this result may be seen by introducing Eq. (8b) to give

$$M_R=\left\langle\eta_a\chi_b^{(-)}\left|\left[V_{ab}+V_{ab}\frac{1}{E-K_a-K_b-U_a-U_b-V_{ab}+i\epsilon}V_{ab}\right]\right|\chi_a^{(+)}\eta_b\right\rangle. \quad (12)$$

The quantity in the bracket is just the effective interaction, or t matrix, representing scattering of a by b in the presence of the core. If the second term is neglected, one obtains the so-called distorted-wave Born approximation:

$$M_R^{DWBA}=\langle\eta_a\chi_b^{(-)}|V_{ab}|\chi_a^{(+)}\eta_b\rangle. \quad (13)$$

The interaction between a and b is treated to first order, while the interactions with the core in both the initial and final states are treated exactly. The distorted wave functions $\chi^{(+)}$ and $\chi^{(-)}$ may be found using standard methods. In problems that are more general than the example treated here, Eq. (13) may be used whenever the effect of the core may be represented by a single-particle potential.

If all distorting effects are neglected, so that the scattering functions are replaced by their lowest Born approximations, Eq. (11) becomes

$$M_R^{BA}=\langle\eta_a\phi_b|V_{ab}|\phi_a\eta_b\rangle. \quad (14)$$

[6] If the core is not infinitely massive, the derivation of Eq. (10) breaks down. Center-of-mass motion, corresponding to the fact that particle b may be shaken off by the recoil of the core under the impact of a, prevents the division of the total kinetic energy operator into parts K_a and K_b which commute with U_b and U_a, respectively. Equation (11) may still be used if U_a in Eq. (5) is replaced by a potential U_i which is so chosen that it depends on the distance to the center of mass of the target rather than to the position of the core. This can be done, for instance, by defining U_i as the average of V_i over the internal coordinates of the bound-state wave function η_b. U_f in Eq. (6) can be similarly chosen. If V_{ab} in Eq. (8) is replaced by V_i-U_i, while V_{ab} appearing explicitly in Eq. (11) is replaced by V_f-U_f, then Eq. (11) applies to all rearrangement collisions.

[7] We would like to emphasize that this result is essentially contained in earlier work, e.g., R. H. Bassel and E. Gerjuoy, Phys. Rev. **117**, 749 (1960) and references contained therein. However, its general validity does not seem to be very widely recognized.

This differs from the *usual* expression, which includes the core interaction $\langle\eta_a\phi_b|U_b|\phi_a\eta_b\rangle$ as well. This usual result is obtained if, in the first term of Eq. (10) (which we know to be identically zero), we replace $\chi_a^{(+)}$ by its Born approximation ϕ_a. In this approximation, this term is no longer zero, since η_a and ϕ_a are eigenfunctions of different Hamiltonians. The result is just the extra term found by perturbation techniques. It is clear that this term should *not* be present, and that only the inadequacies of the perturbation approach have led to its appearance. Some of the virtues of an approach in which the exact expression is obtained before approximations are made can be seen from this example.

We may also observe that the so-called "post-prior" discrepancy has evaporated. This discrepancy arises when the ordinary (but incorrect) perturbation result is used, since then either U_a or U_b may enter the expression for M_R^{BA}. While formally they give identical results, that is $\langle\eta_a\phi_b|U_a|\phi_a\eta_b\rangle=\langle\eta_a\phi_b|U_b|\phi_a\eta_b\rangle$; when approximate bound-state wave functions are introduced into these matrix elements, the equality no longer holds. We now see that this term should not be present at all and that no problem exists.

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Asymptotic Behavior and Subtractions in the Mandelstam Representation*

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It is proved that a two-body reaction amplitude involving scalar particles and satisfying Mandelstam's representation is bounded by expressions of the form $Cs\ln^2 s$ at the forward and backward angles, and $Cs^{\frac{3}{4}}\ln^3 s$ at any other fixed angle in the physical region, C being a constant, s being the total squared c.m. energy. This corresponds to cross sections increasing at most like $\ln^2 s$. These restrictions limit the freedom of choice of the subtraction terms to six arbitrary single spectral functions and one subtraction constant.

I. INTRODUCTION

SINCE the time Mandelstam[1] discovered his representation for two-body reaction amplitudes, there has been in general a little confusion about the question of asymptotic behavior of the different quantities as the energy variables go to infinity.

We shall point out in this paper a number of facts, which, we hope, will help to clarify these questions.

In Sec. II, we derive, from the Mandelstam representation and from a very weakened form of the unitarity condition, an upper bound on the asymptotic behavior of the amplitude in the physical regions.

In Sec. III, we show that these results cannot give us any indication on the behavior of the double spectral function.

In Sec. IV, we write down a general form for the subtracted double dispersion relation, which will prove convenient for the following.

In Sec. V, we investigate the question whether the subtraction constants and the single spectral functions can be determined from the asymptotic conditions which we derived in Sec. II.

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

II. ASYMPTOTIC PROPERTIES OF THE AMPLITUDE IN THE PHYSICAL REGION

We consider a reaction of the type $a+b\to c+d$ among scalar particles. We denote by p_1, p_2, $-p_3$, $-p_4$ the momenta of the particles a, b, c, and d, respectively. We introduce the notations $s=(p_1+p_2)^2$; $t=(p_2+p_3)^2$; $u=(p_3+p_1)^2$. Then

$$s+t+u=p_1^2+p_2^2+p_3^2+p_4^2.$$

We shall assume that all masses are equal to the unit of mass as we deal only with asymptotic properties, where the difference between the masses is negligible. Then: $s+t+u=4$. We call channel s the above reaction $a+b\to c+d$, channel t the reaction $b+\bar{c}\to\bar{a}+d$ and channel u the reaction $a+\bar{c}\to\bar{b}+d$. In the channel s, the momentum of one particle in the c.m. system is given by $\mathbf{q}_s^2=(s-4)/4$ and the reaction angle will be defined by $\cos\theta_s=1+(t/2q_s^2)$. The physical region for channel s will be given by

$$\mathbf{q}_s^2>0,\quad |\cos\theta_s|<1;\quad \text{or}\quad s>4,\quad t\leq 0,\quad u\leq 0.$$

We define the notations in the other channels by a circular permutation among (s,t,u).

In order that the double dispersion integrals make sense, we have to require that the double spectral functions be tempered distributions, and similarly, we require that the single spectral functions be also tempered distributions.