Correlations between Cusps in Differential Cross Sections and in Polarizations

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Experiments have shown that partial waves with $l>1$ appear in $\pi^-+p\rightarrow\Lambda+K^0$ at the ΣK thresholds. This necessitates a reconsideration of the criteria sufficient to determine the $\Sigma\Lambda$ parity $P(\Sigma\Lambda)$ by the method of cusps. In this paper we start from the usual assumption that contributions which show a cusp in either the differential cross section or the polarization may be ignored beyond some optimal power in $\cos\theta$. On this sole basis, previously stated criteria are rendered inadequate due to the occurrence of Minami and other ambiguities. It is shown that under suitable circumstances there exist unambiguous correlations between certain properties of cross-section cusps and of polarization cusps. These correlations could possibly be of use to determine $P(\Sigma\Lambda)$ and give information as to which states contribute significantly to the ΛK production at $\sim$900 Mev. The finite separation between Σ^0K^0 and Σ^-K^+ thresholds is taken into account. The results are summarized in a table of cusp properties.

I. INTRODUCTION

It was first observed by Wigner[1] that scattering and reaction cross sections may show cusps at the threshold energies for competing channels. In the past few years the importance of cusps for particle physics has come to be increasingly realized.[2,3] These days, the attention mainly centers on the cusps in the reaction $\pi^-+p\rightarrow\Lambda+K^0$ at the thresholds for ΣK production. The discussion of the present paper will be given with these reactions as terms of reference, although the arguments to be presented are quite general.

Experimental studies of the cusp in the ΛK reaction are now in progress and preliminary results have been published.[4–7] There are good indications that the cusp phenomenon actually occurs. However, at the time of writing it has not yet been found possible to reach the principal goal of these experiments: the determination of the relative $\Sigma\Lambda$ parity, $P(\Sigma\Lambda)$. A main stumbling block lies in the fact, established by both the Berkeley and the Columbia groups, that in the cusp region the differential cross section for ΛK production contains $x=\cos\theta$ to powers higher than the second (θ is the c.m. angle of production). Hence at these energies an analysis in terms of S and P waves only is inadequate.

In order to see the questions of interpretation which arise, let us return for a moment to the assumption, now shown to be fictitious, that we deal with a problem in which the only contributing partial waves are $S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$, and $P_{\frac{3}{2}}$. Then by the well-known argument of Baz' and Okun',[3] $P(\Sigma\Lambda)$ is odd if the terms x^0, x^1, and x^2 in $d\sigma/d\Omega$ show cusps; if only x^0 and x^1 show cusps the parity is even. It should be noted that this assumption about contributing states is a particular case where, for given maximum J (namely $\frac{3}{2}$), there occurs only a specific one of the two possible corresponding l values (namely, 1 and not 2). Whenever this assumption can be shown to be valid, the Baz'-Okun' argument is sufficient to determine $P(\Sigma\Lambda)$, whatever J is. [For example, a cusp in x^3 means odd $P(\Sigma\Lambda)$ in the absence of F waves.]

In the absence of further information it was certainly natural to assume the presence of S and P waves only. The actual occurrence of higher partial waves complicates the situation in a nontrivial way. Naturally, we need to know which powers of $\cos\theta$ are significant. But furthermore, as has been emphasized by Schwartz,[6] this knowledge in itself is insufficient to determine $P(\Sigma\Lambda)$.

The reason for this insufficiency is essentially the Minami ambiguity[8] which was originally uncovered in π-nucleon scattering, but which of course applies as well to ΛK production. In the present context, the Minami transformation implies that if we only know which powers of $\cos\theta$ in the cross section and polarization do and which do not show a cusp, there exist alternative sets of states which explain this limited information equally well. One set would correspond to even, the other to odd $P(\Sigma\Lambda)$. In Sec. II. (A) we present a compact formalism which is particularly suited to deal with this ambiguity and its role in the $P(\Sigma\Lambda)$ question. There we also discuss another potential source of ambiguity noted by Schwartz,[6] which arises when we compare a given scattering amplitude with its complex conjugate.

Detailed information on the amplitudes of these states which actually participate at the energy in question would remove all ambiguities. Hence, a way out of these complications would present itself if by other dynamical arguments one could determine for which J and l values

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[6] M. Schwartz, Revs. Modern Phys. (to be published).
[7] F. Crawford, Revs. Modern Phys. (to be published).
[8] S. Minami, Progr. Theoret. Phys. (Kyoto) **11**, 213 (1954); S. Hayakawa, M. Kawaguchi, and S. Minami, *ibid.* **11**, 332 (1954); **12**, 355 (1954); H. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, New York, 1955), Vol. 2, pp. 75 and 80.

TABLE I. Cusp properties.

Case No.	f^{l+}	f^{l-}	$f^{(l-1)+}$	$f^{(l-1)-}$	$P(\Sigma\Lambda)$ even First corr.	$P(\Sigma\Lambda)$ even Second corr.	$P(\Sigma\Lambda)$ even "ρ^2/σ"	$P(\Sigma\Lambda)$ odd First corr.	$P(\Sigma\Lambda)$ odd Second corr.	$P(\Sigma\Lambda)$ odd "ρ^2/σ"	Only nonvanishing amplitudes for the example $l=2$				
1					None	None	1	O	O	x^2	$D_{\frac{5}{2}}$	$D_{\frac{3}{2}}$	$P_{\frac{3}{2}}$	$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
2	0				E	None	x	O	None	x		$D_{\frac{3}{2}}$	$P_{\frac{3}{2}}$	$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
3		0			$[(l+1)/l]O$	None	1	O	O	x^2	$D_{\frac{5}{2}}$		$P_{\frac{3}{2}}$	$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
4			0		None	E	1	O	Gap	x^2	$D_{\frac{5}{2}}$	$D_{\frac{3}{2}}$		$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
5				0	None	$[l/(l-1)]O$	1	O	O	x^2	$D_{\frac{5}{2}}$	$D_{\frac{3}{2}}$	$P_{\frac{3}{2}}$		$S_{\frac{1}{2}}$
6	0		0		E	E	x^2	None	None	1		$D_{\frac{3}{2}}$		$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
7	0			0	E	$[l/(l-1)]O$	x	O	None	x		$D_{\frac{3}{2}}$	$P_{\frac{3}{2}}$		$S_{\frac{1}{2}}$
8		0	0		$[(l+1)/l]O$	E	1	O	Gap	x^2	$D_{\frac{5}{2}}$			$P_{\frac{1}{2}}$	$S_{\frac{1}{2}}$
9		0		0	$[(l+1)/l]O$	$[l/(l-1)]O$	1	O	O	x^2	$D_{\frac{5}{2}}$		$P_{\frac{3}{2}}$		$S_{\frac{1}{2}}$
10			0	0	None	Gap	1	O	Gap	x^2	$D_{\frac{5}{2}}$	$D_{\frac{3}{2}}$			$S_{\frac{1}{2}}$
11	0		0	0	E	Gap	x^2	None	None	1		$D_{\frac{3}{2}}$			$S_{\frac{1}{2}}$
12		0	0	0	$[(l+1)/l]O$	Gap	1	O	Gap	x^2	$D_{\frac{5}{2}}$				$S_{\frac{1}{2}}$

the corresponding partial amplitudes contribute significantly to ΛK production at $\sim$900 Mev. Thus it has been surmised that the presence of higher partial waves in the ΛK reaction can be linked to the properties of the so-called third resonance in π-nucleon scattering which occurs at practically the same energy as do the ΣK thresholds. A study of the reactive effects of this resonance on the ΛK reaction may therefore produce the desired information. Of course, it remains to be seen if such an analysis is sufficiently free from theoretical ambiguity to be decisive for the parity question.

In the face of these dilemmas we were led to reconsider the question as to the amount of information that can be extracted from a cusp experiment without recourse to extraneous dynamical information. We have found that not only a knowledge of the powers of x in which cusps occur and do not occur in the differential cross section and in the polarization constitutes useful information. In addition, it turns out that in many instances there exist correlations between the type and magnitude of the cusp term in a certain power of x in the cross section on the one hand, and the type and magnitude of a corresponding term for the polarization on the other.

Figure 1 shows the four types of cusps, denoted by C_1 to C_4, which may in principle occur either in the differential cross section $d\sigma/d\Omega$ or in the polarization $P(d\sigma/d\Omega)$. In the discussion given in Sec. II. (C) we shall see that, depending on the nature of the contributing states, there may exist very characteristic correlations which one may call allowed cusp pairs: To a given cusp type in $d\sigma/d\Omega$ there may occur only one particular cusp type in $P(d\sigma/d\Omega)$. Moreover, whenever this is the

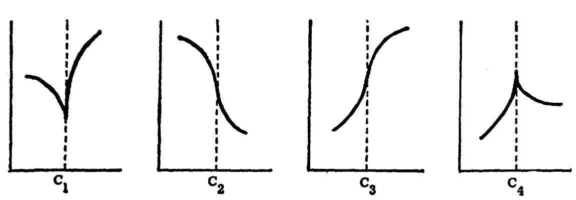

FIG. 1. The four types of cusps for a single threshold.

case, the magnitude of the respective numerical coefficients in these pairs of cusps satisfy very simple relations. The results are summarized in Table I. Section II. C contains a self-contained set of definitions of all the symbols which occur in the table. In Sec. II. B it is noted that these results are independent of the parity of the K^0 relative to the hyperon-nucleon system.

The main idea of the paper is the following. We shall start from the one assumption that the cusp in the differential cross section and in the polarization can be represented by a finite polynomial in $\cos\theta$. In this way we do not commit ourselves on what the participating states are. We shall then inquire whether specific cusp correlations could tell us what $P(\Sigma\Lambda)$ is. It will turn out that there exist a considerable number of cases in which not only $P(\Sigma\Lambda)$ can be determined by means of cusp correlations, but where in addition the cusp experiment itself may serve to reveal to a large extent the characteristics of the states which contribute significantly to the ΛK production at the energy under consideration.

A further comment concerning our assumption is in order. It is of course physically inconceivable that the $\cos\theta$ series for the cusp in the differential cross section rigorously terminates at some optimal power. All one can hope for is a sharp drop beyond a certain power. In Sec. II. C we shall show by example how one may correct the answers correspondingly.

While in Sec. II we treat the case of a single cusp, Sec. III deals with the more realistic problem where the finite separation between the Σ^0K^0 and Σ^-K^+ thresholds is taken into account. In the 72-in. bubble chamber experiment this separation has already been effected clearly.[7] It is pointed out that Table I applies also to the double cusp.

II. SINGLE CUSP

A. Pseudoscalar K^0

Let M_{PS} denote the transition amplitude of $\pi^-+p\rightarrow\Lambda+K^0$ in the center-of-mass system for pseudoscalar K^0 (the Λ-nucleon parity is even, by convention). We

write M_{PS} as[9]

$$M_{PS}=f+g(\boldsymbol{\sigma}\cdot\mathbf{k}')(\boldsymbol{\sigma}\cdot\mathbf{k}). \tag{1}$$

Note that in this representation $(f+gx)$ and $-i(1-x^2)^{\frac{1}{2}}g$ correspond to the conventional non-spin-flip and spin-flip amplitudes, respectively; $x=\mathbf{k}'\cdot\mathbf{k}$, where $\mathbf{k}$ and $\mathbf{k}'$ are unit vectors along the incident and final momenta, respectively. The amplitudes f and g are given by

$$f=\sum[f^{(l-1)+}-f^{(l+1)-}](dP_l/dx), \tag{2}$$

$$g=\sum[f^{l-}-f^{l+}](dP_l/dx), \tag{3}$$

where $f^{\pm l}$ are the transition amplitudes for orbital angular momentum l with total angular momentum $j=l\pm\frac{1}{2}$,[10] and P_l are the Legendre polynomials. The differential cross sections and the polarization are then given by

$$d\sigma/d\Omega=|f|^2+|g|^2+2x\,\mathrm{Re}(f^*g), \tag{4}$$

$$P(d\sigma/d\Omega)=2(1-x^2)^{\frac{1}{2}}\,\mathrm{Im}(f^*g), \tag{5}$$

where $\mathbf{P}=P\mathbf{n}$ is the polarization vector and

$$(1-x^2)^{\frac{1}{2}}\mathbf{n}=\mathbf{k}\times\mathbf{k}'.$$

The Minami ambiguity amounts to the statement that if for a given J we interchange the amplitudes for $l=J+\frac{1}{2}$ and $l=J-\frac{1}{2}$, that is,

$$f^{l+}\rightarrow f^{(l+1)-} \quad\text{and}\quad f^{l-}\rightarrow f^{(l-1)+}, \tag{6}$$

then the differential cross section remains unchanged while the polarization changes sign. This can be easily verified by noticing that as a consequence of the Minami transformation we have

$$f\rightarrow g \quad\text{and}\quad g\rightarrow f. \tag{7}$$

This simple and useful form of the Minami transformation is the main reason for choosing the representation for M given in Eq. (1).

We note that the transformation of complex conjugation,

$$f^{l\pm}\rightarrow f^{l\pm *}, \tag{8}$$

affects $d\sigma/d\Omega$ and $(P\,d\sigma/d\Omega)$ in the same way as does the Minami transformation, namely, $d\sigma/d\Omega$ is again unchanged upon application of Eq. (8) while $(P\,d\sigma/d\Omega)$ again changes sign. Hence, the product of a Minami transformation and a complex conjugation:

$$f^{l+}\rightarrow f^{(l+1)-*},\quad f^{l-}\rightarrow f^{(l-1)+*} \tag{9}$$

or, correspondingly,

$$f\rightarrow g^*,\quad g\rightarrow f^* \tag{10}$$

leaves both $d\sigma/d\Omega$ and $P(d\sigma/d\Omega)$ unchanged.

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[9] See, e.g., G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[10] The amplitudes $f^{l\pm}$ are related to the elements of the unitary S matrix by $f^{l\pm}=S^{l\pm}/2ik$, where k is the magnitude of the incident momentum.

It will be one of our main tasks to study the influence of the transformation (6) and (8) on the question of what one may hope to learn from cusps. In that context one must be careful to state what complex conjugation means, as we shall presently see.

Consider now the ΛK^0 production in the region close to a ΣK threshold; for definiteness we take Σ^0K^0. Denote all ΛK^0 production amplitudes at the Σ^0K^0 threshold with the subscript t. From unitarity and the threshold analytic properties of the s matrix, we have[3]

$$P(\Sigma\Lambda)\ \text{even:}\quad f^{0+}=f_t^{0+}+iq\alpha,\quad \text{(a)} \tag{11}$$

$$P(\Sigma\Lambda)\ \text{odd:}\quad f^{1-}=f_t^{1-}+iq\alpha,\quad \text{(a)} \tag{12}$$

where q is the c.m. momentum in the Σ^0K^0 channel above threshold, and α is equal to the product of the transition amplitudes of $\pi^-+p\rightarrow\Sigma^0+K^0$ and of $\Sigma^0+K^0\rightarrow\Lambda^0+K^0$ at the Σ^0K^0 threshold. The notation (a) will always mean that the equation refers to the small region above threshold where an expression up to terms linear in q is adequate. Similarly, (b) shall refer to the region just below threshold, where Eqs. (11) and (12) have to be subjected to the well-known analytic continuation,[3]

$$q\rightarrow i|q| \quad\text{for}\quad \text{(a)}\rightarrow\text{(b)}. \tag{13}$$

All amplitudes other than those occurring in Eqs. (11) or (12) are to be replaced by their threshold values.

Define

$$(d\sigma/d\Omega)-(d\sigma/d\Omega)_t=2\rho, \tag{14}$$

$$(P\,d\sigma/d\Omega)-(P\,d\sigma/d\Omega)_t=2(1-x^2)^{\frac{1}{2}}\delta, \tag{15}$$

and denote by ρ_e (ρ_0) and δ_e (δ_0) the cusp part of the differential cross section and polarization, respectively, for even (odd) $\Sigma\Lambda$ parity. Then from Eqs. (2) to (5) and (11) and (12) we see that

$$\rho_e=q\,\mathrm{Im}[\alpha^*(f_t+xg_t)],\qquad (a)$$

$$\rho_e=-|q|\,\mathrm{Re}[\alpha^*(f_t+xg_t)],\qquad (b); \tag{16}$$

$$\delta_e=-q\,\mathrm{Re}[\alpha^*g_t],\qquad (a)$$

$$\delta_e=-|q|\,\mathrm{Im}[\alpha^*g_t],\qquad (b); \tag{17}$$

$$\rho_o=q\,\mathrm{Im}[\alpha^*(g_t+xf_t)],\qquad (a)$$

$$\rho_o=-|q|\,\mathrm{Re}[\alpha^*(g_t+xf_t)],\qquad (b); \tag{18}$$

$$\delta_o=q\,\mathrm{Re}[\alpha^*f_t],\qquad (a)$$

$$\delta_o=|q|\,\mathrm{Im}[\alpha^*f_t],\qquad (b). \tag{19}$$

All information about cusps is contained in ρ and δ. We now examine how these quantities behave under the transformations (6) and (8) and their product (9). In particular, we must ask the following question. Suppose that $P(\Sigma\Lambda)$ is even and that among all f^{l+} and f^{l-} a specific set of amplitudes participates. Now apply either the transformation (6) or (8), or both. This transforms

the set into another set. Is this new set with $P(\Sigma\Lambda)$ odd distinguishable from the old set with $P(\Sigma\Lambda)$ even?

First consider the Minami transformation. According to Eqs. (7) and (16)–(19) we have

$$\rho_e \to \rho_o, \qquad \rho_o \to \rho_e, \qquad (a) \text{ and } (b); \tag{20}$$

$$\delta_e \to -\delta_o, \quad \delta_o \to -\delta_e, \quad (a) \text{ and } (b). \tag{21}$$

Next, we apply complex conjugation Eq. (8) above threshold and find

$$\rho_e(a) \to \rho_e(a), \qquad \rho_o(a) \to \rho_o(a); \tag{22}$$

$$\delta_e(a) \to -\delta_e(a), \quad \delta_o(a) \to -\delta_o(a). \tag{23}$$

Hence, above threshold the product transformation (9) merely interchanges ρ_e and ρ_o, and also δ_e and δ_o. Thus we have proved the following: It is impossible to determine $P(\Sigma\Lambda)$ from cusp information above threshold alone, unless we have additional information on the participating set of states.

Consider now the situation below threshold. It is here that we must be cautious with the transformation (8). It is instructive to do it first the wrong way. Take for example Eq. (11) and its analytic continuation

$$f^{0+} = f_t^{0+} + iq\alpha \quad (a); \quad f^{0+} = f_t^{0+} - |q|\alpha \quad (b), \tag{24}$$

in accordance with Eq. (13). The complex conjugate of these expressions is

$$f^{0+} \to f_t^{0+*} - iq\alpha^* \quad (a); \quad f^{0+} \to f_t^{0+*} - |q|\alpha^* \quad (b). \tag{25}$$

If we should use Eq. (25) below threshold we would obtain results similar to Eqs. (22) and (23), with corresponding consequences. However, Eq. (25) is the incorrect analytic continuation, namely, $q \to -i|q|$.

The correct procedure is clearly the following. We must first apply Eq. (8) to f^{0+} above threshold and thereupon analytically continue it. Instead of Eq. (25) we then get

$$f^{0+} \to f^{0+*} - iq\alpha^* \quad (a); \quad f^{0+} \to f_t^{0+*} + |q|\alpha^* \quad (b). \tag{26}$$

Applying Eq. (26) to Eqs. (16)–(19), we get

$$\rho_e(b) \to -\rho_e(b), \quad \rho_o(b) \to -\rho_o(b), \tag{27}$$

$$\delta_e(b) \to \delta_e(b), \qquad \delta_o(b) \to \delta_o(b). \tag{28}$$

Below threshold, the product transformation (9) now interchanges ρ_e with $-\rho_o$ and also δ_e with $-\delta_o$. Equations (23) and (27) imply that for either $P(\Sigma\Lambda)$ the absolute signs of $\delta(a)$ and of $\rho(b)$ are irrelevant, always in the absence of extraneous information. Note, however, that for either $P(\Sigma\Lambda)$ the ratios

$$\rho(a)/\delta(b) \quad \text{and} \quad \rho(b)/\delta(a) \tag{29}$$

do *not* change under the properly defined complex conjugation so that these two ratios are meaningful even in the face of the ambiguities implied by Eq. (8). Here we are precisely at the root of the physical results we shall obtain. We shall show that upon expansion in x of ρ and of δ the ratios of their leading terms $\rho(a)/\delta(b)$ and $\rho(b)/\delta(a)$ exhibit characteristic differences for the two parity choices. These differences are precisely due to the fact that the ratios mentioned in Eq. (29) *do* change sign when we compare even with odd $P(\Sigma\Lambda)$, as follows from Eqs. (20) and (21).

We now discuss these expansions. Substituting Eqs. (2)–(3) in Eqs. (16)–(19), we obtain

$$\rho_e = \left\{ \begin{matrix} q\,\mathrm{Im}\alpha^* \\ -|q|\,\mathrm{Re}\alpha^* \end{matrix} \right\} \sum [(l+1)f_t^{l+} + l f_t^{l-}] P_l, \tag{30}$$

$$\delta_e = \left\{ \begin{matrix} -q\,\mathrm{Re}\alpha^* \\ -|q|\,\mathrm{Im}\alpha^* \end{matrix} \right\} \sum (2l+1)[f_t^{l-} - f_t^{l+} + h_l] P_{l-1}, \tag{31}$$

$$\rho_0 = \left\{ \begin{matrix} q\,\mathrm{Im}\alpha^* \\ -|q|\,\mathrm{Re}\alpha^* \end{matrix} \right\} \sum [(l+1)f_t^{(l+1)-} + l f_t^{(l-1)+}] P_l, \tag{32}$$

$$\delta_0 = \left\{ \begin{matrix} q\,\mathrm{Re}\alpha^* \\ |q|\,\mathrm{Im}\alpha^* \end{matrix} \right\} \sum (2l+1) \times [f_t^{l+} + f_t^{(l+2)+} - f_t^{(l-2)-} - h_{l+2}] P_l. \tag{33}$$

Here

$$h_l = \sum f_t^{(l'+2)-} - f_t^{(l'+2)+},$$

and the summation goes over $l' = l,\ l+2,\ \cdots$.

At this point it may be instructive to recall the trend of the Baz'-Okun' argument.[3] Let us suppose that the only states which contribute are $S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$, and $P_{\frac{3}{2}}$. Keeping only the highest power in x we have, according to Eqs. (2) and (3): $f\sim x$, $g\sim 1$, hence $\rho_e\sim x$, $\delta_e\sim 1$, while $\rho_o\sim x^2$, $\delta_o\sim x$. Because a prescribed set of states is involved which is not invariant under the transformation (6), the determination of $P(\Sigma\Lambda)$ is therefore free of Minami ambiguity. Had we also included $D_{\frac{3}{2}}$ this would no longer have been true. Thus, in general, the sole knowledge of the highest power of x in ρ or δ does not constitute sufficient information for the determination of $P(\Sigma\Lambda)$.

We now start the more general discussion, which is exclusively based on the assumption that we may neglect cusp effects in ρ beyond some fixed optimal power. As we shall presently see, our assumption is equivalent to putting

$$f^{l'\pm} = 0 \quad \text{for all} \quad l' > l,$$

where l is now the optimal power in question. It may directly be noted that this includes both alternatives of Minami invariant situations [$f^{l+}=0$, provided further that $f^{(l-1)+}\neq 0$] and of noninvariant ones ($f^{l+}\neq 0$).

Writing out the series in Eqs. (30)–(33) in decreasing order of the Legendre polynomials we obtain, starting

with the highest l,

$$\rho_e=\begin{Bmatrix} q\ \mathrm{Im}\alpha^* \\ -|q|\ \mathrm{Re}\alpha^* \end{Bmatrix}[(l+1)f_t{}^{l+}+lf_t{}^{l-}]Pl$$
$$+[lf_t{}^{(l-1)+}+(l-1)f_t{}^{(l-1)-}]P_{l-1}+\cdots, \quad (34)$$

$$\delta_e=\begin{Bmatrix} -q\ \mathrm{Re}\alpha^* \\ -|q|\ \mathrm{Im}\alpha^* \end{Bmatrix}(2l+1)[f_t{}^{l-}-f_t{}^{l+}]P_{l-1}$$
$$+(2l-3)[f_t{}^{(l-1)-}-f_t{}^{(l-1)+}]P_{l-2}$$
$$+(2l-5)[f_t{}^{(l-2)-}-f_t{}^{(l-2)+}$$
$$+f_t{}^{l-}-f_t{}^{l+}]P_{l-3}+\cdots, \quad (35)$$

$$\rho_0=\begin{Bmatrix} q\ \mathrm{Im}\alpha^* \\ -|q|\ \mathrm{Re}\alpha^* \end{Bmatrix}[(l+1)f_t{}^{l+}]P_{l+1}+[lf_t{}^{(l-1)+}]P_l$$
$$+[lf_t{}^{l-}+(l-1)f_t{}^{(l-2)+}]P_{l-1}+\cdots, \quad (36)$$

$$\delta_0=\begin{Bmatrix} q\ \mathrm{Re}\alpha^* \\ |q|\ \mathrm{Im}\alpha^* \end{Bmatrix}(2l+1)f_t{}^{l+}P_l+(2l-1)f_t{}^{(l-1)+}P_{l-1}$$
$$+(2l-3)[f_t{}^{(l-2)+}+f_t{}^{l+}-f_t{}^{l-}]P_{l-2}+\cdots. \quad (37)$$

B. Scalar K^0

Our foregoing discussion for pseudoscalar K^0 is also valid for scalar K^0 provided we interpret the value of l in the transition amplitudes $f^{l\pm}$ as the orbital angular momentum of the final ΛK^0 state. The corresponding initial πN state has orbital angular momentum $l'=l\pm1$ while both states, of course, have the same total angular momentum $J=l\pm\frac{1}{2}=l'\mp\frac{1}{2}$. To prove this assertion we note that the transition amplitude for scalar K^0 can be written in the form[11]

$$M_S=M_{PS}(\boldsymbol{\sigma}\cdot\mathbf{k}), \quad (38)$$

where M_{PS} is the transition amplitude for pseudoscalar K^0 given in Eq. (1). The operator $(\boldsymbol{\sigma}\cdot\mathbf{k})$ is rotationally invariant and of odd parity. Substituting Eq. (1) in Eq. (28), we obtain

$$M_S=f(\boldsymbol{\sigma}\cdot\mathbf{k})+g(\boldsymbol{\sigma}\cdot\mathbf{k}'). \quad (39)$$

Since $(\boldsymbol{\sigma}\cdot\mathbf{k})$ operating on an initial state changes only its orbital parity, f and g are still expressed in the form given by Eqs. (2)–(3), but now the superscript l in $f^{l\pm}$ refers to the orbital angular momentum of the final state only, as we stated earlier. The angular distribution and the polarization obtained from M_S as well as the dependence on q of f^{0+} and f^{1-} near the Σ^0K^0 threshold for $P(\Sigma\Lambda)$ even and odd, respectively, are the same as for pseudoscalar K^0 [see Eqs. (4)–(12)]. It follows that the remaining discussion and equations in Sec. II.A also apply equally well for scalar K^0.[12] In particular, the implications of Eqs. (34)–(37) hold true for either K^0-parity.

C. The Table of Cusp Properties

In order to distinguish the various possibilities to which Eqs. (34)–(37) give rise, it is necessary to focus attention first of all on which of the four quantities $f_t{}^{l(+)}$, $f_t{}^{(l)-}$, $f_t{}^{(l-1)+}$, and $f_t{}^{(l-1)-}$ vanish and which do not. This is indicated in the first four columns where a zero means that the quantity in question vanishes; otherwise the f's are supposed to be nonzero and for the rest, arbitrary. Of course $f^{(l)+}$ and $f^{(l)-}$ cannot simultaneously be zero.

The columns marked "first correlation" refer to the relation between the cusp in the leading power of x in ρ and the cusp in the leading power of x in δ. Similarly, the columns marked "second correlation" refer to the relation between the cusp of the next-to-leading power of x in ρ and the cusp in the next-to-leading power of x in δ.

In order to explain in some detail the meaning of the first and second correlations, we first define the following four symbols:

$$C_1(\beta,\gamma)=(\beta|q|,\gamma q),$$
$$C_2(\beta,\gamma)=(\beta|q|,\ -\gamma q),$$
$$C_3(\beta,\gamma)=(-\beta|q|,\ \gamma q),$$
$$C_4(\beta,\gamma)=(-\beta|q|,\ -\gamma q),$$

where β, $\gamma>0$. These symbols describe the four corresponding pictures in Fig. 1, with one added specification: The first (second) argument of C_i denotes the slope of the cusp below (above) threshold. Next we define the 16 correlation symbols

$$(C_i,C_j)=[C_i(\beta,\gamma),C_j(\gamma,\beta)],$$

where the first C in the bracket always refers to ρ and the second C to δ. The symbols E and O which appear in the table each denote four possible choices of correlation symbols, namely

$$E=\text{either } (C_1,C_3),\ (C_2,C_1),\ (C_3,C_4),\text{ or } (C_4,C_2), \quad (40)$$

$$O=\text{either } (C_1,C_2),\ (C_2,C_4),\ (C_3,C_1),\text{ or } (C_4,C_3). \quad (41)$$

Example: If E enters under "first correlation," then (C_1,C_3) means that if the highest power of x in ρ has a cusp $C_1(\beta,\gamma)$, then the highest power in δ has a cusp $C_3(\beta,\gamma)$.

It is characteristic for each correlation symbol which appears in E or O that the ratio of the magnitude of the slope in ρ above (below) the threshold and the magnitude of the slope in δ below (above) the threshold is equal to unity. The factor $(l+1)/l$ or $l/(l-1)$ which multiplies O at certain places in the table means that

[11] An alternatively simple form is given by $M=(\boldsymbol{\sigma}\cdot\mathbf{k}')M_{PS}=g(\boldsymbol{\sigma}\cdot\mathbf{k})+f(\boldsymbol{\sigma}\cdot\mathbf{k}')$. In this case the l value in $f^{l\pm}$ refers to the orbital angular momentum in the initial πN state. The expressions for the cross section and the polarization obtained from M_S are still given by Eqs. (4) and (5) except for an additional minus sign in the polarization Eq. (5).

[12] In order to distinguish scalar and pseudoscalar K in this reaction, it is necessary to use polarized targets.

these ratios should instead be $(l+1)/l$ or $l(l-1)$, respectively.

An entry "none" means that under the stated conditions on the f's there is no prescribed correlation (of the first and/or the second kind).

An entry "gap" under "second correlation" means that the power of x which is one less than the leading one is missing in both ρ and δ.

The entries in the column "ρ^2/σ" mean the following. If the leading term in ρ is $\sim x^m$ and the leading power in $(d\sigma/d\Omega)_t$ is x^n, then the entry is x^{2m-n}.

In the last column we have written out for the example $l=2$ the states for which the amplitudes are in general nonvanishing. As we go down the table, more and more gaps develop in the set of states until in the last two cases rather peculiar combinations occur. We would not have bothered with cases 11 and 12, for example, if it were not for the fact that the significant occurrence of partial waves with $l>1$ in ΛK production is in itself a somewhat peculiar phenomenon at the relatively low lab energy of 900 Mev. Thus it may actually be true that some gaps do exist of a kind found in the table. It should further be noted that in principle there may be further correlations, provided that the set of nonvanishing amplitudes shows even wider gaps. It seems to us to be reasonable to ignore such possibilities for the present.

Next we make the following comments on what one may hope to learn from experiment with the help of this table.

(1) The bothersome cases are those which have "none" entries for both the first and the second correlations. The reason is, of course, that a "none" case in a given column may by accident fake the result in another place in the same column which has a prescribed correlation. It should be noted, though, that a "double accident" is necessary for this to happen. Indeed, the real and the imaginary parts, separately, of certain linear combinations of at least two scattering amplitudes must have just the right values before a fake result can take place which satisfies all the specifications mentioned above. Thus, if any particular correlation prescribed by the table is actually found we cannot say with certainty that $P(\Sigma\Lambda)$ has been determined, but it seems fair to us to say that such an experimental finding would carry a strong presumption.

(2) The cases "none" both in the first and second correlation always have a "ρ^2/σ" value equal to 1. If "ρ^2/σ" can be found experimentally, it is therefore only possible to have a fake result for cases with "ρ^2/σ"$=1$.

(3) The case "none" both in the first and second correlation occurs for even as well as for odd $P(\Sigma\Lambda)$. However, for odd $P(\Sigma\Lambda)$ there must be characteristic gaps in the set of contributing states if this case is to obtain.

(4) Only case 2 is fully Minami invariant. For this case, the first correlation is sufficient to resolve all ambiguities.

(5) The sets of correlation symbols in E and O are disjoint. Thus, barring the above-mentioned accidents, which can occur only when "ρ^2/σ"$=1$, the table contains 14 mutually exclusive instances of prescribed correlations which could possibly fix $P(\Sigma\Lambda)$. Only the cases (O,O,x^2) $(O$, none, $x)$ and $(O$, gap, $x^2)$ each occur more than once, but always for odd $P(\Sigma\Lambda)$ only.

(6) If experiment should tell us that we are *not* in any of the cases with prescribed correlations, then we are necessarily in the none-none situation. In that case the table tells us what is sufficient additional information to determine $P(\Sigma\Lambda)$; namely, we have to know if there are gaps in the set of states.

(7) As a general rule, if the leading power in ρ is x^n, then the leading power in δ is x^{n-1}. The only exception can come about by a fortuitous cancelation for those cases where the first correlation is of the "none" type.

(8) In practice, one will compare the cross-section cusp with the polarization cusp at energies above and below threshold for which $q=|q|$.

(9) It has been customary to ask the question: Given the participating states, what are the cusp properties? The table shows that circumstances may arise where (always barring the double accidents mentioned above) it may be possible to turn the problem around and derive information about contributing states from the correlations between cusps in ρ and δ.

Finally, we would like to show by one example the effects of corrections due to the fact that an "absent" state is never truly absent but has a relatively small amplitude. Take the case of even $P(\Sigma\Lambda)$, $D_{\frac{3}{2}}$ present, $D_{\frac{5}{2}}$ absent. The first correlation is then of type E. Now introduce a $D_{\frac{5}{2}}$ amplitude which is small compared to $D_{\frac{3}{2}}$:

$$f^{(2)+}=re^{i\phi}f^{(2)-},\quad |r|\ll 1,\quad 0<\phi<2\pi.$$

Denote by $\rho(a)$ $(\rho(b))$ the coefficient of P_2 in ρ above (below) threshold for this case, and introduce likewise $\delta(a)$ $(\delta(b))$. We have

$$\begin{aligned}\frac{\rho(a)}{\delta(b)}&=\frac{A[1+\frac{3}{2}r\cos\phi]+B\times\frac{3}{2}r\sin\phi}{-A[1-\frac{3}{2}r\cos\phi]+B\times\frac{3}{2}r\sin\phi}\rightarrow -1\quad\text{for}\quad r\rightarrow 0,\\ \frac{\rho(b)}{\delta(a)}&=\frac{-B[1+\frac{3}{2}r\cos\phi]+A\times\frac{3}{2}r\sin\phi}{-B[1-\frac{3}{2}r\cos\phi]-A\times\frac{3}{2}r\sin\phi}\rightarrow +1\quad\text{for}\quad r\rightarrow 0,\\ A&=\mathrm{Im}[\alpha^*f^{(2)-}],\quad B=\mathrm{Re}[\alpha^*f^{(2)-}].\end{aligned}\tag{42}$$

The limit values of these two ratios are just equivalent to the statement that for $D_{\frac{5}{2}}$ absent we are in class E. Consider some numerical values as examples. For $\phi=0$, $r=\frac{1}{10}$, we get deviations from the $r=0$ values which are ~ 0.25 in magnitude. As another limiting case take

$\phi=\pi/2$. Now the corrections depend as well on the relative magnitudes of A and B. For $A\approx B$ we get practically the same numerical results as were quoted just before. If A and B differ considerably in magnitude, the corrections get much smaller for one, much larger for the other ratio. Thus for $A\gg B$ the value of $\rho^{(a)}/\delta^{(b)}$ stays quite close to the limit value -1, while the other ratio may undergo drastic changes. For increasing r we move toward the cases $D_{\frac{5}{2}}$, $D_{\frac{3}{2}}$, $\cdots$ which have "none" as first correlation. Our example for even $P(\Sigma\Lambda)$ was actually a less favorable one; it is readily verified that there is relatively less disturbance for odd $P(\Sigma\Lambda)$. These orienting remarks may at least suffice until detailed experimental information becomes available.

III. DOUBLE CUSP

In this section we discuss the problem of the finite separation of the Σ^0K^0 and Σ^-K^+ thresholds due to the observed $\Sigma^0\Sigma^-$ and K^0K^+ mass differences.[7] We confine ourselves to the case of even relative (K^+K^0) parity[13] and retain those relations between transition amplitudes obtained from charge independence, neglecting mass differences. Furthermore, we assume that these thresholds are sufficiently close so that the approximation of keeping only terms linear in momentum in either of the ΣK channels is valid in an interval containing both thresholds. In this case we obtain, by a simple extension of the single cusp arguments,[3,14]

$$P(\Sigma\Lambda)\text{ even:}\quad f^{0+}=f_t^{0+}+i[q_0+2q_+]\alpha, \tag{43}$$

$$P(\Sigma\Lambda)\text{ odd:}\quad f^{1-}=f_t^{1-}+i[q_0+2q_+]\alpha, \tag{44}$$

where q_0 and q_+ are the respective c.m. momenta in the Σ^0K^0 and Σ^-K^+ channels, to be replaced by $i|q_0|$ and $i|q_+|$ below their respective thresholds. α is the product of the transition amplitudes $\pi+N\rightarrow\Sigma+K$ and $\Sigma+K\rightarrow\Lambda+K$ for total isotopic spin $\frac{1}{2}$. The factor 2 in Eqs. (29) and (30) is the relative probability of Σ^-K^+ to Σ^0K^0 in a ΣK state of isotopic spin $\frac{1}{2}$. The amplitudes f_t^{0+} and f_t^{1-} are constants, but unlike the single cusp case these do not correspond to threshold amplitudes. Instead, we have to satisfy the following relations for the amplitudes $f_{t_0}^{0+}$ $(f_{t_0}^{1-})$ and $f_{t_+}^{0+}$ $(f_{t_+}^{1-})$ at the Σ^0K^0 (subscript t_0) and Σ^-K^+ (subscript t_+) thresholds:

$$P(\Sigma\Lambda)\text{ even:}\quad \begin{aligned} f_{t_0}^{0+}&=f_t^{0+}-2|q_+|\alpha,\\ f_{t_+}^{0+}&=f_t^{0+}+iq_0\alpha;\end{aligned} \tag{45}$$

$$P(\Sigma\Lambda)\text{ odd:}\quad \begin{aligned} f_{t_0}^{1-}&=f_t^{1-}-2|q_+|\alpha,\\ f_{t_+}^{1-}&=f_t^{1-}+iq_0\alpha.\end{aligned} \tag{46}$$

If we substitute Eqs. (43) and (44) in Eqs. (4) and (5) we obtain again Eqs. (16)–(19), where now q and $|q|$ are given by

$$\begin{aligned} q&=q_0+2q_+ \quad &(a),\\ |q|&=|q_0|+2|q_+| \quad &(b),\end{aligned} \tag{47}$$

and where the notation (a) and (b) now means

$$\begin{aligned} (a)&=\text{above both }\Sigma^0K^0\text{ and }\Sigma^-K^+\text{ threshold},\\ (b)&=\text{below both }\Sigma^0K^0\text{ and }\Sigma^-K^+\text{ threshold}.\end{aligned} \tag{48}$$

For practical evaluations of Eqs. (16)–(19) the relations (45) and (46) should be used with the understanding that terms of higher order than the first in q_0 and/or q_+ should be dropped, consistent with the linear approximation to these quantities which we have used throughout.

In addition, however, there exists now, also, a third region which we denote by (ab):

$$(ab)=\text{above }\Sigma^0K^0\text{ but below }\Sigma^+K^+\text{ threshold},$$

where

$$\begin{aligned} \rho_e&=-|q_0|\,\mathrm{Re}[\alpha^*(f_t+xg_t)]+2q_+\,\mathrm{Im}[\alpha^*(f_t+xg_t)],\\ \delta_e&=-|q_0|\,\mathrm{Im}[\alpha^*g_t]-2q_+\,\mathrm{Re}[\alpha^*g_t],\end{aligned} \tag{49}$$

$$\begin{aligned} \rho_0&=-|q_0|\,\mathrm{Re}[\alpha^*(gt+xf_t)]+2q_+\,\mathrm{Im}[\alpha^*(gt+xf_t)],\\ \delta_o&=|q_0|\,\mathrm{Im}[\alpha^*f_t]+2q_+\,\mathrm{Re}[\alpha^*f_t].\end{aligned} \tag{50}$$

Substituting in Eqs. (49)–(50) the expansions of f_t and g_t in angular momentum amplitudes Eqs. (2)–(3) and assuming as before that $f'^{l\pm}=0$ for all $l'>l$, we obtain

$$\begin{aligned} \rho_e=\{&-|q_0|\,\mathrm{Re}\alpha^*+2q_+\,\mathrm{Im}\alpha^*\}[(l+1)f_t^{l+}+lf_t^{l-}]P_l\\ &+[lf_t^{(l-1)+}+(l-1)f_t^{(l-1)}]P_{l-1}+\cdots,\end{aligned} \tag{51}$$

$$\begin{aligned} \delta_e=\{&-|q_0|\,\mathrm{Im}\alpha^*-2q_+\,\mathrm{Re}\alpha^*\}\\ &\times(2l-1)[f_t^{l-}-f_t^{l+}]P_{l-1}\\ &+(2l-3)[f_t^{(l-1)-}-f^{(l-1)+}]P_{l-2}\\ &+(2l-5)[f_t^{(l-2)-}-f_t^{(l-2)+}\\ &\qquad+f_t^{l-}-f_t^{l+}]P_{l-3}+\cdots,\end{aligned} \tag{52}$$

$$\begin{aligned} \rho_o=\{&-|q_0|\,\mathrm{Re}\alpha^*+2q_+\,\mathrm{Im}\alpha^*\}\\ &\times[(l+1)f_t^{l+}]P_{l+1}+[lf_t^{(l-1)+}]P_{l-}\\ &+[lf_t^{l-}+(l-1)f_t^{(l-2)+}]P_{l-1}+\cdots,\end{aligned} \tag{53}$$

$$\begin{aligned} \delta_o=\{&|q_0|\,\mathrm{Im}\alpha^*+2q_+\,\mathrm{Re}\alpha^*\}\\ &\times[(2l+1)f_t^{l+}]P_l+[(2l-1)f^{(l-1)+}]P_{l-1}\\ &+(2l-3)[f^{(l-2)+}+f^{l+}-f^{l-}]P_{l-2}+\cdots.\end{aligned} \tag{54}$$

It is clear that above and below both ΣK thresholds our discussion of the single cusp case applies here as well, provided we identify q in terms of q_0, and q_+ by the expressions given in Eq. (47). In particular, the table of cusp properties can be used without further modifications. As in the single cusp, one will again compare the cross-section cusp at an energy in the region a (or b) with the polarization cusp in at an energy in the region

[13] Dr. Frank Crawford has pointed out that a direct way of obtaining $P(K^+,K^0)$ is to measure the cross section for the process $\pi^-+p\rightarrow\Sigma^0+K^0$ near the Σ^-K^+ threshold. Since the Σ^0K^0 and Σ^-K^+ thresholds are very close, one expects only S waves in the Σ^0K^0 channel at the Σ^-K^+ threshold, and therefore the presence (absence) of a cusp indicates $P(K^+,K^0)$ even (odd). In the ΛK^0 channel this cusp is a "cusp within a cusp" which is a nonlinear effect.

[14] J. Sucher, G. A. Snow, and T. B. Day, Phys. Rev. **122**, 1645 (1961).

b (or a), where the two energies are chosen to satisfy $q=|q|$.

The region (ab) has, of course, no parallel in the single-cusp problem and has been included here only for the sake of completeness. No simple cusp correlation properties in this region have been found.

The finite separation of the two ΣK thresholds, is, of course, an effect which violates charge independence. It may therefore be asked if it is justified to assign the above value 2 to the Σ^-K^+/Σ^0K^0 ratio. Small deviations from the value 2 can be easily incorporated in our considerations; however, large departures from 2 (including the possiblity of complex numbers) would lead to a much more intricate situation.

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Broken Symmetries and Bare Coupling Constants*

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There are known cases of symmetry laws valid for one kind of interaction but broken by another. Each symmetry is then supposed to be exact for bare masses and coupling constants but only approximate for the renormalized quantities, like neutron and proton masses. We ask how the equality of unrenormalized constants can be rephrased as a statement about measurable quantities. This question is particularly important in connection with proposed strong-interaction symmetries that are supposed to be badly broken. The answer appears to involve the limits of ratios of experimental quantities at very high momenta. We discuss first the connection between wave-function renormalizations and weak and electromagnetic form factors. Then we take up the measurement of strong-interaction vertex renormalization factors by the study of scattering amplitudes at energies and momentum transfers large compared to all masses. The last part of the work is based in part on indications from the perturbation development of pseudoscalar meson theory, but we hope it will point the way to similar results in a better theory.

I. INTRODUCTION

THERE is no question that broken symmetries are of the highest importance in particle physics. We are familiar with the conservation of the isotopic spin current, which is violated by electromagnetism, and the conservation of the strangeness or hypercharge current, which is violated by the weak interactions. In both of these cases, the violations are small.

Recently it has been suggested that there may be other conservation laws that are badly broken but nevertheless correct in some limit. Some examples of proposed "partially-conserved currents" are the following:

(a) The axial vector currents in the weak interactions.[1-3] Here, the conservation law is broken by the masses of some particles if by nothing else.

(b) Strangeness-changing vector currents.[4-6] Partial conservation has been suggested in this case not only for the weak interactions, but also as a manifestation of a symmetry of the strong interactions higher than charge independence. The violation takes place through the mass differences of the various baryons and of the various mesons and perhaps through some strong interactions as well. (In the global symmetry scheme, the culprit was supposed to be the K-meson coupling.)

Such proposals of partially-conserved currents are incomplete without some statement of how, in principle, the limit of exact conservation can be explored experimentally. The same is true, really, of the conservation of isotopic spin and strangeness, although in those cases the smallness of the violation makes it clear that there is *some* sense to the conservation law, even without a precise statement of the limit in which the conservation is exact.

The conservation of isotopic spin is usually stated as follows: The bare masses of neutron and proton, say,

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