$b$ (or $a$), where the two energies are chosen to satisfy $q=|q|$.

The region ($ab$) has, of course, no parallel in the single-cusp problem and has been included here only for the sake of completeness. No simple cusp correlation properties in this region have been found.

The finite separation of the two $\Sigma K$ thresholds, is, of course, an effect which violates charge independence. It may therefore be asked if it is justified to assign the above value 2 to the $\Sigma^- K^+/\Sigma^0 K^0$ ratio. Small deviations from the value 2 can be easily incorporated in our considerations; however, large departures from 2 (including the possiblity of complex numbers) would lead to a much more intricate situation.

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# Broken Symmetries and Bare Coupling Constants*

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There are known cases of symmetry laws valid for one kind of interaction but broken by another. Each symmetry is then supposed to be exact for bare masses and coupling constants but only approximate for the renormalized quantities, like neutron and proton masses. We ask how the equality of unrenormalized constants can be rephrased as a statement about measurable quantities. This question is particularly important in connection with proposed strong-interaction symmetries that are supposed to be badly broken. The answer appears to involve the limits of ratios of experimental quantities at very high momenta. We discuss first the connection between wave-function renormalizations and weak and electromagnetic form factors. Then we take up the measurement of strong-interaction vertex renormalization factors by the study of scattering amplitudes at energies and momentum transfers large compared to all masses. The last part of the work is based in part on indications from the perturbation development of pseudoscalar meson theory, but we hope it will point the way to similar results in a better theory.

## I. INTRODUCTION

THERE is no question that broken symmetries are of the highest importance in particle physics. We are familiar with the conservation of the isotopic spin current, which is violated by electromagnetism, and the conservation of the strangeness or hypercharge current, which is violated by the weak interactions. In both of these cases, the violations are small.

Recently it has been suggested that there may be other conservation laws that are badly broken but nevertheless correct in some limit. Some examples of proposed "partially-conserved currents" are the following:

(a) The axial vector currents in the weak interactions.[1–3] Here, the conservation law is broken by the masses of some particles if by nothing else.

(b) Strangeness-changing vector currents.[4–6] Partial conservation has been suggested in this case not only for the weak interactions, but also as a manifestation of a symmetry of the strong interactions higher than charge independence. The violation takes place through the mass differences of the various baryons and of the various mesons and perhaps through some strong interactions as well. (In the global symmetry scheme, the culprit was supposed to be the $K$-meson coupling.)

Such proposals of partially-conserved currents are incomplete without some statement of how, in principle, the limit of exact conservation can be explored experimentally. The same is true, really, of the conservation of isotopic spin and strangeness, although in those cases the smallness of the violation makes it clear that there is *some* sense to the conservation law, even without a precise statement of the limit in which the conservation is exact.

The conservation of isotopic spin is usually stated as follows: The bare masses of neutron and proton, say,

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[1] M. Gell-Mann and M. Lévy, Nuovo cimento **16**, 705 (1960).

[2] M. Gell-Mann, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960).

[3] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo cimento **17**, 757 (1960).

[4] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

[5] J. Schwinger, Ann. Phys. **2**, 407 (1957).

[6] M. Gell-Mann (to be published).

are equal. The bare coupling constants of $n$ and $p$ to $\pi^0$, say, are equal and opposite. Under the influence of electromagnetic interactions, however, the physical masses $m_n$ and $m_p$ and the renormalized (squared) coupling constants $g_1^2(n)$ and $g_1^2(p)$ are slightly different. It would be desirable to be able to check experimentally the equality of $m_{0n}$ and $m_{0p}$ and of $g_0^2(n)$ and $g_0^2(p)$. If these quantities are meaningless, as sometimes claimed, then a new statement of the principle of charge independence is required.

In the case of badly broken symmetries, a statement of where we must look to find the symmetry in pure form is not only desirable, but absolutely necessary, since otherwise we have no way of testing whether the partial conservation law has any meaning or not.

The clues we have at present all point in the same direction, namely toward the realm of high frequencies. Suppose the correct statement of a symmetry or universality principle involves bare quantities. We know that in our present field theories there is a close connection between bare quantities and the behavior of certain renormalized quantities at high energies or momentum transfers. We might expect, then, that the equality of two bare quantities could be restated in terms of the limit of the ratio of two renormalized quantities approaching unity at high frequencies.

In the case of the axial vector $\Delta S=0$ current $P_\alpha$, it has been shown[3,7] that if the matrix elements of $\partial_\alpha P_\alpha$ between the vacuum and states of mass $M$ approach zero rather rapidly as $M\rightarrow\infty$, that would provide an explanation of the Goldberger-Treiman formula for the rate of charged pion decay. Thus it is conjectured that in a sense $\partial_\alpha P_\alpha\rightarrow 0$ at high frequency; in other words, the axial vector current is conserved in the limit of high frequencies.

In general, if it is supposed that certain conservation laws are broken by masses or mass differences, then it should be possible to find cases of exact conservation in the limit where all relevant energies are large compared to all masses. Our argument about bare quantities indicates that even when the conservation laws are broken by interactions, it may be possible to find high-energy limits in which universality becomes exact.

In this article we shall discuss some examples of ways in which ratios of bare coupling constants may be explored experimentally in the high-energy limit. We shall make use of existing, probably wrong, field theories and we shall be guided in some cases by the probably misleading perturbation expansions of those theories, but we believe that some of the results may be valuable in pointing the way toward the study of exact universality in the high-energy domain.

[7] Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

## II. RENORMALIZATION CONSTANTS AND FORM FACTORS

As an example of a renormalizable field theory with strong interactions, let us consider nucleons and pions with pseudoscalar coupling. We shall consider electromagnetic and weak interactions in first order only; in other words, we shall be concerned only with matrix elements of the various currents within the strong interaction theory. In our approximation, isotopic spin conservation, for example, is exact.

The bare coupling constant $g_0$ for the Yukawa coupling is given by the relation

$$g_0=g_1Z_1Z_2^{-1}Z_3^{-\frac{1}{2}}, \tag{2.1}$$

where $g_1$ is the renormalized coupling constant, $Z_1^{-1}$ is the vertex renormalization factor, and $Z_2$ and $Z_3$ are the renormalization factors for the nucleon and pion propagators, respectively. In this section, we shall discuss the physical significance of $Z_2$, $Z_3$, and some related quantities, leaving $Z_1$ for the next section.

In the perturbation expansion of the field theory, the $Z$'s are power series in $g_1^2/4\pi$ with coefficients depending logarithmically on a cutoff $\Lambda$, taken large compared to all masses. We have for each $Z$ a function $Z(\Lambda^2/m^2)$ of the cutoff. (Dependence on $g_1^2/4\pi$ and on the various mass-ratios is not explicitly indicated.)

We know, however, that the same asymptotic functions $Z$ occur also in the expressions for renormalized quantities.[8] For instance, the renormalized meson propagator is[9]

$$\Delta_{F1}(k^2)=[d(k^2/m^2)/k^2+m_\pi^2], \tag{2.2}$$

and asymptotically[10]

$$d(k^2/m^2)\approx Z_3^{-1}(k^2/m^2), \tag{2.3}$$

where the function $Z_3$ now has a momentum instead of a cutoff as its argument. Similarly,

$$S_{F1}(k)=[s(k^2/m^2)+(\boldsymbol{k}/m)\epsilon(k^2/m^2)/-\boldsymbol{k}+m], \tag{2.4}$$

where asymptotically

$$s(k^2/m^2)\approx Z_2^{-1}(k^2/m^2) \tag{2.5}$$

and $\epsilon$ is negligible.

While we have related the functions $Z$ to asymptotic forms of renormalized quantities, we have not yet connected them with quantities that are directly measurable.

Let us now consider the matrix elements of the electric current $j_\mu$. The matrix element between physical nucleon states has an electric term

$$\gamma_\mu[F_1{}^S(k^2/m^2)+F_1{}^V(k^2/m^2)\tau_z], \tag{2.6}$$

[8] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[9] We adopt the following notation: $p^2=p_\mu p_\mu=\mathbf{p}^2-p_0^2$ is positive for a spacelike four-vector $p_\mu$. The Dirac $\gamma$ matrices are defined so that $\gamma_\mu$ and $\gamma_5$ are all Hermitian, and we use the shorthand $\boldsymbol{p}=-i\gamma_\mu p_\mu$.

[10] Note that for $k^2$ large and timelike (i.e., negative) the argument of $Z_3^{-1}$ is negative and $Z_3$ is a complex function.

where $F_1{}^S(k^2)$ is the electric isoscalar form factor and $F_1{}^V(k^2)$ is the electric isovector form factor for the nucleon.

Now just as the asymptotic propagators are related to the renormalization factors $Z_2$ and $Z_3$, so the asymptotic form factors are related to vertex renormalization constants. Here we have an electric vertex, however, so it is the electric vertex renormalizations that enter. Let the renormalization factor for $\gamma_\mu$ be $Z_{ES}{}^{-1}(\Lambda^2/m^2)$ and that for $\gamma_\mu\tau_Z$ be $Z_{EV}{}^{-1}(\Lambda^2/m^2)$. Then asymptotically[11]

$$\begin{aligned}F_1{}^S(k^2/m^2)&\approx Z_{ES}(k^2/m^2),\\ F_1{}^V(k^2/m^2)&\approx Z_{EV}(k^2/m^2).\end{aligned}\tag{2.7}$$

But for the electric vertex there is the famous Ward identity connecting $Z_E$ with $Z_2$. It holds separately for the isoscalar and isovector form factors and identically in the cutoff for $\Lambda^2 \gg m^2$. Thus

$$Z_{ES}(\Lambda^2/m^2)\approx Z_{EV}(\Lambda^2/m^2)\approx Z_2(\Lambda^2/m^2),\tag{2.8}$$

and so we have connected the asymptotic form factors with $Z_2$:

$$F_1{}^S(k^2/m^2)\approx F_1{}^V(k^2/m^2)\approx Z_2(k^2/m^2).\tag{2.9}$$

In exactly the same way, we can relate $Z_3$ to the form factor $F_\pi$ of the charged pion, which is directly measurable in $\pi-e$ scattering at very high energies and indirectly measurable in a number of ways using "polology." We have

$$F_\pi(k^2/m^2)\approx Z_3(k^2/m^2).\tag{2.10}$$

We can draw some interesting conclusions from the asymptotic equalities we have discussed. We recall that $Z_3$ is the asymptotic form of $d^{-1}(k^2/m^2)$ and make use of the parametric representation[8,11] of $d$:

$$d(k^2/m^2)=1+(k^2/m_\pi{}^2)\times\int[dM^2\,\rho(M^2)/k^2+M^2-i\epsilon],\tag{2.11}$$

where $\rho$ is real and positive. For positive (i.e., spacelike) $k^2$, the function is positive and increasing. If $d\to\infty$ as $k^2/m^2\to\infty$, then we say that $Z_3=\lim_{\Lambda^2\to\infty}Z_3(\Lambda^2/m^2)$ is zero. If not, then $Z_3$ is positive and less than one. If field theory is inconsistent with itself, suffering from the pathology called "ghosts" by Källén and Pauli[12] and "zeroness of the charge" by Landau and collaborators,[13] then a calculation of $d$ will reveal a physically impossible behavior, viz.: rising to $+\infty$ at some positive value of $k^2$ and then coming up from $-\infty$ above that value.

Landau and his associates[13] claim that field theory actually exhibits this pathology. Landau[14] states, however, that perhaps if we use dispersion relations to define theory and stick to measurable quantities, then the ghosts will not trouble us.

We do not agree. We do not know if field theory is pathological, but if $d(k^2/m^2)$ *does* have an unphysical behavior, then the theory is wrong no matter how it is expressed, since asymptotically $d(k^2/m^2)\approx F_\pi(k^2/m^2)$ and is measurable. We shall, in what follows, assume that there are no ghosts. We can then write a dispersion relation for $d^{-1}$, as follows[15]:

$$d^{-1}(k^2/m^2)=1-(k^2+m_\pi{}^2)\times\int\frac{dM^2}{k^2+M^2-i\epsilon}\frac{\rho(M^2)}{|d(-M^2/m^2)|^2}.\tag{2.12}$$

Thus $F_\pi\approx Z_3\approx d^{-1}$ is asymptotic to a form factor with uniformly negative weight function $\rho(M^2)\times|d(-M^2/m^2)|^{-2}$. This is an interesting result, since we do not otherwise know anything about the sign of the weight function in the dispersion relation for the form factor. If $Z_3=0$, then $F_\pi\to 0$ at $\infty$ and the form factor obeys a dispersion relation with no subtractions.

The same argument holds for the nucleon form factors $F_1{}^S$ and $F_1{}^V$. They are asymptotically equal and obey a dispersion relation with asymptotically negative weight function. If $Z_2=0$, they tend to zero at infinity.

In the conserved current theory of the vector interaction in $\beta$ decay, $F_1{}^V$ is the form factor of the vector weak current also. We may define in this case too a vertex renormalization $Z_V{}^{-1}$, which by Ward's identity equals $Z_2{}^{-1}$. The renormalization factor $G_V/G$ relating the Fermi constant $G_V$ in $\beta$ decay to the unrenormalized constant $G$ (which presumably equals the Fermi constant $G_\mu$ in $\mu$ decay) is then

$$(G_V/G)=(Z_2/Z_V)=\lim_{k^2\to\infty}[F_1{}^V(k^2/m^2)/F_1{}^V(k^2/m^2)]=1,\tag{2.13}$$

as is well known.

For the axial vector current, which is not conserved (although it may be conserved at high frequencies), we can obtain a less trivial result.[16] We define a vertex renormalization $Z_A{}^{-1}$ and a form factor $F_A(k^2/m^2)$ for the axial vector current. Then the renormalization factor $-G_A/G$ (we assume the same $G$ as for the vector case) is given by the relation

$$(-G_A/G)=(Z_2/Z_A)=\lim_{k^2\to\infty}[F_1{}^V(k^2/m^2)/F_A(k^2/m^2)].\tag{2.14}$$

[11] G. Källén, Helv. Phys. Acta, **25**, 417 (1952). G. Källén, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 5, Part 1, pp. 358–363. See also, Appendix A.

[12] G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 7 (1955).

[13] L. D. Landau, A. Abrikosov, and I. Khalatnikov, Doklady Akad. Nauk U.S.S.R. **95**, 773 (1954).

[14] L. D. Landau, lecture at the Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (unpublished).

[15] G. Källén, reference 10, and P. Redmond, Phys. Rev. **112**, 1404 (1958).

[16] K. Symanzik, Nuovo cimento **11**, 269 (1959).

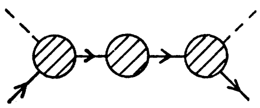

FIG. 1. General form of the Feynman graphs included in the uncrossed single integral terms for $\pi N$ scattering.

Here the form factor $F_A$ can be studied experimentally in the reaction $\bar{\nu}+p \rightarrow e^+ + n$ at high energies. It will be very interesting to see whether the ratio $F_1^V/F_A$ approaches 1.25 at some attainable energy.

The ratio $F_1^V/F_1^S$, which must approach unity at large $k^2$, is actually close to unity at small $k^2$ (i.e., the charge form factor of the neutron is very small). It will be useful here too to know how the ratio behaves experimentally at larger $k^2$. We need to have some reliable information on how large a value of $k^2/m^2$ is needed for the asymptotic relations to become valid in practice (assuming, of course, that they are correct).

In the next section, we shall extend our work to include the vertex renormalization factor $Z_1$ for the Yukawa interaction. Assume for the moment that $Z_1(k^2/m^2)$ is measurable too. Then we can say something about the problem of universality for the strong interactions.

A principle of higher symmetry for the strong interactions may state that $g_0$ for nucleons equals $g_0$ for $\Xi$ particles. The renormalized coupling constants $g_1$ will not be exactly equal because of the $\Xi - N$ mass difference and possibly some interactions that break the higher symmetry. We want a statement of the universality principle in terms of the limit of measurable quantities. Referring to Eq. (2.1), we see that the statement is

$$\frac{g_{1N}}{g_{1\Xi}} = \lim_{k^2\to\infty} \frac{F_{1N}(k^2/m^2)}{F_{1\Xi}(k^2/m^2)} \frac{Z_{1\Xi}(k^2/m^2)}{Z_{1N}(k^2/m^2)}, \tag{2.15}$$

where $F_1$ is either $F_1^S$ or $F_1^V$.

We have supposed that the pion is a fundamental particle and that the principle of higher symmetry is stated in terms of *its* bare couplings. It may be, however, that the pion is composite and that the true Yukawa particles are vector mesons.[6,17]

Unfortunately, the present theory of vector mesons (at least those with isotopic spin greater than zero) is in an unsatisfactory state and seems to show unrenormalizable divergences. Thus we are unable, at the moment, to generalize our results in a convincing way to vector mesons. Improving the theory of vector particles is now a major challenge to theoreticians.

## III. PION-NUCLEON SCATTERING. SINGLE INTEGRAL TERMS

In order to investigate experimental procedures for the determination of the vertex renormalization $Z_1$ of the pion-nucleon interaction, we turn our attention next to pion-nucleon scattering. Although we shall confine our detailed discussions to this specific process, it will be evident that the same type of argument is applicable to many others.

The pion-nucleon scattering amplitude is conventionally decomposed into four scalar amplitudes $A^\pm$ and $B^\pm$ by the expression

$$T_{\sigma_2\sigma_1}(p_2q_2,p_1q_1) = \frac{1}{(16E_1E_2\omega_1\omega_2)^{\frac{1}{2}}}\left(\bar{u}_{p_2}\left|\left\{A^+\delta_{\sigma_2\sigma_2}+A^-\frac{[\tau_{\sigma_2},\tau_{\sigma_1}]}{2}\right\}+\left\{B^+\delta_{\sigma_2\sigma_1}+B^-\frac{[\tau_{\sigma_2},\tau_{\sigma_1}]}{2}\right\}\left\{\frac{\mathbf{q}_2+\mathbf{q}_1}{2}\right\}\right|u_{p_1}\right). \tag{3.1}$$

$T$ is the amplitude for scattering a meson with charge $\sigma_1$ and 4-momentum $q_1$ from a nucleon with 4-momentum $p_1$ into the corresponding final state. The functions $A^\pm$ and $B^\pm$ are scalar functions of the two independent variables describing the scattering, which may conveniently be chosen to be $s$ and $t$, where $s=-(p_1+q_1)^2$ is the square of the total center-of-mass energy, and $t=-(p_1-p_2)^2$ is the invariant momentum transfer. It is also useful to define a "crossed momentum transfer" $u=-(p_1-q_2)^2$; we then have $s+t+u=2m^2+2\mu^2$, when $m$ and $\mu$ are the nucleon and meson masses, respectively.

We shall be concerned with high energies in what follows; that is to say with large values of $s$, $t$, and $u$. Because of the kinematical factor $(\mathbf{q}_1+\mathbf{q}_2)/2$ in Eq. (3.1) the $B$ term will dominate the elastic cross section in the limit of large $s, t, u$ provided that $t|A^\pm|^2 \ll su|B^\pm|^2$. Perturbation theory indicates that this inequality is true, and we shall henceforth assume it. (See, however, the objections below to the behavior of perturbation theory.)

It is interesting to remark that dominance of the $B$ term corresponds to helicity conservation at high energies.

The $B$ amplitudes, on which we may now concentrate, satisfy the Mandelstam representation[18]:

$$\begin{aligned} B^\pm(s,t) = {} & \frac{g_1^2}{s-m^2} \mp \frac{g_1^2}{u-m^2} \\ & + \frac{1}{\pi}\int \frac{b_1^\pm(s')}{s'-s}ds' \mp \frac{1}{\pi}\int \frac{b_1^\pm(u')}{u'-u}du' \\ & + \frac{1}{\pi^2}\int\int \frac{B_{13}^\pm(s',t')}{(s'-s)(t'-t)}ds'dt' \\ & \mp \frac{1}{\pi^2}\int\int \frac{B_{13}^\pm(u',t')}{(t'-t)(u'-u)}dt'du' \\ & + \frac{1}{\pi^2}\int\int \frac{B_{12}^\pm(u',s')}{(u'-u)(s'-s)}du'ds'. \end{aligned} \tag{3.2}$$

[17] Fermi and Yang, Phys. Rev. **76**, 1739 (1949); E. Teller, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), J. J. Sakurai, Ann. Phys. **11**, 1 (1960).

[18] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

The choice of subtractions here—namely none—is made on the basis of indications from perturbation theory. It seems unlikely, however, that this number of subtractions is sufficient to allow the diffraction type behavior in the limit of large $s$, with a constant total cross section and a diffraction peak depending only on the momentum transfer $t$. Such a behavior requires $\mathrm{Im}B(s,t)\to$ function only of $t$ (as $s\to\infty$ for fixed small negative $t$). While the experimental total $\pi N$ cross section seems to be very closely constant at high energies, it is conceivable that it actually decreases slowly to zero; in such a case (3.2) would be consistent as it stands. In any case, we shall assume (3.2) in accord with the philosophy that we are only looking for indications as to what properties might be true in the high-energy limit of a real theory, and do not pretend that the conventional $\gamma_5$ meson theory is correct.

Since there are no subtractions in (3.2), it is clear that the Feynman graphs encompassed by the single integral terms are all of the general form shown in Fig. 1, and conversely. The double integrals contain only graphs with singularities in *both* independent variables, and in the absence of subtractions there is no ambiguity allowing pieces of a single Feynman graph to appear both in the single and in the double integrals. The pole and single integral terms may therefore be written in terms of the renormalized vertex function and the renormalized propagator:

$$\left(\frac{q_1+q_2}{2}\right)\left\{\left(\frac{g_1^2}{s-m^2}+\frac{1}{\pi}\int\frac{b_1^+(s')}{s'-s}ds'\right)\delta_{\sigma_2\sigma_1}+\left(\frac{g_1^2}{s-m^2}+\frac{1}{\pi}\int\frac{b_1^-(s')}{s'-s}ds'\right)\frac{[\tau_{\sigma_2},\tau_{\sigma_1}]}{2}\right\}$$
$$=g_1^2\Gamma_1(s)\gamma_5\tau_{\sigma_2}S_{F1}(s)\Gamma_1(s)\gamma_5\tau_{\sigma_1}. \quad (3.3)$$

Let us use the high-energy behavior known for the propagator and vertex—namely, that as $s\to\infty$,[8]

$$-S_{F1}(s)\approx[Z_2^{-1}(-s/m^2)/s](p_1+q_1)$$

and

$$\Gamma_1(s)\approx Z_1(-s/m^2).$$

(A sketch proof of the second limit is given in Appendix A; otherwise see Källén.[11]) It follows that

$$\frac{g_1^2}{s-m^2}+\frac{1}{\pi}\int\frac{b_1^\pm(s')}{s'-s}ds' \approx g_1^2Z_1^2(-s/m^2)Z_2^{-1}(-s/m^2)(1/s) \quad (3.4)$$

as $s\to\infty$. Similarly for the crossed terms,

$$\frac{g_1^2}{u-m^2}+\frac{1}{\pi}\int\frac{b_1^\pm(u')}{u'-u}du' \approx g_1^2Z_1^2(-u/m^2)Z_2^{-1}(-u/m^2)(1/u) \quad (3.5)$$

as $u\to\infty$.

First let $u$ have a fixed (negative) value. Then let $s$ (and therefore also $-t$ since $t=2m^2+2\mu^2-u-s$) approach infinity. Since any unsubtracted dispersion integral of the form $\int[f(s')ds'/(s'-s)]$ always vanishes as $s\to\infty$, the limit $s$, $-t\to\infty$ ($u$ fixed) removes all the double integrals and the single integral in $s$. Next, let $-u$ get large; this produces the asymptotic form of the single integral in $u$. Thus we get,[19] for large $u$ and infinite $s$,

$$B^\pm(s,t)\approx g_1^2Z_2^{-1}(-u/m^2)Z_1^2(-u/m^2)(1/u). \quad (3.6)$$

It may be worth discussing this limiting procedure a bit further. In the c.m. system, for large $s$, $u=-(s/2)\times(1+x)$, where $x$ is the cosine of the c.m. scattering angle. Hence, if $s\to\infty$ for fixed $u$, $x=-1-2u/s\to-1$, and the first stage of the limiting procedure is therefore to approach backward angles more and more closely at ever increasing total energies. The parameter $u$ controls the rapidity with which the backward direction is approached with increasing $s$, so the second stage (namely $u$ large) of the limit requires looking at situations where $x=-1$ is approached more and more slowly as $s$ grows. To summarize, then, one measures the differential cross section at an energy $s$ and an angle with cosine equal to $-1-2u/s$, for very large $s$, for fixed $u<0$. One does this for larger and larger values of the parameter $u$; for sufficiently large $u$, the asymptotic region is reached and the amplitude is given by (3.6).

Equation (3.6) may be used as a means of measuring $Z_1$. In order to determine whether a particular symmetry exists, then, one forms the "bare coupling constant" function

$$g_0(-u/m^2)=g_1Z_1(-u/m^2)Z_2^{-1}(-u/m^2)Z_3^{-\frac{1}{2}}(-u/m^2),$$

and compares this function for different processes in the limit of large $-u$, as mentioned in Sec. II.

All the main features which have been discussed for $\pi N$ scattering are also true of, for example, $NN$ scattering. There is again an "unrenormalized pole," one in $t$ and one in $u$ this time, so the limiting procedure which picks it out is now $s$, $-u\to\infty$ and $-t$ large or equally well $s$, $-t\to\infty$ and $-u$ large. The limit now involves $g_1^2Z_1^2(t/m^2)Z_3^{-1}(t/m^2)$.

## IV. DOUBLE INTEGRAL TERMS. THE "RENORMALIZATION GROUP"[20]

Let us now see what can be said about the double integral terms in $B$ from an inspection of the perturbation series. Let us ignore, for the time being, all virtual $\pi$-$\pi$ scattering and forget other virtual particles, such as $K$ mesons, etc.

[19] Note that in the physical region $u$ is negative, and the $Z$'s in Eq. (3.6) are therefore real.

[20] The following remarks are the extension of the work of Gell-Mann and Low (reference 8) to scattering processes, and are based on the same criterion, namely that if $m$ is set equal to zero in the unrenormalized perturbation series, no infrared divergences appear.

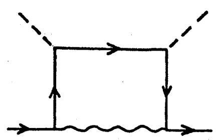

FIG. 2. A diagram destroying the symmetry between $\pi N$ and $\pi\Xi$ scattering, which contributes to the double integral terms in $\pi\Xi$ scattering. The wavy line represents a neutral meson.

The renormalization theory tells us we can write the "uncrossed amplitude," that is, the sum of all uncrossed Feynman graphs, in the form

$$B_{\text{uncrossed}}=\frac{1}{s}Z_2(\Lambda^2/m^2,g_0)Z_3(\Lambda^2/m^2,g_0)\times F(t/s,\Lambda^2/s,m^2/s,g_0) \quad (3.7)$$

(we suppress the dependence on the ratio of pion and nucleon masses), by summing the unrenormalized perturbation series. The perturbation series indicates that in the limit $s, -t\rightarrow\infty$ no infrared divergence results from setting the masses $m^2=0$ in the function $F$. Furthermore, we know that if $g_0$ in Eq. (3.7) is replaced by $g_0(\Lambda^2/m^2,g_1)$, the left-hand side is independent of $\Lambda^2$. Hence $\Lambda^2$ may be replaced by $-s$ on the right-hand side. Thus

$$B_{\text{uncrossed}}=(1/s)Z_2(-s/m^2)Z_3(-s/m^2)\times F(t/s,\,-1,\,0,\,g_0(-s/m^2,\,g_1)), \quad (3.8)$$

where we remark that what we have denoted all along by $Z_{2,3}(-s/m^2)$ is the same as $Z_{2,3}(-s/m^2,\,g_0(-s/m^2,\,g_1))$. Exactly the same is, of course, true of the crossed amplitude with $s$ replaced by $u$.

Equation (3.8) shows that in the high energy limit the entire energy dependence of the scattering amplitude appears through the renormalization quantities $Z_2$, $Z_3$, and $g_0$. Aside from the dependence on $s$ through these, $B_{\text{uncrossed}}$ is a function of angle alone. Furthermore, $Z_2$ and $Z_3$ appear only as explicit multiplying factors of the entire amplitude, the remainder involving only $g_0$. If we expand $F$ in a series in $g_0$, the first term is just $g_0^2$, and we again obtain Eq. (3.4).

These general features can be explicitly verified through 4th-order perturbation theory, as is outlined in Appendix B. It seems that in the high energy limit, the entire perturbation series approaches the sum of only the "skeleton" Feynman graphs with $g$ replaced by $g_0(-s/m^2)$ or $g_0(-u/m^2)$ accordingly as the skeleton graph is uncrossed or crossed, and with the uncrossed series multiplied with $Z_2(-s/m^2)Z_3(-s/m^2)$, the crossed one with $Z_2(-u/m^2)Z_3(-u/m^2)$. Furthermore, according to the above, each skeleton graph, aside from the $g_0$, is a function only of angle.

It is no doubt the case that in *every* amplitude, there is a factor $Z^{\frac{1}{2}}$ for each external particle, where the argument of the $Z$ depends on the energy of the process.

Since we have shown that the entire amplitude for large $s$, $t$, and $u$ involves only $g_0^2$, (apart from the functions $Z_2$ and $Z_3$) it might seem that in studying broken symmetries there is no need to take the limit $s, t, u\rightarrow\infty$ in such a way as to isolate the single integral terms. As an example, let us take a symmetry between the nucleon and the $\Xi$ particle, as mentioned in Sec. II. Ignoring experimental feasibility, one might look at the limit of the ratio of the $\pi N$ and $\pi\Xi$ scattering cross sections. If the broken symmetry really exists, and if we correct for the different $Z_2$'s of the nucleon and $\Xi$, we would expect the symmetry to show up for large $s$, $t$, $u$ regardless of how the limit is taken.

Such may, in fact, be the case if the symmetry is somehow broken by mass difference terms and not by interactions. However, if there are symmetry-violating couplings (analogous to the electromagnetic interactions that violate charge independence), then the double integral terms present an extra complication and it is desirable to eliminate them by taking the special limit.

To illustrate the complication, suppose there is a neutral meson coupled to $\Xi$ but not to $N$. Then in $\pi-\Xi$ scattering, among the double integral terms there is a contribution from the diagram in Fig. 2, which has no counterpart for $\pi N$ scattering. Therefore, the symmetry-violating interaction affects the form of the double integral terms otherwise than simply through the $Z$'s. The single integral terms, however, continue to involve nothing but $Z$'s and the effect of the neutral meson is just to make the $Z_2$ for the $\Xi$ particle different from that of the nucleon. But the ratio of $Z_2$'s is measurable from the ratio of the charge form factors (we continue to choose our examples without regard to ease of measurement) and thus the single integral term permits a check on the broken symmetry even when it is violated by an interaction.

## APPENDIX A

It was mentioned in the text that the vertex function $\Gamma_1(s)\rightarrow Z_1(-s/m^2)$ as $s\rightarrow\infty$. This is a fairly well-known statement[11]; nevertheless for convenience we should like to give a brief pseudo-proof of it. The renormalized vertex operator $\Gamma_1$ satisfies a dispersion relation of the form

$$\Gamma_1(s)=1+(s-\mu^2/\pi)\int[\bar{\rho}(s')/(s'-\mu^2)(s'-s)]ds'. \quad (A.1)$$

If we compare the unrenormalized perturbation series for the unrenormalized vertex function $\Gamma(s,\Lambda^2/m^2,g_0)$ with the renormalized series we infer also that

$$\Gamma(s,\Lambda^2/m^2,g_0)=1+\frac{1}{\pi}\int\frac{\rho(s',\Lambda^2/m^2,g_0)}{s'-s}\left(\frac{\Lambda^2}{s'+\Lambda^2}\right)ds', \quad (A.2)$$

$\Lambda^2$ is a cutoff; we know a cutoff is needed on the unsubtracted dispersion integral, hence the factor $\Lambda^2/$

$(s'+\Lambda^2)$. The standard definition of $Z_1$ is that

$$Z_1(\Lambda^2/m^2,g_0)^{-1}=\Gamma(\mu^2,\Lambda^2/m^2,g_0)$$
$$=1+\frac{1}{\pi}\int\frac{ds'}{s'-\mu^2}\rho(s',\Lambda^2/m^2,g_0)\left(\frac{\Lambda^2}{s'+\Lambda^2}\right)ds'. \quad (A.3)$$

The renormalization theory assures us that

$$\Gamma_1(s)=Z_1(\Lambda^2/m^2,g_0(\Lambda^2/m^2,g))\times\Gamma(s,\Lambda^2/m^2,g_0(\Lambda^2/m^2,g)), \quad (A.4)$$

and $\Gamma_1(s)$ is independent of the cutoff $\Lambda^2$ for large $\Lambda^2$. Using this relation, and comparing (A.1) with (A.2) gives

$$\bar{\rho}(s')=\rho(s',\Lambda^2/m^2,g_0(\Lambda^2/m^2,g))Z_1(\Lambda^2/m^2)\Lambda^2/(s'+\Lambda^2)$$
$$\rightarrow\rho(s',\Lambda^2/m^2,g_0(\Lambda^2/m^2,g))Z_1(\Lambda^2/m^2)$$

as $\Lambda^2\rightarrow\infty$. Since the left side is independent of $\Lambda^2$, we may set $\Lambda^2=-s$ on the right side. Then the asymptotic form of $\Gamma_1(s)$ is

$$\Gamma_1(s)\rightarrow 1+\frac{1}{\pi}\int\frac{ds'}{s'-\mu^2}\frac{s'}{s'-s}\times\rho(s',-s/m^2,g_0(-s/m^2,g))Z_1(-s/m^2)$$
$$=Z_1(-s/m^2), \quad (A.5)$$

according to (A.3).

## APPENDIX B

In fourth-order perturbation theory the relevant Feynman graphs for $\pi N$ scattering are shown in Fig. 3. We wish to compute their contribution to the $\pi N$ scattering amplitude in the limit $s,t,u\rightarrow\infty$. In order to do this, it is necessary to compute only the asymptotic form of the spectral functions in the Mandelstam representation, provided that the order of the limits $s,t,u\rightarrow\infty$ cannot be interchanged with the dispersion integral. This is certainly the case for the $\pi N$ problem in the usual $\gamma_5$ theory. The asymptotic form of the spectral functions is most easily computed using the techniques of Cutkosky[21]; the method is straightforward and the results are:

$$b_1^{\pm}(s)\approx\frac{1}{2}\frac{g_1^4}{16\pi}\frac{1}{s},$$
$$B_{13}^{+}(s,t)\approx-3\frac{g_1^4}{16}\frac{1}{s+t}, \quad (B.1)$$
$$B_{13}^{-}(s,t)\approx+\frac{g_1^4}{16}\frac{1}{s+t}.$$

[21] R. E. Cutkosky, Phys. Rev. Letters 4, 624 (1960).

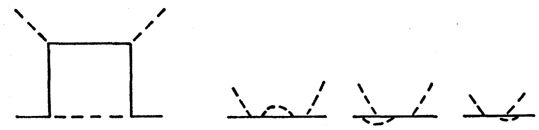

FIG. 3. Feynman graphs for fourth-order $\pi N$ scattering. Only uncrossed graphs are shown.

There is no contribution to $B_{12}^{\pm}(u,s)$ to 4th order. Next, the asymptotic forms of $B^{\pm}$ which follow from (B.1) are

$$B^{+}(s,t)\approx g_1^2\left(\frac{1}{s}-\frac{1}{u}\right)+\left(\frac{g_1^2}{4\pi}\right)^2\left[-\frac{1}{2}\frac{\ln(-s/m^2)}{s}+\frac{1}{2}\frac{\ln(-u/m^2)}{u}\right]+\left(\frac{g_1^2}{4\pi}\right)^2\left[3\frac{\ln^2(-t/s)+i\pi\ln(-t/s)}{t+s}-3\frac{\ln^2(t/u)}{t+u}\right], \quad (B.2)$$

and

$$B^{-}(s,t)\approx g_1^2\left(\frac{1}{s}+\frac{1}{u}\right)+\left(\frac{g_1^2}{4\pi}\right)^2\left[-\frac{1}{2}\frac{\ln(-s/m^2)}{s}-\frac{1}{2}\frac{\ln(-u/m^2)}{u}\right]+\left(\frac{g_1^2}{4\pi}\right)^2\left[-\frac{\ln^2(-t/s)+i\pi\ln(-t/s)}{t+s}-\frac{\ln^2(t/u)}{t+u}\right]. \quad (B.3)$$

These results are in complete conformity with the statement made in Sec. III, since we may recall that the expressions for $Z_1$ and $Z_2$, through fourth order, are

$$Z_1=1-\frac{g_1^2}{16\pi}\ln(\Lambda^2/m^2),$$
$$Z_2=1+\frac{3}{2}\frac{g_1^2}{16\pi}\ln(\Lambda^2/m^2). \quad (B.4)$$

Hence the single integral terms do combine to give

$$g_1^2Z_1^2(-s/m^2)Z_2^{-1}(-s/m^2)(1/s)\pm\text{crossed term}.$$

Furthermore, the double integral terms, corresponding to the skeleton graph of Fig. 3(a), are functions only of angle, aside from the over-all dimensional factor $1/s$. Finally, the limit as $s,t,u\rightarrow\infty$, $u/s$, $u/t\rightarrow 0$ clearly is just the crossed single integral term.