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Radiation in a Plasma. I. Čerenkov Effect*

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We start with the linearized plasma equations containing an isotropic pressure term, plus extra source terms $\mathbf{J}^s$ and ρ^s in the Maxwell equations. The fields of $(\mathbf{J}^s,\rho^s)$ can be decomposed into two modes. The electromagnetic (EM) mode has all the magnetic field and no charge accumulation; it is the ordinary EM field of $(\mathbf{J}^s,\rho^s)$ in a dispersive medium of relative dielectric constant $\epsilon_r=1-(\omega_p/\omega)^2$. The plasma (P) mode has all the charge accumulation and no magnetic field; at great distances from the source, it becomes a longitudinal (radial) plasma wave with the usual dispersion relation for plane plasma waves. Various potentials for the EM and P modes are given by the inhomogeneous Klein-Gordon equation. The fields of a uniformly moving charged particle are found by a Lorentz transformation. When $(u/v_0)<1$ (u=particle velocity, v_0=rms thermal velocity), the EM and P fields are exponentially screened outside oblate spheroids foreshortened in the direction of motion. When $(u/v_0)>1$, the P field exists only within the Mach (Čerenkov) cone trailing the particle. The frequency and angular spectra of the Čerenkov radiation are found, and the total radiated energy is found by assuming an arbitrary high-frequency cutoff due to Landau damping. The expression for total radiated energy agrees with that given by Pines and Bohm, except for the logarithmic terms.

I. INTRODUCTION

THIS paper is part I of a series[1] in which a linear continuum theory of radiation in a plasma is developed and applied to several situations. The main departure from usual practice is the addition of source terms to the equations. The source so represented is arbitrarily set by an external agent. The source generates the disturbances in the plasma but is itself unchanged by the reaction of the fields.

In Secs. II and III of this paper we discuss the basic equations, which are the usual linearized Maxwell and Euler equations for the electron fluid, with source terms. We show that the disturbance can be broken into two components. One contains all the magnetic field and no charge accumulation; it is an ordinary electromagnetic field in a dispersive medium of relative dielectric constant $\epsilon_r=(1-\omega_p^2/\omega^2)$. The other component has no magnetic field and all the charge accumulation. It is an electrified sound-wave field which, at great distances from the source, becomes a radial plasma wave with the usual dispersion relation found in plane plasma waves.

Various potential functions for these two components satisfy the inhomogeneous Klein-Gordon equation of quantum-field theory. This equation is peculiarly well-suited to plasmas, for screening phenomena occur simply and automatically. It has also been used by others. Schatzman,[2] in particular, used its property of being invariant to a Lorentz transformation, when discussing the charge distribution around a slow particle. In Sec. IV we discuss the slow particle, by using the Lorentz transformation. In Sec. IV we also find the Čerenkov field of a fast particle, which exists in the form of a plasma-wave component confined to a cone trailing the particle. In Sec. V we discuss the frequency and angular spectra of the Čerenkov radiation, and the total radiated energy.

Gould[3] has also considered the radiation from sources in a plasma. He worked with the total fields, rather than breaking them into the two simple components as we have done, and so his potential functions and equations are more complicated than ours. The Klein-Gordon operator does appear in his equations, however.

Of the results in Sec. IV and V, concerning uniformly

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[1] M. H. Cohen, Phys. Rev. (to be published).

[2] E. Schatzman, Suppl. Nuovo cimento **13**, 166 (1959).

[3] R. Gould, Technical Report No. 4, Electron Tube and Microwave Laboratory, California Institute of Technology, November, 1955 (unpublished).

moving particles, only the Čerenkov spectra appear to be new. In particular, Pines and Bohm[4] have computed the screening fields and the total radiated Čerenkov energy. Our result concerning the spheroidal screening is identical with that of Pines and Bohm; and our expression for total energy differs only in a minor way, in a logarithm which depends on an arbitrary frequency cutoff.

Our point of view in making these calculations is that the plasma is a continuous fluid. Pines and Bohm, on the other hand, start with a particle point of view, and work with Fourier components of electron-density fluctuations. In Sec. VI we discuss their work briefly. We show that their "random phase approximation" essentially reduces their equation to one similar to ours.

II. BASIC EQUATIONS

We assume that we deal with a plasma containing a bounded electric source consisting of current $\mathbf{J}^s$ and charge ρ^s, where $\nabla\cdot\mathbf{J}^s+\partial\rho^s/\partial t=0$. The source is independent of the plasma and may be arbitrarily prescribed. It exerts an electrical force on the plasma, which responds in some fashion. The problem is to compute the response.

We make the following simplifying assumptions. The plasma is homogeneous and neutral, and the ions are uniformly distributed and stationary. The source is so weak that the linearized equations are applicable, and all magnetic forces are negligible. There are no electron-ion collisions. Finally, the electrons behave as a continuous fluid, and the effect of all electron interactions may be represented by an isotropic pressure.

The system then obeys the four linearized Maxwell equations, which we write in mks rationalized units as

$$\nabla\times\mathbf{E}=-\mu_0(\partial\mathbf{H}/\partial t), \tag{2.1}$$

$$\nabla\times\mathbf{H}=\epsilon_0(\partial\mathbf{E}/\partial t)-en_0\mathbf{v}+\mathbf{J}^s, \tag{2.2}$$

$$\nabla\cdot\mathbf{H}=0, \tag{2.3}$$

$$\epsilon_0\nabla\cdot\mathbf{E}=-en_1+\rho^s. \tag{2.4}$$

The source terms $\mathbf{J}^s$ and ρ^s have been inserted in their usual places in Eqs. (2.2) and (2.4). n_0 is the mean electron density and n_1 is the systematic variation in density due to the action of the source; $\mathbf{v}$ is the systematic velocity imparted to the electrons by the source; and $-e$ is the charge of an electron.

The system also obeys the linearized Euler equations. The continuity equation is a consequence of Eqs. (2.2) and (2.4):

$$n_0\nabla\cdot\mathbf{v}+\partial n_1/\partial t=0. \tag{2.5}$$

We write the force equation as follows:

$$n_0m(\partial\mathbf{v}/\partial t)=-n_0e\mathbf{E}-mv_0^2\nabla n_1, \tag{2.6}$$

where m is the electron mass and v_0 is a velocity connected with the thermal motion of the electrons. For v_0 we shall use the rms velocity (assuming a Maxwellian distribution):

$$v_0^2=3kT/m. \tag{2.7}$$

This is the commonly used value in cases where one assumes that the pressure variations are adiabatic, collisions are infrequent, and one is dealing with plane waves, with a one-dimensional compression.[5] The waves with which we are dealing, however, are spherically diverging, and close to the source more than one degree of freedom may be excited. This lack of justification points up further the approximate nature of the pressure term in Eq. (2.6).

A generalization of Poynting's theorem may be obtained by manipulating the quantity $\nabla\cdot(\mathbf{E}\times\mathbf{H})$ as shown by Field.[6] The result is

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}+mv_0^2n_1\mathbf{v})+\partial/\partial t[\tfrac{1}{2}\epsilon_0E^2+\tfrac{1}{2}\mu_0H^2 +\tfrac{1}{2}n_0mv^2+\tfrac{1}{2}n_0mv_0^2(n_1/n_0)^2]=-\mathbf{E}\cdot\mathbf{J}^s. \tag{2.8}$$

Equation (2.8) displays the symmetry between electrical and mechanical energy storage and power flow terms. The right-hand side represents the rate at which the source works on the plasma, per unit volume. An integral over a volume containing the source gives the total power expended by the source. The $\partial/\partial t$ term gives the rate of accumulation of energy in the volume, in electric, magnetic, kinetic, and potential forms. The integral of $\mathbf{E}\times\mathbf{H}$ gives the power flow out of the volume in electromagnetic waves, and the integral of $mv_0n_1\mathbf{v}$ gives the power flow out of the volume in mechanical waves.

III. SEPARATION OF THE FIELD INTO COMPONENTS

A. Definitions

We are dealing with the three vector fields $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{v}$, and the scalar field n_1. It turns out that we can effect a great simplification by separating these fields into two groups. For reasons which will appear obvious, we call them the electromagnetic (EM) and plasma (P) components, and denote them by the subscripts e and p. The EM component consists of the quantities $(\mathbf{E}_e,\mathbf{H},\mathbf{v}_e)$; i.e., parts of $\mathbf{E}$ and $\mathbf{v}$, and all of $\mathbf{H}$. The P component consists of the quantities $(\mathbf{E}_p,\mathbf{v}_p,n_1)$; i.e., the rest of $\mathbf{E}$ and $\mathbf{v}$, and all of n_1.

Let

$$\mathbf{E}=\mathbf{E}_e+\mathbf{E}_p,\quad \mathbf{v}=\mathbf{v}_e+\mathbf{v}_p. \tag{3.1}$$

We shall ultimately specify the divergence and curl of each component; this uniquely specifies $\mathbf{E}$ and $\mathbf{v}$, provided we require all components to vanish at infinity. Since the P mode has no magnetic field, we set

$$\nabla\times\mathbf{E}_p=0, \tag{3.2}$$

and, since the EM mode has no charge accumulation,

[4] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).

[5] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 59.

[6] G. B. Field, Astrophys. J. **124**, 555 (1956).

we set

$$(\partial^2/\partial t^2+\omega_p{}^2)\nabla\cdot\mathbf{E}_e=(\partial^2/\partial t^2)(\rho^s/\epsilon_0),\qquad(3.3)$$

where ω_p is the plasma frequency, $\omega_p{}^2=n_0e^2/\epsilon_0 m$. The motivation for this definition can be seen by examining the oscillating case. When $\partial/\partial t=-i\omega$, $\nabla\cdot\mathbf{E}_e=\rho^s/\epsilon_0\epsilon_r$, where $\epsilon_r(=1-\omega_p{}^2/\omega^2)$, is the relative dielectric constant appropriate for plane electromagnetic waves in the plasma.

In accordance with the above definitions, the EM mode is now defined by a sequence of equations obtained from Eqs. (2.1)–(2.6):

$$\nabla\times\mathbf{E}_e=-\mu_0\frac{\partial\mathbf{H}}{\partial t},\qquad(3.4)$$

$$\nabla\times\mathbf{H}=\epsilon_0\frac{\partial\mathbf{E}_e}{\partial t}-en_0\mathbf{v}_e+\mathbf{J}^s,\qquad(3.5)$$

$$\nabla\cdot\mathbf{H}=0,\qquad(3.6)$$

$$\epsilon_0\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)\nabla\cdot\mathbf{E}_e=\frac{\partial^2\rho^s}{\partial t^2},\qquad(3.7)$$

$$n_0\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)\nabla\cdot\mathbf{v}_e=-\frac{\omega_p{}^2}{e}\frac{\partial\rho^s}{\partial t},\qquad(3.8)$$

$$n_0m\frac{\partial\mathbf{v}_e}{\partial t}=-n_0e\mathbf{E}_e.\qquad(3.9)$$

The P mode is defined by

$$\nabla\times\mathbf{E}_p=0,\qquad(3.10)$$

$$0=\epsilon_0\frac{\partial\mathbf{E}_p}{\partial t}-en_0\mathbf{v}_p,\qquad(3.11)$$

$$\epsilon_0\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)\nabla\cdot\mathbf{E}_p=-e\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)n_1+\omega_p{}^2\rho^s,\qquad(3.12)$$

$$n_0\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)\nabla\cdot\mathbf{v}_p=-\left(\frac{\partial^2}{\partial t^2}+\omega_p{}^2\right)\frac{\partial n_1}{\partial t}+\frac{\omega_p{}^2}{e}\frac{\partial\rho^s}{\partial t},\qquad(3.13)$$

$$n_0m\frac{\partial\mathbf{v}_p}{\partial t}=-n_0e\mathbf{E}_p-mv_0{}^2\nabla n_1.\qquad(3.14)$$

It may readily be verified that these sets of equations are self-consistent, and that the components which satisfy them add to form total fields satisfying Eqs. (2.1)–(2.6). There is, however, one exception: The description is not valid at the plasma frequency, and we must exclude any time dependence of the form $\exp(-i\omega_pt)$. The plasma is resonant and can support a standing wave for $\omega=\omega_p$.

Equation (3.4) defines $\nabla\times\mathbf{E}_e$; Eq. (3.12) defines $\nabla\cdot\mathbf{E}_p$ in terms of ρ^s and n_1, for which we shall later have an inhomogeneous differential equation. The velocity components, $\mathbf{v}_e$ and $\mathbf{v}_p$, are defined by Eqs. (3.5) and (3.11). This completes the specification of the electromagnetic and plasma components. They are generated by the same source, but otherwise are independent.

B. Electromagnetic Component

By Fourier analysis, any source may be regarded as the sum of oscillating components. With the time dependence $\exp(-i\omega t)$, Eqs. (3.4)–(3.9) reduce to

$$\nabla\times\mathbf{E}_e=i\omega\mu_0\mathbf{H},\qquad\nabla\cdot\mathbf{H}=0,\qquad(3.15)$$
$$\nabla\times\mathbf{H}=-i\omega\epsilon_0\epsilon_r\mathbf{E}_e+\mathbf{J}^s,\quad\epsilon_0\epsilon_r\nabla\cdot\mathbf{E}_e=\rho^s,$$

where

$$\epsilon_r=1-(\omega_p/\omega)^2.$$

We see that the EM mode is the ordinary electromagnetic field that the source would radiate into a homogeneous dispersive dielectric medium, of relative dielectric constant ϵ_r. This field is transverse at great distances from the source. The solutions to the homogeneous version of Eqs. (3.15) include the usual plane electromagnetic waves which can propagate in a plasma.[7]

We may develop Poynting's theorem for the set of Eqs. (3.15), and thereby find that the power flow connected with the EM mode is given by the vector $\mathbf{E}_e\times\mathbf{H}$. We shall show below that the P mode has a radial electric field at great distances from the source. Therefore, at great distances from the source, the radial component of the electromagnetic power flow in Eq. (2.8) is contributed entirely by the EM mode. Moreover, since $\mathbf{v}_e$ is parallel to $\mathbf{E}_e$, the radial component of the mechanical power flow term in Eq. (2.8) must be contributed entirely by the P mode. At great distances from the source, we have

$$\mathbf{r}\cdot\mathbf{E}\times\mathbf{H}=\mathbf{r}\cdot\mathbf{E}_e\times\mathbf{H},\quad mv_0{}^2n_1\mathbf{v}\cdot\mathbf{r}=mv_0{}^2n_1\mathbf{v}_p\cdot\mathbf{r}.\qquad(3.16)$$

C. Plasma Component

For the time dependence $\exp(-i\omega t)$, Eqs. (3.10)-(3.14) may be rearranged to yield

$$-i\omega\epsilon_0\mathbf{E}_p=en_0\mathbf{v}_p,\qquad(3.17)$$

$$\epsilon_0\epsilon_r\mathbf{E}_p=(ev_0{}^2/\omega^2)\nabla n_1,\qquad(3.18)$$

$$(\nabla^2+k_p{}^2)n_1=-(\omega_p{}^2/ev_0{}^2)\rho^s,\qquad(3.19)$$

where

$$k_p{}^2=(\omega^2-\omega_p{}^2)/v_0{}^2.\qquad(3.20)$$

The homogeneous version of Eq. (3.19) has solutions which include the usual plane plasma waves with a dispersion relation given by Eq. (3.20).[7] The particular solution of Eq. (3.19) is proportional to $\int\int\int dv\,\rho^se^{ik_pr}/r$. The electric field is proportional to the gradient of this expression by Eq. (3.18), and for r sufficiently big the radial component of the gradient

[7] J. F. Denisse and J. L. Delcroix, *Théorie des ondes dans les plasmas* (Dunod, Paris, 1961).

will be dominant. We therefore conclude that, at great distances from the source, the P mode is a simple longitudinal (radial) plasma wave, with the usual dispersion relation for plane plasma waves.

The velocity field of the P mode is proportional to $\mathbf{E}_p$ by Eq. (3.17). This field carries the mechanical energy away from the source, as shown in Eq. (3.16).

From Eqs. (3.4)–(3.14), we see that we have defined the components in such a way that the field of a static charge distribution is called a plasma, and not an electromagnetic, mode. If a source has $\rho^s=0$, it will not excite the plasma mode. A uniform neutral ring of current is a source of this type. Such a current ring is the usual idealization of a small loop antenna. We might expect that a small loop antenna in a plasma will be weakly coupled to a plasma wave field, and will behave very differently from a dipole antenna. The radiation from an antenna, however, is a boundary value problem, and one cannot extrapolate readily from the free-space case. This will be discussed in a later part[1] of this series.

D. Evaluation of the Fields

We now derive the inhomogeneous differential equations for the fields and their potential functions. By eliminating $\mathbf{v}_p$ between Eqs. (3.11) and (3.14) we obtain

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\mathbf{E}_p=-\frac{v_0^2 e}{\epsilon_0}\nabla n_1. \qquad (3.21)$$

When n_1 is known, $\mathbf{E}_p$ may be found from Eq. (3.21) by Fourier analysis. The function n_1 essentially acts as a scalar potential for $\mathbf{E}_p$, since each frequency component of $\mathbf{E}_p$ is proportional to the gradient of the corresponding component of n_1. The equation for n_1 itself is found by taking the divergence of Eq. (3.14), eliminating $\mathbf{v}_p$ with (3.11), and eliminating $\mathbf{E}_p$ with (3.12). This gives

$$\left(\nabla^2-\frac{1}{v_0^2}\frac{\partial^2}{\partial t^2}-\frac{1}{D^2}\right)n_1=-\frac{\rho^s}{eD^2}, \qquad (3.22)$$

where

$$D^2=v_0^2/\omega_p^2=3kT\epsilon_0/n_0e^2=3\lambda_D^2, \qquad (3.23)$$

by Eq. (2.7) and the customary definition for the Debye length λ_D.

Equation (3.22) is the inhomogeneous Klein-Gordon equation of quantum field theory, with v_0 replacing the velocity of light, and D replacing the Compton wavelength. This equation has as the solution for a static point source the Yukawa potential[8]; in our case this becomes the screened Coulomb potential $[\exp(-r/D)]/r$. We shall also make use of the property that the Klein-Gordon equation is invariant with respect to a Lorentz transformation, in finding the field of a slowly moving charge by transforming the static solution to a moving coordinate system.

It is convenient to define potentials for the electromagnetic components. In view of Eqs. (3.6) and (3.4), we set

$$\mu_0\mathbf{H}=\nabla\times\mathbf{A}, \quad \mathbf{E}_e=-\partial\mathbf{A}/\partial t-\nabla\phi. \qquad (3.24)$$

Substitute Eqs. (3.24) into (3.5), and eliminate $\mathbf{v}_e$ with (3.9). Then let

$$\nabla\cdot\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\phi=0, \qquad (3.25)$$

where c is the velocity of light. Equation (3.25) specifies the divergence of $\mathbf{A}$, which so far has been arbitrary. Apart from a static term, this gives

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\mathbf{A}=-\mu_0\mathbf{J}^s. \qquad (3.26)$$

The equation for ϕ is obtained by eliminating $\mathbf{A}$ in Eq. (3.25) by using (3.26). This gives

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\omega_p^2}{c^2}\right)\phi=-\frac{1}{\epsilon_0}\frac{\partial^2\rho^s}{\partial t^2}. \qquad (3.27)$$

E. Coupling Between the Modes

The independence of the two modes follows from our original assumption of linearity. But, in fact, wherever the disturbance is not infinitesimal, there will be some coupling between the modes. One way of regarding this coupling is to say that the electromagnetic wave scatters off the variations in electron density caused by the plasma wave. This furnishes a criterion: If, in some volume of interest, such scattering is negligible, then the two modes may be regarded as independent.

We have also assumed that the unperturbed medium is homogeneous. A real plasma, however, will have gradients of electron density and temperature as well as fluctuations in density due to thermal motions. These variations will produce a coupling between the two modes. The processes by which the coupling occurs are of great interest in radio astronomy, since there is reason for expecting some solar bursts to originate as plasma waves in the corona. These must couple to electromagnetic waves, which can propagate to the earth. There have already been a number of papers discussing the coupling arising from plane discontinuities or gradients in electron density,[6,9] and from random fluctuations of electron density.[3,10]

In later parts of this series[1] we shall consider the problems of reflection at a plane metal boundary, and

[8] S. Schweber, H. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1956), Vol. I, p. 116.

[9] D. A. Tidman, Phys. Rev. **117**, 366 (1960); A. H. Kritz and D. Mintzer, *ibid.* **117**, 382 (1960).

[10] V. L. Ginsburg and V. V. Zheleznіakov, Soviet Astron.—AJ (translation) **2**, 653 (1958); **3**, 235 (1959); *Paris Symposium on Radio Astronomy*, edited by R. N. Bracewell (Stanford University Press, Stanford, California, 1959), p. 574.

scattering by a small plasma bubble. In both these cases we shall find coupling between the modes.

IV. FIELD OF A UNIFORMLY MOVING CHARGED PARTICLE

Let the source consist of a single particle of charge q constrained to move with uniform velocity u along the z axis. This source, in principle, is different from the test particle used in other papers, for the latter is a free plasma particle and subject to acceleration. There are, however, many problems where the free motion is nearly uniform, and one then assumes that the test particle is a source in our sense.

A. Zero Velocity

First, consider a stationary charge at the origin of the xyz coordinate system: $\rho^s=q\delta(x)\delta(y)\delta(z)$. With $\partial/\partial t=0$, Eq. (3.22) becomes

$$\left(\nabla^2-\frac{1}{D^2}\right)n_1=-\frac{1}{eD^2}q\delta(x)\delta(y)\delta(z). \tag{4.1}$$

The solution to Eq. (4.1) is well known:

$$n_1=\frac{q}{4\pi eD^2}\frac{e^{-r/D}}{r}, \tag{4.2}$$

where

$$r^2=x^2+y^2+z^2.$$

From Eq. (3.21) we see that the electric field also contains the factor $e^{-r/D}$. The stationary charge is thus screened, in a distance on the order of a Debye length, by a distribution of charge of opposite sign. (By definition, n_1 is an excess of negative charge.)

This result has been derived from statistical considerations by Pines and Bohm[4] and others. From the point of view of randomly moving particles, the stationary charge is surrounded by a cloud of particles of opposite sign which, on the average, effectively neutralizes the charge at distances greater than the Debye length. From the point of view of the continuum theory, the Coulomb force displaces the fluid until the electrostatic force is balanced by the pressure. The excess fluid density then has the exponential dependence, and the total electric field, of source plus excess fluid, dies out as $e^{-r/D}$.

B. Nonzero Velocity

Equation (3.22) is invariant with respect to a Lorentz transformation, modified by exchanging v_0 for c. The field of the uniformly moving charge can therefore be found from the known solution for a static charge by transforming to a moving coordinate system.

Let

$$\rho^s=q\delta(x)\delta(y)\delta(z-ut). \tag{4.3}$$

This represents a point charge moving with velocity u along the z axis. For the case $u^2<v_0^2$, let

$$x'=x,\quad z'=(z-ut)(1-\beta^2)^{-\frac{1}{2}}, \tag{4.4}$$
$$y'=y,\quad t'=(t-zu/v_0^2)(1-\beta^2)^{-\frac{1}{2}},$$

where $\beta^2=(u/v_0)^2$. This is the Lorentz transformation, with inverse

$$\begin{aligned} x&=x', & z&=(z'+ut')(1-\beta^2)^{-\frac{1}{2}},\\ y&=y', & t&=(t'+z'u/v_0^2)(1-\beta^2)^{-\frac{1}{2}}. \end{aligned} \tag{4.5}$$

Under the transformation (4.5), Eq. (3.22) reduces to the static equation

$$\left(\nabla'^2-\frac{1}{D^2}\right)n_1(x'y'z')=-\frac{q}{eD^2}\delta(x')\delta(y')\delta[z'(1-\beta^2)^{\frac{1}{2}}], \tag{4.6}$$

which has the solution

$$n_1(x'y'z')=\frac{q}{4\pi eD^2}\frac{1}{(1-\beta^2)^{\frac{1}{2}}}\frac{e^{-\rho/D}}{\rho}, \tag{4.7}$$

where

$$\rho^2=x^2+y^2+(z-ut)^2/(1-\beta^2).$$

Thus, the excess electron density still has the shielded Coulomb form, but the equal-density surfaces have been compressed in the direction of motion and are oblate spheroids. The identical result has also been found by Pines and Bohm.[4] Schatzman[2] has also used the Lorentz transformation in dealing with a slow particle in a plasma.

There will also be an electromagnetic mode which comoves with the particle. We may write the current density as

$$J_x{}^s=J_y{}^s=0,\quad J_z{}^s=qu\delta(x)\delta(y)\delta(z-ut),$$

so that, by integrating Eq. (3.26) in the manner just used, we find

$$A_z=-\frac{\mu_0 u}{4\pi}\frac{1}{[1-(u/c)^2]^{\frac{1}{2}}}\frac{\exp(-\omega_p\rho_c/c)}{\rho_c} \tag{4.8}$$

where

$$\rho_c{}^2=x^2+y^2+(z-ut)^2/[1-(u/c)^2].$$

The vector potential, and therefore the total electromagnetic component, is also spheroidally shielded. The shielding is different from that for the plasma mode however, for c replaces v_0. The spheroids are nearly spheres, and the shielding distance is $c/\omega_p\ll D$.

The shielding of the electromagnetic mode comes about because the electrons move in response to the electric field of the electromagnetic mode. This motion, however, results in zero accumulation of charge; and the shielding is effective only at great distances, because the electrons can not squeeze in upon the source, as they do when shielding the plasma wave.

C. $\beta^2 > 1$

Equation (4.3) still represents the source, but now the particle velocity u is greater than the rms thermal velocity. The Čerenkov field is usually not found from a Lorentz transformation, because the character of the transformed equation changes.[11] In our case, however, this method yields the desired results as quickly as a direct integration of Eq. (3.22).

It is necessary to modify the Lorentz transformation. Let

$$\begin{aligned} x' &= x, \quad z' = (z-ut)(\beta^2-1)^{-\frac{1}{2}}, \\ y' &= y, \quad t' = (t-zu/v_0^2)(\beta^2-1)^{-\frac{1}{2}}, \end{aligned} \tag{4.9}$$

which has the inverse transformation

$$\begin{aligned} x &= x', \quad z = -(z'+ut')(\beta^2-1)^{-\frac{1}{2}}, \\ y &= y', \quad t = -(t'+z'u/v_0^2)(\beta^2-1)^{-\frac{1}{2}}. \end{aligned} \tag{4.10}$$

Under the transformation (4.9), Eq. (3.22) reduces to the static equation

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}-\frac{\partial^2}{\partial z'^2}-\frac{1}{D^2}\right)n_1(x'y'z') = -\frac{q}{eD^2}\delta(x')\delta(y')\delta[-z'(\beta^2-1)^{\frac{1}{2}}]. \tag{4.11}$$

Equation (4.11) is a two-dimensional Klein-Gordon equation. As shown in Appendix A, the appropriate Green's function is

$$\begin{aligned} g(\mathbf{r}'-\mathbf{r}_0) &= -\frac{2\cos\{[(z'-z_0)^2-(x'-x_0)^2-(y'-y_0)^2]^{\frac{1}{2}}/D\}}{[(z'-z_0)^2-(x'-x_0)^2-(y'-y_0)^2]^{\frac{1}{2}}} \\ &\qquad \text{for} \quad (z'-z_0)^2 > (x'-x_0)^2+(y'-y_0)^2; \\ &= 0 \quad \text{for} \quad (z'-z_0)^2 < (x'-x_0)^2+(y'-y_0)^2, \end{aligned} \tag{4.12}$$

where x_0, y_0, z_0 are the coordinates of the Green's function source. The solution to Eq. (4.11) is, therefore,

$$n_1(x'y'z') = \frac{q}{4\pi eD^2}\int\int\int g(\mathbf{r}'-\mathbf{r}_0) \times\delta(x_0)\delta(y_0)\delta[-z_0(\beta^2-1)^{\frac{1}{2}}]dx_0dy_0dz_0,$$

or

$$\begin{aligned} n_1(x'y'z') &= \frac{q}{2\pi eD^2}\frac{\cos[(z'^2-x'^2-y'^2)^{\frac{1}{2}}/D]}{(\beta^2-1)^{\frac{1}{2}}(z'^2-x'^2-y'^2)^{\frac{1}{2}}} \\ &\qquad \text{for} \quad z'^2 > x'^2+y'^2, \\ &= 0 \quad \text{for} \quad z'^2 < x'^2+y'^2. \end{aligned}$$

The final result is obtained by transforming back to the

[11] W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 309.

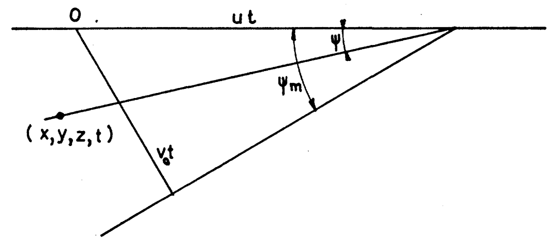

FIG. 1. Čerenkov cone.

$(xyzt)$ coordinate system:

$$\begin{aligned} n_1(xyzt) &= \frac{q}{2\pi eD^2}\frac{\cos\{1/D[(z-ut)^2/(\beta^2-1)-x^2-y^2]^{\frac{1}{2}}\}}{[(z-ut)^2/(\beta^2-1)-x^2-y^2]^{\frac{1}{2}}} \\ &\qquad \text{for} \quad (z-ut)^2/(\beta^2-1) > x^2+y^2, \\ &= 0 \quad \text{for} \quad (z-ut)^2/(\beta^2-1) < x^2+y^2. \end{aligned} \tag{4.13}$$

The solution (4.13) is very different from that obtained above for a slow particle. The field consists of damped waves which die out as z^{-1} for z^2 very large, and at a fixed point die out as t^{-1} for t^2 very large. The Debye length now controls the wavelength, rather than the space rate of decay.

At great distances, the field is oscillatory and dies out as z^{-1}. This allows for the radiation of power. This is the Čerenkov radiation, in the form of plasma waves whose phase velocities are slower than the particle velocity. The familiar confinement of the field to a cone trailing the particle is explicitly contained in Eq. (4.13). The field is zero unless

$$(1/\beta^2)(z-ut)^2 > (x^2+y^2)\cos^2\psi_m, \tag{4.14}$$

where

$$\sin^2\psi_m = v_0^2/u^2.$$

Now, add $(z-ut)^2$ to both sides of (4.14); a little rearrangement gives

$$\cos^2\psi > \cos^2\psi_m,$$

where

$$\cos^2\psi = (z-ut)^2/[x^2+y^2+(z-ut)^2].$$

The retarded solution is

$$\psi < \psi_m. \tag{4.15}$$

The cone is depicted in Fig. 1.

The cone in Fig. 1 is, of course, the Mach cone containing the disturbance of the fluid, for n_1 is zero outside the cone. Our particle is supersonic and runs ahead of its disturbance, in the usual way. The cone is determined by the requirement that the front face of the disturbance travel with the velocity v_0, which is the fastest group velocity for plane plasma waves.

The particle will also be surrounded by an EM component, but since $u < c$, this component is still given by Eq. (4.8). This field is shielded and dies out exponentially outside a spheroid which is convected along with the particle. When the particle velocity

approaches the velocity of light, the spheroid becomes a disk, with radius c/ω_p.

V. ČERENKOV SPECTRUM

A. Frequency Spectrum

To find the spectrum of the plasma mode Čerenkov radiation, we follow, as far as possible, the treatment for the ordinary Čerenkov effect in optics.[11] We revert to the original equation for n_1 and use the method of Fourier analysis.

Let

$$\rho^s(xyzt)=\int_{-\infty}^{\infty}\rho_\omega e^{-i\omega t}d\omega,$$
$$\rho_\omega(xyz\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho^s e^{i\omega t}dt. \quad (5.1)$$

From Eq. (4.3),

$$\rho_\omega(xyz\omega)=-\frac{q}{2\pi u}e^{i\omega z/u}\delta(x)\delta(y).$$

The density n_1 also can be expressed as a Fourier integral as in Eq. (5.1). From Eq. (3.22), the Fourier components n_ω satisfy the equation

$$(\nabla^2+k_p^2)n_\omega=\frac{q}{2\pi ueD^2}e^{i\omega z/u}\delta(x)\delta(y). \quad (5.2)$$

Equation (5.2) has the particular solution

$$n_\omega=-\frac{q}{8\pi^2ueD^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-L}^{L}e^{i\omega z_0/u}\times\delta(x_0)\delta(y_0)\frac{e^{ik_pR}}{R}dx_0dy_0dz_0,$$

where

$$R^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2,$$

and where the integration on z_0 is taken from $-L$ to $+L$ to avoid a singularity in the field[11]; we require $\omega L/u\gg1$. Since we are interested in the radiated power only, we may evaluate this integral by using the far-field approximation:

$$[x^2+y^2+(z-z_0)^2]^{\frac{1}{2}}\approx r-(z-z_0)\cos\theta, \quad (5.3)$$

where θ is the polar angle from the positive z axis, and the term $(z-z_0)\cos\theta$ may be neglected in the denominator. The integral is readily evaluated, giving

$$n_\omega=-\frac{qL}{4\pi ueD^2}\frac{e^{ik_pr}}{r}\frac{\sin(\omega/u-k_p\cos\theta)L}{(\omega/u-k_p\cos\theta)L}. \quad (5.4)$$

In the far field, the r^{-1} term of the electric field is dominant, so we have, from Eq. (3.21),

$$\mathbf{E}_\omega=\frac{ik_pev_0^2}{\epsilon_0(\omega^2-\omega_p^2)}n_\omega\mathbf{r}, \quad (5.5)$$

where $\mathbf{r}$ is a unit radial vector.

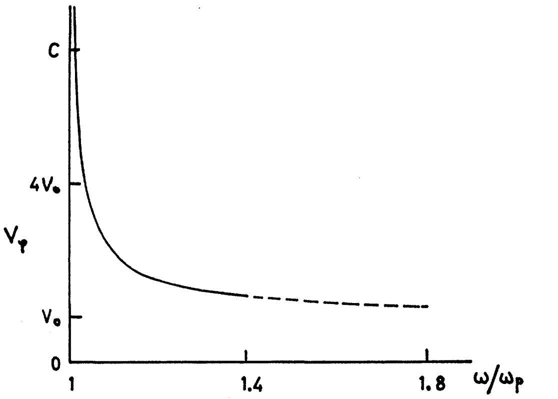

FIG. 2. Dispersion relation for longitudinal plasma waves $c/v_0=7$. In the dashed region, Landau damping inhibits the wave propagation.

Since $\omega L/u\gg1$, n_ω has a sharp maximum when $\omega/u=k_p\cos\theta$, or $\cos\theta=v_\varphi/u$, where v_φ is the phase velocity for plane plasma waves of frequency ω. This is the familiar result from optics. There is a difference from the optical case, however, which comes from the fact that the medium is very dispersive. The dispersion relation is shown in Fig. 2. At a frequency a little greater than ω_p, $v_\varphi=u$, and the radiation is forward. Higher frequencies are projected at various angles, and, if the particle velocity is very high, the radiation persists nearly to 90°. This angular spectrum is discussed below.

As discussed in Sec. IIIB, the energy flux density in the plasma mode is given by $\mathbf{S}_p=mv_0^2n_1\mathbf{v}_p$ w/m². The usual Fourier transform relation for the total radiated energy [Panofsky and Phillips,[11] Eq. (13–36)] must be modified for our present use. The result we need, derived in Appendix B, is that the radial energy flux density is

$$\int_{-\infty}^{\infty}\mathbf{S}_p(t)dt=\mathbf{r}4\pi\epsilon_0v_0\int_0^{\infty}E_\omega E_\omega^*(1-X)^{\frac{1}{2}}/Xd\omega, \quad (5.6)$$

where $X=\omega_p^2/\omega^2$. Integrate Eq. (5.6) over a sphere to obtain the total radiated energy

$$W_L=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}4\pi\epsilon_0v_0E_\omega E_\omega^*(1-X)^{\frac{1}{2}}/Xr^2d\omega d\Omega.$$

Now, define the spectrum I_ω as the total radiated energy per unit frequency interval per unit length of path by

$$W_L=\int_0^{\infty}2LI_\omega d\omega, \quad (5.7)$$

so that

$$I_\omega=2\pi\epsilon_0v_0L^{-1}\int_0^{2\pi}\int_0^{\pi}E_\omega E_\omega^*(1-X)^{\frac{1}{2}}/Xr^2d\Omega.$$

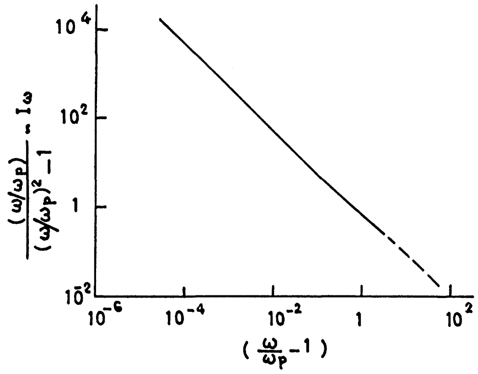

FIG. 3. Čerenkov frequency spectrum. In the dashed region, Landau damping inhibits the wave propagation. The spectrum has a low-frequency cutoff at $\omega_c=\omega_p/[1-(v_0/u)^2]^{\frac{1}{2}}$.

Substitution from Eqs. (5.5) and (5.4) gives

$$I_\omega=\frac{e^2}{4\pi\epsilon_0}\frac{\omega_p}{u^2}\frac{(\omega/\omega_p)}{(\omega/\omega_p)^2-1}B, \tag{5.8}$$

where

$$B=\pi^{-1}\int_a^b t^{-2}\sin^2 t dt,$$

$a=(\omega L/u)(1-u/v_\varphi)$, $b=(\omega L/u)(1+u/v_\varphi)$, and $v_\varphi=v_0[1-(\omega_p/\omega)^2]^{-\frac{1}{2}}$=phase velocity. When $(\omega L/u)\gg 1$, the function B has the value $\frac{1}{2}$ if $u=v_\varphi$; if $(u/v_\varphi-1)$ goes positive, B very rapidly approaches unity, and if $(u/v_\varphi-1)$ goes negative, B very rapidly approaches zero. It has the character of a unit step function

$$B\approx 1 \quad \text{for} \quad u>v_\varphi;$$
$$B\approx 0 \quad \text{for} \quad u<v_\varphi.$$

This is another statement of the fact seen above, that the particle excites only those frequencies whose phase velocities are less than the velocity of the particle.

The spectrum, Eq. (5.8), can finally be written as

$$I_\omega=\frac{e^2\omega_p}{4\pi\epsilon_0 u^2}\frac{(\omega/\omega_p)}{(\omega/\omega_p)^2-1} \quad \text{for} \quad \omega>\omega_c,$$
$$=0 \qquad\qquad \text{for} \quad \omega<\omega_c, \tag{5.9}$$

where $\omega_c=\omega_p/[1-(v_0/u)^2]^{\frac{1}{2}}$. In cgs units, this is

$$I_\omega=1.62\times 10^{-31}\left(\frac{f_p}{10^8}\right)\left(\frac{c^2}{u^2}\right)\frac{(\omega/\omega_p)}{(\omega/\omega_p)^2-1} \quad \text{ergs/cps cm}, \tag{5.10}$$

where $f_p=\omega_p/2\pi$.

The spectrum is shown in Fig. 3. It has a maximum at the cutoff frequency ω_c. When $(v_0/u)^2\ll 1$, ω_c is very close to the plasma frequency $\omega_c\approx\omega_p[1+\frac{1}{2}(v_0/u)^2]$.

B. Total Radiated Energy

The total energy radiated, per unit length of path, is written from Eq. (5.7) as $W=\int_0^\infty I_\omega d\omega$. If we use the spectrum as it stands in Eq. (5.9) we derive an infinite radiated energy, because the spectrum drops off only as ω^{-1} as $\omega\to\infty$. We now avoid this situation by assuming that the Landau damping inhibits the propagation of a plasma wave when its phase velocity is close enough to v_0.[7] We assume, arbitrarily, that the spectrum is zero unless $v_\varphi\geqslant\sqrt{2}v_0$, or equivalently, unless $\omega\leqslant\omega_{\max}=\sqrt{2}\omega_p$. Since $v_\varphi<u$, this also implies $u\geqslant\sqrt{2}v_0$. The regions where the Landau damping is effective are shown dashed in Figs. 2 and 3.

The energy thus becomes

$$\begin{aligned} W&=\frac{e^2\omega_p}{4\pi\epsilon_0 u^2}\int_{\omega_c}^{\omega_{\max}}\frac{(\omega/\omega_p)}{(\omega/\omega_p)^2-1}d\omega,\\ W&=\frac{e^2\omega_p^2}{8\pi\epsilon_0 u^2}\ln\left[\left(\frac{u}{v_0}\right)^2-1\right]. \end{aligned} \tag{5.11}$$

In cgs units, this result is

$$W=5.1\times 10^{-23}\left(\frac{f_p}{10^8}\right)^2\left(\frac{c}{u}\right)^2\ln\left[\left(\frac{u}{v_0}\right)^2-1\right] \quad \text{ergs/cm}. \tag{5.12}$$

The logarithmic factor in Eq. (5.11) varies very slowly with $\omega_{\max}$, so that the arbitrary nature of our selection for $\omega_{\max}$ is not too important.

Note that the radiated energy is approximately inversely proportional to the square of the particle velocity. This point may be of importance in the discussion of solar radio bursts. It means that low-energy particles, which the sun produces in abundance, are more efficient in producing electrical waves than are relativistic particles.

Pines and Bohm[4] have shown that a charged particle with velocity greater than the rms thermal velocity will excite a Čerenkov field. The expression they deduced for radiated energy per unit length is $(e^2\omega_p^2/2u^2)\ln(1+2u^2/v_0^2)$ (cgs) [their Eq. (58)]. Our result is the same, except for the logarithmic terms. When $(u/v_0)^2\gg 1$, the difference is negligible. The calculations of Pines and Bohm are discussed in Sec. VI.

C. Angular Spectrum

We now discuss the angular distribution of the radiated energy. Since the radiation is symmetric around the z axis, we define the angular spectrum, $I_{\theta,\varphi}$, by $2\pi I_{\theta,\varphi}\sin\theta d\theta=I_\omega d\omega$. From Eq. (5.4) we have the relation between θ and ω. At the sharp peak of the function n_ω, $\cos\theta=(v_0/u)[1-(\omega_p/\omega]^{-\frac{1}{2}}$, so that

$$\frac{d\omega}{d\theta}=\frac{\sin\theta}{\cos^3\theta}\frac{\omega^3}{\omega_p^2}\left(\frac{v_0}{u}\right)^2.$$

Substitution from Eq. (5.9) gives, after a little rear-

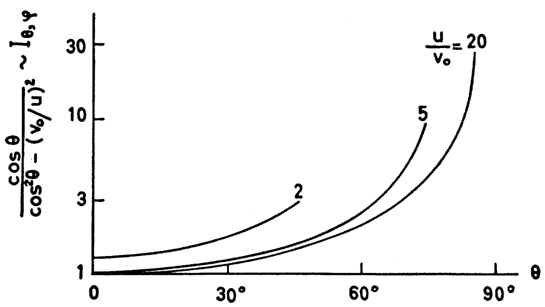

FIG. 4. Čerenkov angular spectrum. The high-angle cutoff results from Landau damping.

rangement,

$$I_{\theta,\varphi}=\frac{e^2\omega_p^2}{8\pi^2\epsilon_0 u^2}\frac{\cos\theta}{\cos^2\theta-(v_0/u)^2}\text{ joule/sr m.} \quad (5.13)$$

The angular spectrum has a singularity at $\cos\theta=(v_0/u)$. This occurs at infinite frequency, and corresponds to the infinity in the expression for total radiated energy, which is discussed above. By using the same arbitrary upper frequency cutoff, $\omega_{\max}=\sqrt{2}\omega_p$, we find an angle cutoff; $\theta\leqslant\theta_{\max}$ where

$$\cos\theta_{\max}=\sqrt{2}v_0/u.$$

The angular spectrum, for three values of (u/v_0), is shown in Fig. 4. This figure may be contrasted with Fig. 3. The *frequency* spectrum (Fig. 3) has a maximum at the lowest frequency, where the radiation is forward. The *angular* spectrum (Fig. 4) has a maximum at its largest angle, where the frequency is highest. When $(u/v_0)\gg1$, both spectra are sharply peaked; the frequency spectrum has a sharp maximum near the plasma frequency, and the angular spectrum has a sharp maximum near 90°.

This peculiar situation comes about because the low-frequency part of the frequency spectrum is spread very thinly in angle, so that there is actually a minimum at $\theta=0$, corresponding to the maximum in the frequency spectrum. The high-frequency tail of the spectrum is concentrated very much in angle, giving the maximum of the angular spectrum.

The angular spectrum is different from that obtaining in the optical Čerenkov effect, because the dispersion formulas are different. In the optical case, the visible light is emitted in a cone determined by the index of refraction. To keep the energy finite, one assumes that the index of refraction drops to unity in the ultraviolet region, so that the high-frequency components are concentrated near $\theta=0°$.

D. A Numerical Example

To see the order of magnitude of some of these quantities, we consider a high-energy particle traversing the solar corona. Take $u\approx c$, $v_0=6.7\times10^8$ cm/sec (for $T=10^6$°K), and $f_p=10^8$ cps. The spectrum has a low-frequency cutoff at $\omega/\omega_p=1+2.5\times10^{-4}$, i.e., at a frequency only 25 kc/sec above the plasma frequency. The spectrum is reduced in intensity by a factor of 10 at a frequency 250 kc/sec above the plasma frequency. The high-frequency Landau cutoff, according to our assumption, occurs at $\sqrt{2}f_p$, so that the total spectrum has a bandwidth of 40 Mc/sec.

At the peak of the spectrum, the radiated energy is 3×10^{-28} ergs/cps cm. At the high-frequency cutoff, the intensity is reduced by a factor of 1400. The total radiated energy is 4×10^{-22} ergs/cm. If we assume a total path of 10^{10} cm, the total radiated energy is 4×10^{-12} ergs, or about 3 ev. This estimate is rough, for the spectrum changes as the electron density decreases, but it should give the order of magnitude. The Čerenkov radiation thus will have a negligible effect on the slowing down of a single energetic particle in the corona. The situation may be different when there is a beam of particles, as discussed in the next section.

In the above example, the angular spectrum has a flat minimum at $\theta=0°$, and a sharp maximum at the cutoff $\theta=88.2°$. The angular spectrum is reduced in intensity by a factor of 10 in 7.5°, at $\theta=80.7°$.

In the ionosphere, v_0^2 is smaller than the value for the corona by about 10^3. The low-frequency cutoff for the Čerenkov spectrum for a relativistic particle will, therefore, occur for $\omega/\omega_p=(1+2.5\times10^{-7})$. This value clearly is unreasonable, since the fluctuations in electron density are bigger than 1 part in 10^7, and the medium cannot be regarded as homogeneous on this scale. In cold plasmas, a proper discussion of the Čerenkov spectrum near the low-frequency cutoff must take account of the fluctuations. In any event, such effects will not affect strongly the angular spectrum and the total radiated energy, since these are mainly derived from the large part of the spectrum well away from the plasma frequency.

E. Bunching

Equation (5.11) gives the energy radiated per unit length by a single charged particle constrained to move with uniform velocity. If the particle is free, it will be decelerated by the radiation reaction; but Eq. (5.11) is still approximately true if $u\gg v_0$. If, however, there are n free particles, forming a beam with velocity $\mathbf{u}$, it will, in general, not be even approximately correct to say that the total radiated energy is n times the value given by Eq. (5.11). The beam will be unstable, and the particles will bunch.

The first particle of the beam is in a homogeneous plasma and radiates according to the picture developed in the previous sections. The trailing particles, however, are affected by the plasma wave fields of the particles that precede them. These fields are convected along with the radiating particles, and so appear constant to any trailing particle within the Čerenkov cone, provided the trailing particle has some velocity.

The total field acting on a constrained trailing particle is the sum of all the fields of the front particles; each component is constant, and, in general, the sum will not be zero. The trailing particle is, thus, subject to a constant force. If the trailing particle is free, instead of being constrained to move with velocity $\mathbf{u}$, it will oscillate about a potential minimum. If *all* the trailing particles are free, the total field will reach some equilibrium configuration, with nearly all the particles oscillating about potential minima. The bunched particles will then radiate in a partially coherent fashion, greatly increasing the total radiated energy.

This bunching process, with the possible great increase in radiated energy, is a manifestation of the twin stream instability.[3,12] Our discussion shows why there is a sharp difference between beams with $u>v_0$ and those with $u<v_0$. The latter do not excite large-scale plasma oscillations because there is no Čerenkov radiation for $u<v_0$.

Ginsburg and Zhelezniakov[10] have discussed the generation of solar radio bursts by coherent and incoherent emission of Čerenkov plasma waves. They have also discussed the reabsorption of the plasma waves by the beam.

VI. DISCUSSION OF PINES AND BOHM

Our results, where comparable, have agreed with those of Pines and Bohm,[4] although the methods have been very different. We assumed at the beginning that the plasma was a fluid containing an isotropic pressure. Pines and Bohm analyzed the Fourier components of electron density fluctuations and introduced approximations as needed to simplify the mathematics. Their "central approximation" is the "random phase approximation," in which a certain sum is assumed to be negligible because the terms have phase factors depending on particle positions. It is just this random phase approximation which reduces their analysis to one which is essentially the same as ours.

After they first make the random phase approximation, they have their Eq. (9), which we rewrite here as follows:

$$\frac{d^2\rho_k}{dt^2}=-\sum\nolimits_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i}-\omega_p{}^2\rho_k, \tag{6.1}$$

where $\mathbf{x}_i$ and $\mathbf{v}_i$ are the position and velocity of the ith electron, and ρ_k is the kth Fourier component of the electron density distribution. (The ions are assumed stationary and uniformly distributed.) We recover the fluctuation in electron density distribution, n_1, by multiplying Eq. (6.1) by $e^{i\mathbf{k}\cdot\mathbf{x}}$ and summing over k, leaving out the term $k=0$:

$$\frac{d^2n_1}{dt^2}=-\sum_{k\neq 0}\sum_i(\mathbf{k}\cdot\mathbf{v}_i)^2e^{-i\mathbf{k}\cdot\mathbf{x}_i}-\omega_p{}^2n_1. \tag{6.2}$$

Since $\rho_k=\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}$ and $\nabla^2\rho_k=-k^2\rho_k$, Eq. (6.2) is very close to Eq. (3.22), our Klein-Gordon equation for n_1. The results one obtains from Eq. (6.2) may thus be more or less close to what one obtains from Eq. (3.22), depending on what one assumes for the velocity distribution function, and how one approximates the double sum. The subsequent approximations which Pines and Bohm did make were such that their results for the screening field were identical with ours, and the total Čerenkov energy was nearly the same.

Pines and Bohm found the radiated Čerenkov energy by integrating along the path of the particle itself, whereas we integrated over a distant sphere. We essentially have used the opposite sides of the volume integral of Eq. (2.8).

ACKNOWLEDGMENTS

The author is indebted to J. F. Denisse and G. B. Field for many helpful discussions.

APPENDIX A. TWO-DIMENSIONAL KLEIN-GORDON EQUATION

The Green's function we seek satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\kappa^2\right)g_2 =4\pi\delta(x-x_0)\delta(y-y_0)\delta(t-t_0). \tag{A.1}$$

The equation is two-dimensional, but we may regard the field as existing in a three-dimensional space, and having for its source a line through the point (x_0,y_0) parallel to the z axis. The solution for g_2 then is the three-dimensional solution for a line source, and is obtained by integrating the three-dimensional Green's function along the line.[13]

The Green's function for the three-dimensional Klein-Gordon equation is given by Morse and Feshbach[13] as follows:

$$g_3=\frac{\delta(\tau-R/c)}{R}-\frac{\kappa}{[\tau^2-(R/c)^2]^{\frac{1}{2}}} \times J_1\{\kappa c[\tau^2-(R/c)^2]^{\frac{1}{2}}\}u(\tau-R/c), \tag{A.2}$$

where

$$\tau=t-t_0,\quad R^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2,$$
$$u(x)=1 \quad\text{for}\quad x>0,$$
$$=0 \quad\text{for}\quad x<0.$$

Equation (A.2) is the retarded solution. The advanced solution is equally valid, and we must allow for it also. This is necessary because, at this point, we do not know which solution will ultimately correspond to the retarded potential in our final coordinate system. The choice must be reserved for Eq. (4.19). The

[12] A. V. Haeff, Phys. Rev. 74, 1532 (1948).

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), pp. 842–844.

advanced Green's function is obtained from Eq. (A.2) by replacing $\delta(\tau-R/c)$ with $\delta(\tau+R/c)$, and by replacing $u(\tau-R/c)$ with $1-u(\tau+R/c)$.

Our desired function g_2 is given by

$$g_2=\int_{-\infty}^{\infty} g_3 dz_0. \tag{A.3}$$

The first term in Eq. (A.2) is the Green's function for the three-dimensional wave equation; when integrated it gives the Green's function for the two-dimensional wave equation[13]:

$$\begin{aligned} T_1 &= \int_{-\infty}^{\infty} \frac{\delta(\tau-R/c)}{R} dz_0 \\ &= -\frac{2c}{(c^2\tau^2-P^2)^{\frac{1}{2}}} \quad \text{for} \quad (c\tau)^2>P^2, \\ &= 0 \quad \text{for} \quad (c\tau)^2<P^2, \end{aligned} \tag{A.4}$$

where

$$P^2=(x-x_0)^2+(y-y_0)^2.$$

In writing down Eq. (A.4) we have generalized and allowed for both retarded and advanced solutions.

The second term from Eq. (A.2) gives the integral

$$T_{2r}=-\kappa c\int_{-\infty}^{\infty} \frac{J_1[\kappa(c^2\tau^2-P^2-\zeta^2)^{\frac{1}{2}}]}{(c^2\tau^2-P^2-\zeta^2)^{\frac{1}{2}}} u(\tau-R/c)d\zeta,$$

where

$$\zeta=z_0-z.$$

The integrand is zero unless $\zeta_a<\zeta<\zeta_b$, where

$$\zeta_b=-\zeta_a=(c^2\tau^2-P^2)^{\frac{1}{2}}; \quad c\tau>P.$$

Consequently, $T_{2r}=0$ for $c\tau<P$; and, for $c\tau>P$,

$$\begin{aligned} T_{2r} &= -2\kappa c\int_0^{\zeta_b} \frac{J_1[\kappa(c^2\tau^2-P^2-\zeta^2)^{\frac{1}{2}}]}{(c^2\tau^2-P^2-\zeta^2)^{\frac{1}{2}}} d\zeta \\ &= 2\kappa c\int_0^{\kappa(c^2\tau^2-P^2)^{\frac{1}{2}}} \frac{J_1(\xi)d\xi}{[\kappa^2(c^2\tau^2-P^2)-\xi^2]^{\frac{1}{2}}} \end{aligned}$$

This integral is in a standard form.[14] We thus have

$$\begin{aligned} T_{2r} &= \frac{2c}{(c^2\tau^2-P^2)^{\frac{1}{2}}}[1-\cos\kappa(c^2\tau^2-P^2)^{\frac{1}{2}}] \quad \text{for} \quad c\tau>P, \\ &= 0 \quad \text{for} \quad c\tau<P. \end{aligned} \tag{A.5}$$

When we evaluate the corresponding formula for the advanced potential, we have that the integrand again is zero unless $\zeta_a<\zeta<\zeta_b$, where now $\zeta_b=-\zeta_a=(c^2\tau^2-P^2)^{\frac{1}{2}}$, for $-c\tau>P$. The second term for the advanced potential thus is

$$\begin{aligned} T_{2a} &= \frac{2c}{(c^2\tau^2-P^2)^{\frac{1}{2}}}[1-\cos\kappa(c^2\tau^2-P^2)^{\frac{1}{2}}] \text{ for } -c\tau>P, \\ &= 0 \quad \text{for } -c\tau<P. \end{aligned} \tag{A.6}$$

[14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. II, p. 18.

Combine Eqs. (A.4)–(A.6) to obtain the final result

$$\begin{aligned} g_2 &= -\frac{2c\cos[\kappa(c^2\tau^2-P^2)^{\frac{1}{2}}]}{(c^2\tau^2-P^2)^{\frac{1}{2}}} \quad \text{for} \quad (c\tau)^2>P^2, \\ &= 0 \quad \text{for} \quad (c\tau)^2<P^2. \end{aligned} \tag{A.7}$$

Equation (4.12) is obtained from Eq. (A.7) by setting $c=1$, $t=z$, and $\kappa=D^{-1}$.

APPENDIX B. RADIATED ENERGY

The radial component of the mechanical energy flux density is given by Eq. (3.16)

$$\mathbf{S}_p(t)=mv_0^2 n_1 \mathbf{v}_p, \tag{B.1}$$

and we need an expression for the total radiated energy per unit area $\int_{-\infty}^{\infty}\mathbf{S}_p(t)dt$, in terms of the Fourier components $\mathbf{E}_\omega$ for the plasma mode electric field. From Eq. (3.12) we may write $n_1=-(\epsilon_0/e)\nabla\cdot\mathbf{E}_p$, since $\rho^s=0$ in the far field. Substituting this expression, and the value of $\mathbf{v}_p$ from Eq. (3.11), into (B.1) gives

$$S_p(t)=-\frac{\epsilon_0 v_0^2}{\omega_p^2}(\nabla\cdot\mathbf{E}_p)\left(\frac{\partial\mathbf{E}_p}{\partial t}\right).$$

Now

$$\mathbf{E}_p(t)=\int_{-\infty}^{\infty}\mathbf{E}_\omega e^{-i\omega t}d\omega,$$

so that

$$\partial\mathbf{E}_p(t)/\partial t=\int_{-\infty}^{\infty}-i\omega\mathbf{E}_\omega e^{-i\omega t}d\omega,$$

and

$$\nabla\cdot\mathbf{E}_p=\int_{-\infty}^{\infty} ikE_\omega e^{-i\omega t}d\omega,$$

by Eq. (5.5), and the far-field approximation. Thus, the radial energy flux is

$$\int_{-\infty}^{\infty}\mathbf{S}_p(t)dt=-\mathbf{r}\epsilon_0 v_0^2\omega_p^{-2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\omega E_\omega e^{-i\omega t}d\omega \times\int_{-\infty}^{\infty}k(\omega')E_{\omega'}e^{-i\omega' t}d\omega'.$$

Changing the order of integration and performing the integration on t gives

$$\begin{aligned} \int_{-\infty}^{\infty}\mathbf{S}_p(t)dt &= -\mathbf{r}2\pi\epsilon_0 v_0^2\omega_p^{-2}\int_{-\infty}^{\infty}\omega k(-\omega)E_\omega E_{-\omega}d\omega \\ &= \mathbf{r}4\pi\epsilon_0 v_0^2\omega_p^{-2}\int_0^{\infty}\omega k(\omega)E_\omega E_\omega^* d\omega, \end{aligned}$$

since $E_{-\omega}=E_\omega^*$, and $k(-\omega)=-k(\omega)$. The latter is necessary if we are to have outgoing waves. Substitution for $k(\omega)$ now gives Eq. (5.6).