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Nonlinear Time-Dependent Plasma Oscillations*

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The Laplace transform technique employed by Landau to solve the problem of the first-order motions in an unbounded, rarified, electron plasma is modified to solve the problem to arbitrarily high order. The transforms of the nth-order contributions are expressible in terms of convolution integrals involving only terms up to order $n-1$. The method is applied to second order for the case of square-integrable disturbances.

THE problem of the first-order motions of an unbounded, rarified, electron plasma was first solved by Landau.[1] It is interesting to observe that the same methods can be adapted to solve the problem to nth order, where n is arbitrarily large.

Consider an electron (charge $-e$, mass m) plasma with distribution $f(\mathbf{x},\mathbf{v},t)$, governed by

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{e}{m}\frac{\partial\phi}{\partial\mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{v}}=0, \tag{1}$$

$$\frac{\partial^2\phi}{\partial\mathbf{x}^2}=-4\pi e\left(N_0-\int f d^3v\right), \tag{2}$$

where N_0 represents a uniform positive background charge, assumed immobile.

Let us write the solution to (1) and (2) formally as

$$\phi=\sum_{n=1}^{\infty}\phi^{(n)};\quad f=\sum_{n=0}^{\infty}f^{(n)}, \tag{3}$$

where $f^{(0)}=f_0(\mathbf{v})$ and $\int f_0(\mathbf{v})d^3v=N_0$.

It is no loss of generality to assume that at $t=0$, all the $f^{(n)}$ vanish for $n>1$. The expansion parameter implied in (3) is then $|f^{(1)}/f^{(0)}|_{\max}$.

Making this substitution into (1) and (2), taking Fourier transforms in space (indicated by subscripts $\mathbf{k}$) and Laplace transforms in time (indicated by subscripts p), we obtain

$$(p+i\mathbf{k}\cdot\mathbf{v})f_{\mathbf{k}p}{}^{(n)}+\frac{ie}{m}\mathbf{k}\phi_{\mathbf{k}p}{}^{(n)}\cdot\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}} =f_{\mathbf{k}}{}^{(n)}(0)-\frac{e}{m}S_{\mathbf{k}p}{}^{(n)}, \tag{4}$$

$$k^2\phi_{\mathbf{k}p}{}^{(n)}=-4\pi e\int f_{\mathbf{k}p}{}^{(n)}d^3v, \tag{5}$$

where

$$S^{(n)}\equiv\frac{\partial\phi^{(n-1)}}{\partial\mathbf{x}}\cdot\frac{\partial f^{(1)}}{\partial\mathbf{v}}+\cdots+\frac{\partial\phi^{(1)}}{\partial\mathbf{x}}\cdot\frac{\partial f^{(n-1)}}{\partial\mathbf{v}}, \tag{6}$$

and contains all the terms whose superscripts add up to n, but each of whose superscripts is less than n. Observe that $S^{(1)}=0$, and that $f_{\mathbf{k}}{}^{(n)}(0)=0$, for $n>1$.

This system was solved by Landau for $n=1$. For $n>1$, it is solvable at once, since $S^{(n)}$ contains only terms of order less than n. There is an obvious iteration procedure for obtaining the $f^{(n)}$ and $\phi^{(n)}$, limited only by one's patience in evaluating the convolution integrals $S_{\mathbf{k}p}{}^{(n)}$.

For $n>1$, we have

$$\phi_{\mathbf{k}p}{}^{(n)}=\frac{4\pi e^2}{mk^2}\left\{\frac{1}{D_{\mathbf{k}p}}\int\frac{S_{\mathbf{k}p}{}^{(n)}d^3v}{p+i\mathbf{k}\cdot\mathbf{v}}\right\}, \tag{7}$$

$$f_{\mathbf{k}p}{}^{(n)}=\frac{-e/m}{p+i\mathbf{k}\cdot\mathbf{v}}\left\{i\mathbf{k}\cdot\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}}\phi_{\mathbf{k}p}{}^{(n)}+S_{\mathbf{k}p}{}^{(n)}\right\}, \tag{8}$$

$$D_{\mathbf{k}p}\equiv 1-\frac{4\pi ie^2}{mk^2}\mathbf{k}\cdot\int\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}}\frac{d^3v}{p+i\mathbf{k}\cdot\mathbf{v}}, \tag{9}$$

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[1] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).

the $n=1$ solution being given by Eqs. (8) and (9) of reference 1, with a change in sign of e.

One interesting phenomenon that can be studied with a minimum of effort is the second-order contribution to one-dimensional oscillations in the case where $f_{\mathbf{k}}^{(1)}(0)$ is an entire, square-integrable function of $\mathbf{v}$, and $D_{\mathbf{k}p}$ has only simple zeros in the complex p plane. The vector $\mathbf{k}$ can be chosen to lie along the x axis, and the v_y, v_z integrations are inconsequential.

The second-order field is given by $(v_x \to v)$

$$\phi_k^{(2)}(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\,\frac{e^{pt}}{D_{kp}}\left(\frac{4\pi e^2}{mk^2}\right)\int\frac{S_{kp}^{(2)}dv}{p+ikv}, \tag{10}$$

where σ is greater than the real part of the rightmost singularity of the integrand.

For $S_{kp}^{(2)}$, we have the following convolution:

$$S_{kp}^{(2)}=\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} dp'\int_{-\infty}^{\infty} dk'\, i(k-k') \times\phi_{k-k',\,p-p'}^{(1)}\frac{\partial f_{k'p'}^{(1)}}{\partial v}, \tag{11}$$

where η is greater than the real part of the rightmost singularity of $f_{k'p'}^{(1)}$, in the complex p' plane and where $\mathrm{Re}(p-p')$ is greater than the real part of the rightmost singularity of $\phi_{k-k',\,p-p'}^{(1)}$ in the complex $(p-p')$ plane.

We may interchange the orders of integration, deform the p contour to the left, and evaluate the $t\to\infty$ behavior of $\phi_k^{(2)}(t)$ in terms of the singularities of the integrand.

These occur as: (a) the zeros of D_{kp}, which we call $p_i(k)$; (b) the zeros of $D_{k-k',\,p-p'}$, which are at $p=p'+p_i(k-k')$. The pole at $p=-ikv$ does not contribute, since the v contour can be dropped below the real axis to go below the point $v=ip/k$ for $\mathrm{Re}(p)\leq 0$, as done by Landau.

From (a) and (b) we shall get terms which vary (for large t) as $\exp[p_i(k)t]$ and $\exp\{[p_i(k-k')+p']t\}$, respectively. In the second of these, the p' integration is still important, and we determine its effect by now deforming the p' contour to the left, picking up contributions: (c) at $p'+p_i(k-k')=p_j(k)$, or terms which $\to e^{p_j(k)t}$ as $t\to\infty$; (d) at $p'=p_i(k')$, or terms which $\to \int\{\exp[p_i(k')+p_j(k-k')]t\}h(k')dk'$ as $t\to\infty$, where h is some function of k'; (e) at $p'=-ik'v$, where the second-order pole that arises from $(\partial/\partial v)\times[1/(p'+ik'v)]$ can no longer be eliminated by deforming the v contour, but can be evaluated by performing the p' integral exactly, closing the contour around the left half-plane. It is easy to show that for all $\mathrm{Re}(p_i)\leq 0$, this last contribution $\to 0$ as $t\to\infty$.

We are now in a position to make two statements about the $t\to\infty$ behavior of $\phi_k^{(2)}(t)$. First, if the system is "stable" in first order [i.e., all $\mathrm{Re}(p_i)<0$], then it is so in second order. Systems which are "unstable" in first order are "unstable" in second order, only more so. The first-order theory is probably an accurate prediction only for "stable" systems.

An nth order solution can be given in terms of normal modes, as done by Case,[2] for the first order, except that so far it has been impossible to construct a proof of completeness.

[2] K. M. Case, Ann. Phys. 7, 349 (1959).