Self-Consistent Pair Interaction for Many-Fermion System

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(Received August 15, 1960)

The equations which determine the one-particle energy and effective two-body interaction in an interacting Fermi gas are constructed within the approximation which sums up all pair creation-annihilation processes. The equation corresponds to the familiar equation for the K matrix which represents the interaction between particles (or holes) and sums up the particle-particle (or hole-hole) scattering processes. The method of the equation of motion is used in this paper. Our result for the one-particle energy is shown to lead to the result previously obtained by Quinn and Ferrell, and by Rockmore for the case of the electron gas with Coulomb interactions, when we replace screened potentials by bare potential in the self-consistent energy equation. For nuclear matter, it is shown that the presence of an attractive interaction in the equation of motion for number density causes an "enhancement" of exchange forces, whereas in the electron gas repulsive Coulomb interactions lead to "screening" of the exchange force. The strength of the isospin density interaction pseudopotential is enhanced by a factor of two when one solves the self-consistent equation; and a simple estimate shows that the Goldhaber-Teller mode lies about 15% higher than the value $p_F q/m$ previously estimated by Glassgold *et al.* (q: momentum of the oscillation, p_F: Fermi momentum).

I. INTRODUCTION

THE static behavior of nuclear matter has been well described by Brueckner *et al.*,[1] using the generalized Hartree-Fock scheme. This becomes possible because of the rather weak character of the smeared-out interaction (K matrix) and the low density of nuclear matter. The K matrix was obtained by summing up the scattering processes between two particles (or two holes). The existence of collective excitations of various kinds in nuclear matter was first shown by Glassgold, Heckrotte, and Watson,[2] who used a feasible model for the K-matrix interaction to create particle-hole pairs.

To solve the collective excitation problem from first principles is of course a matter of interest, but here we will first try to improve the result obtained by G-H-W. In a previous paper,[3] we have applied the so-called "equation of motion" method to construct the K matrix from a potential by first constructing an approximate scattering eigenmode. We will use the same method here to obtain the self-consistent equations of motion of a particle-hole pair taking into account possible higher order effects in the propagator and vertex parts of interaction between pairs.

In Sec. II, we will describe in detail the method used in this paper, and in Sec. III we will apply the self-consistent equation for the one-particle energy to the electron gas and show that the result of Quinn and Ferrell, and Rockmore[4] is obtained when we replace the screened interaction by the bare interaction in the lowest order. In Sec. IV, we will apply the self-consistent equation for the strength of the pseudopotential in nuclear matter; and estimate the effects of higher order pair interactions on the ground-state energy, the Goldhaber-Teller mode and the symmetry energy.

II. FORMULATION

As an interaction Hamiltonian, we will take the most general form of the two-body interaction:

$$H_{\rm int}=\sum_{\rm p}\sum_{\rm p'}\sum_{\rm q}\sum_j \mathbf{c}_{\rm p+q}{}^*(\mathbf{c}_{\rm p'}{}^*\rho_j\mathbf{c}_{\rm p'+q})\rho_j\mathbf{c}_{\rm p}\tfrac{1}{2}v^j(\mathbf{q}), \quad (1)$$

where the $\mathbf{c}$'s are matrices given by

$$\mathbf{c}_{\rm p}=\begin{pmatrix} c_{\rm p\uparrow} \\ c_{\rm p\downarrow}\end{pmatrix}; \qquad \rho_j=1 \text{ for the electron gas,}$$

$$\mathbf{c}_{\rm p}=\begin{pmatrix} c_{{\rm p};P,\uparrow} \\ c_{{\rm p};P,\downarrow} \\ c_{{\rm p};N,\uparrow} \\ c_{{\rm p};N,\downarrow}\end{pmatrix}; \qquad \rho_j=1,\ \sigma_j,\ \tau_j, \text{ or } (\sigma\tau)_j \text{ for nuclear matter,}$$

and the $\mathbf{c}$'s are annihilation operators of particles. As

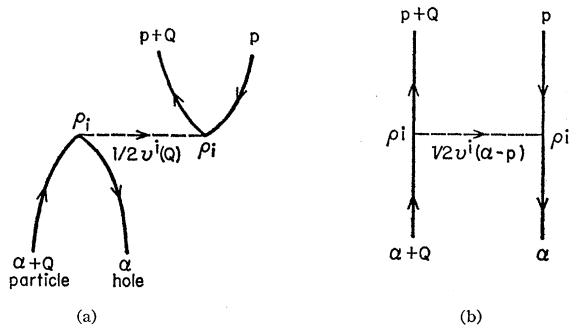

FIG. 1(a). An ordinary pair scattering in the second order of the perturbation expansion. (b) An exchange pair scattering in the second order of the perturbation expansion.

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[2] A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. **6**, 1 (1959). We refer to this as G-H-W. The Goldhaber-Teller modes have an energy $\sim p_F q/m$ (q: momentum of oscillation) in the model used by these authors, simply because the (repulsive) pair-interaction is weak.

[3] K. Sawada, Phys. Rev. **119**, 2090 (1960). We stated our intention to construct a pair eigenmode in Sec. I, Eq. (30) of this reference.

[4] J. J. Quinn and R. A. Ferrell, Phys. Rev. **112**, 812 (1958); R. M. Rockmore, *ibid.* **114**, 941 (1959).

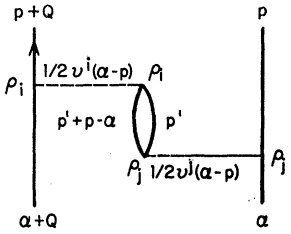

FIG. 2. A scattering diagram of a pair in the fourth order of the perturbation expansion which cannot be represented by an iteration of Fig. 1(a) and Fig. 1(b).

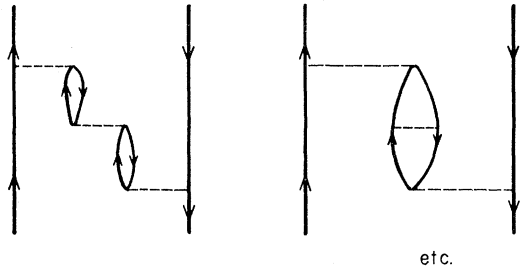

FIG. 3. Higher order pair scatterings in the perturbation expansion.

usual, we separate c into

$$c_p = a_p + b_p^*, \tag{2}$$

where a_p is the annihilation operator for a particle of momentum p (above the Fermi surface) and b_p^* is the creation operator for a hole of momentum p (below the Fermi surface). The total Hamiltonian is

$$H_T = H_0 + H_{\rm int}; \quad H_0 = \sum_{\mathbf{p}} \mathbf{c}_{\mathbf{p}}^* \mathbf{c}_{\mathbf{p}} (\mathbf{p}^2/2m). \tag{3}$$

For nuclear matter, we assume that the bare interaction has already been replaced by K, as was done in G-H-W,[2] and we regard v [in (1)] as a pseudopotential. Consequently in this work we will consider only the effects of the interaction between particle-hole pairs which are not represented in the K matrix.

We would like to give a somewhat qualitative pictorial argument at this point. The lowest order processes which represent the scattering of a pair (hole and particle) are given by Fig. 1, where (a) represents the so-called "ordinary" scattering and (b) represents the "exchange" scattering of a pair. The next order diagram which cannot be represented as an iteration of Figs. (1a) and (b) is represented in Fig. 2 (which will be followed in higher order by Fig. 3, etc.).

Figure 2 represents the correction to Fig. 1(b) and the matrix element for Fig. 2 is

$$2\times\sum_{ij} \rho_i \tfrac{1}{2} v^i(\boldsymbol{\alpha}-\mathbf{p})\, \mathrm{Tr}(\rho_i\rho_j)\tfrac{1}{2}v^j(\boldsymbol{\alpha}-\mathbf{p})\rho_j \sum_{\substack{\mathbf{p}' \\ \mathbf{p}<p_F \\ |\mathbf{p}'+\mathbf{p}-\alpha|>p_F}} \frac{1}{-\Delta-(\epsilon_{\mathbf{p}'+\mathbf{p}-\alpha}-\epsilon_{\mathbf{p}'})}$$
$$= \sum_j \rho_j \tfrac{1}{2} v^j(\boldsymbol{\alpha}-\mathbf{p})\rho_j 4 v^j(\boldsymbol{\alpha}-\mathbf{p}) \sum_{\mathbf{p}'} \frac{1}{-\Delta-(\epsilon_{\mathbf{p}'+\mathbf{p}-\alpha}-\epsilon_{\mathbf{p}'})}.$$

The factor 2 comes from taking account of the upside-down diagram, and Δ is a certain "excitation" energy which arises from the excitation of other particle-hole interactions along with Fig. 2. We should notice here, that if we assume the sign of potentials for nuclear matter to be

$$v^1<0; \quad v^\sigma, v^\tau, v^{\tau\sigma}>0,$$

as is usually supposed, the "exchange" scattering strength for $\rho_j=1$ is enhanced, and the other strengths for $\rho_j=\sigma, \tau, \sigma\tau$ are reduced by this correction, because

$$4v^j(\boldsymbol{\alpha}-\mathbf{p}) \sum_{\mathbf{p}'} \frac{1}{-\Delta-(\epsilon_{\mathbf{p}'+\mathbf{p}-\alpha}-\epsilon_{\mathbf{p}'})} \begin{cases} >0; & \rho_j=1 \\ <0; & \rho_j=\sigma, \tau, \sigma\tau. \end{cases}$$

One can expect the correction to be largest when the "momentum transfer" Q between a pair is small, because in such a case all excitations are confined to the vicinity of the Fermi surface, and Δ takes its smallest value[$\Delta\sim 0(\mathbf{Q})$]. However, because no singularity appears in the potential at small $\mathbf{Q}$ (because of the finite range of nuclear forces) and because the matrix element for Fig. 2 is large only in a small volume of phase space, we only obtain a small correction to the ground-state energy from Fig. 2 and from its higher order terms.

Now in the following sections we shall use Figs. 1(a, b) and sums of diagrams of Fig. 2, and its higher order diagrams, and show the procedures which give "self-consistent" equations for determining both the scattering strength of a pair and the one-particle energy.

The self-consistent method we will use to find the "pair" interaction can be described as follows. We first construct the equation of motion of a "pair," taking the uniform Fermi-sphere as a starting point:

$$[\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\mathbf{c}_\alpha, H_T] = -\{(\epsilon_{\alpha+\mathbf{Q}}+nv^1(0)\textstyle\sum_{\mathbf{p}} n_{\mathbf{p}} - \sum_{\mathbf{q}j} v^j(\mathbf{q}) n_{\alpha+\mathbf{Q}+\mathbf{q}}) - (\epsilon_\alpha + nv^1(0)\sum_{\mathbf{p}} n_{\mathbf{p}} - \sum_{\mathbf{q}j} v^j(\mathbf{q}) n_{\alpha+\mathbf{q}})\}\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\mathbf{c}_\alpha$$
$$-(n_\alpha - n_{\alpha+\mathbf{Q}})\{\textstyle\sum_j v^j(\mathbf{Q})\,\mathrm{Tr}(\rho_j\sigma)\sum_{\mathbf{p}} :\mathbf{c}_{\mathbf{p}+\mathbf{Q}}^*\rho_j\mathbf{c}_{\mathbf{p}}: - \sum_{j\mathbf{q}} v^j(\mathbf{q}) :\mathbf{c}_{\alpha+\mathbf{Q}-\mathbf{q}}^*\rho_j\sigma\rho_j\mathbf{c}_{\alpha-\mathbf{q}}:\}$$
$$-\textstyle\sum_{\mathbf{q};j;\mathbf{p}}\{:(\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\rho_j\mathbf{c}_{\alpha+\mathbf{q}})(\mathbf{c}_{\mathbf{p}+\mathbf{q}}^*\rho_j\mathbf{c}_{\mathbf{p}}): - :(\mathbf{c}_{p+q}^*\rho_j\mathbf{c}_p)(\mathbf{c}_{\alpha+\mathbf{Q}-\mathbf{q}}^*\rho_j\sigma\mathbf{c}_\alpha):\}v^j(\mathbf{q}), \tag{4}$$

where σ represents 1, σ, τ, or $\sigma\tau$;

$$n_p=1 \quad \text{for} \quad \mathbf{p}<p_F$$
$$=0 \quad \text{for} \quad \mathbf{p}>p_F;$$

: : means that the operators a^*, b^*, a, and b appear ordered; n is the number of internal degrees of freedom (2 for the electron gas, 4 for nuclear matter).

Let us rewrite (4) in the following form:

$$[\mathbf{c}_{\alpha+Q}{}^*\sigma\mathbf{c}_\alpha,H_T]=-(\omega_{\alpha+Q}-\omega_\alpha)\mathbf{c}_{\alpha+Q}{}^*\sigma\mathbf{c}_\alpha-(n_\alpha-n_{\alpha+Q})\sum_j \bar{v}^j(\mathbf{Q})\,\mathrm{Tr}(\rho_j\sigma)\sum_p:\mathbf{c}_{p+Q}{}^*\rho_j\mathbf{c}_p:+\Delta_{\alpha;Q}, \tag{5}$$

where

$$\Delta_{\alpha;Q}=\{[\omega_{\alpha+Q}-(\epsilon_{\alpha+Q}+nv^1(0)\textstyle\sum_p n_p-\sum_{qj} v^j(\mathbf{q})n_{\alpha+Q+q})]-(\omega_\alpha-(\epsilon_\alpha+nv^1(0)\textstyle\sum_p n_p-\sum_{qj} v^j(\mathbf{q})n_{\alpha+q})]\}\mathbf{c}_{\alpha+Q}{}^*\sigma\mathbf{c}_\alpha$$
$$+(n_\alpha-n_{\alpha+Q})\textstyle\sum_j\{[\bar{v}^j(\mathbf{Q})-v^j(\mathbf{Q})]\,\mathrm{Tr}(\rho_j\sigma)\sum_p:\mathbf{c}_{p+Q}{}^*\rho_j\mathbf{c}_p:+\sum_p v^j(\alpha-\mathbf{p}):\mathbf{c}_{p+Q}{}^*\rho_j\sigma\rho_j\mathbf{c}_p:\}$$
$$+\textstyle\sum_{qjp}[:(\mathbf{c}_{\alpha+Q}{}^*\sigma\rho_j\mathbf{c}_{\alpha+q})(\mathbf{c}_{p+q}{}^*\rho_j\mathbf{c}_p):-:(\mathbf{c}_{p+q}{}^*\rho_j\mathbf{c}_p)(\mathbf{c}_{\alpha+Q-q}{}^*\rho_j\sigma\mathbf{c}_\alpha):]v^j(\mathbf{q}),$$

and ω and $\bar{v}^j(\mathbf{Q})$ are unknown functions which will be determined later.

In (5), we consider the term $\Delta_{\alpha;Q}$ as the residual "interaction" and try to choose ω and $\bar{v}^j(\mathbf{Q})$ so as to make $\Delta_{\alpha;Q}$ as small as possible.

Let us construct an eigenmode which has the following property:

$$[X^{*n},H_T]=-\omega^nX^{*n}.$$

Then it follows

$$H_TX^{*n}\Psi_0=(E_0+\omega^n)X^{*n}\Psi_0,$$

where Ψ_0 is the ground-state wave function. In order to find such an operator $X_n{}^*$, we multiply both sides of (5) with coefficients $\psi_{\alpha;Q}{}^{n;i}$:

$$[\textstyle\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha,H_T]-$$
$$=\textstyle\sum_\alpha \mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha\{-(\omega_{\alpha+Q}-\omega_\alpha)\psi_{\alpha;Q}{}^{n;i}-n\bar{v}^i(\mathbf{Q})\sum_{\alpha'}(n_{\alpha'}-n_{\alpha'+Q})\psi_{\alpha';Q}{}^{n;i}\}+\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\Delta_{\alpha;Q} \tag{6}$$
$$=-\omega_Q{}^{n;i}\textstyle\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha+Y,$$

where[5]

$$Y=\textstyle\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\Delta_{\alpha;Q}.$$

Then we obtain an equation for $\psi_{\alpha;Q}{}^{n;i}$:

$$[\omega_Q{}^{n;i}-(\omega_{\alpha+Q}-\omega_\alpha)]\psi_{\alpha;Q}{}^{n;i}=n\bar{v}^i(\mathbf{Q})\textstyle\sum_{\alpha'}(n_{\alpha'}-n_{\alpha'+Q})\psi_{\alpha;Q}{}^{n;i}. \tag{7}$$

If we choose Δ as small as possible, $\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha$ represents the operator X^{*n} approximately, and X^{*n}, X^n represent creation and annihilation operators for excitations of energy $\omega^n(>0)$. Namely, the "packet"

$$A_Q{}^{*n;i}=\textstyle\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha(\simeq X^{*n})(\omega_Q{}^{n;i}>0), \tag{8}$$
$$A_{-Q}{}^{n;i}=\textstyle\sum_\alpha\psi_{\alpha;Q}{}^{n;i}\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha(\simeq X^n)(\omega_Q{}^{n;i}<0)$$

gives

$$H_TA_Q{}^{*n;i}\Psi_0\simeq(E_0+\omega_Q{}^{n;i})A_Q{}^{*n;i}\Psi_0,$$
$$H_TA_{-Q}{}^{n;i}\Psi_0\simeq(E_0-|\omega_Q{}^{n;i}|)A_{-Q}{}^{n;i}\Psi_0,$$

if we regard Y as a small quantity; and hence $A_Q{}^{*ni}$, $A_{-Q}{}^{ni}$ represent "creation" and "annihilation" of excitations of energy $|\omega_Q{}^{n;i}|$. Equation (7) gives us a complete orthonormal set of functions; hence we can expand $\mathbf{c}_{\alpha+Q}{}^*\rho_i\mathbf{c}_\alpha$ in A and A^* by applying the inverse transformation to (8).

To make $\Delta_{\alpha;Q}$ in (5) as small as possible, we should remember that a^*, b^*, a, and b appear ordered in the 4-operator term of the form $:(\mathbf{c}^*\sigma\rho\mathbf{c})(\mathbf{c}^*\rho\mathbf{c}):$ in Δ. But since small Δ means that A^* and A given by (8) represent creation and annihilation of excitations, then $a^*\rho b^*$ or $b\rho a$ [contained in $(\mathbf{c}^*\rho\mathbf{c})$] no longer represent creation or annihilation of excitations. Namely, as mentioned above, we can expand $\mathbf{c}^*\rho\mathbf{c}(a^*\rho b^*,b\rho a,$ etc.) in a linear combination of A and A^*. Then $:(\mathbf{c}^*\sigma\rho\mathbf{c})(\mathbf{c}^*\rho\mathbf{c}):$ must be re-ordered about A, A^* in order to minimize its effect. The reordering gives us the form:

$$:(\mathbf{c}^*\sigma\rho\mathbf{c})(\mathbf{c}^*\rho\mathbf{c}):=A^*:(\mathbf{c}^*\sigma\rho\mathbf{c}):+:(\mathbf{c}^*\sigma\rho\mathbf{c}):A+\Delta\omega:\mathbf{c}^*\sigma\mathbf{c}:+\Delta v:\mathbf{c}^*\rho\sigma\rho\mathbf{c}:,$$

where $\Delta\omega$, Δv terms arise because of noncommutativity of A and A^* with a^*, b^*, a, and b. The terms $\Delta\omega$ and Δv can be combined with terms which appeared originally in Δ of (5) and the sums of coefficients should be taken as

[5] Equation (6) corresponds to $[X^*,H_T]=-\omega X^*+Y$ of reference 3.

zero to make the effect of Δ small. In this way, we can get the equation which determines the one-particle energy ω and the self-consistent potential $\bar{v}^i(Q)$. Then the equation for the A's takes the form

$$[A^*,H_T]_-=-\omega A^*+(A^*:\mathbf{c}^*\sigma\rho\mathbf{c}:+:\mathbf{c}^*\sigma\rho\mathbf{c}:A)$$
$$=-\omega A^*+Y$$

[from (6)]; Δ is now of the form $A^*:\mathbf{c}^*:\rho\mathbf{c}:+:\mathbf{c}^*\sigma\rho\mathbf{c}:A$ and the residual "interaction" Y becomes a nonlinear coupling between A's. We can expand $:\mathbf{c}^*\sigma\rho\mathbf{c}:$ again into A's, but no further re-ordering is necessary because A^* and A already stand to the extreme left and right in Y.

The above procedure provides us with an approximate pair eigenmode A^* self-consistently. Of course, to obtain "exact" self-consistent equations for $\bar{v}^i$ and ω (one-particle energy), we should assume

$$\bar{v}^i(\mathbf{Q})\rightarrow\bar{v}^i(\mathbf{Q};\boldsymbol{\alpha},\mathbf{p}),$$

but, for simplicity, we assume the simplest possible dependence on momentum for the $\bar{v}^i$ [in some sense of "average," we expect to be able to include in the simple form $\bar{v}^i(Q)$ the effect of the "exchange" term (the fourth term in (5)) and the generally momentum-dependent Δv which arises from re-ordering of 4-operator term in $\Delta_{\alpha;\mathrm{Q}}$].

To solve the equation for ψ, Eq. (7), we take[3]

$$\psi_{\alpha;\mathrm{Q}}{}^{n;i}=\begin{bmatrix}\Phi_{\alpha;\mathrm{Q}}{}^{n;i}|\boldsymbol{\alpha}+\mathbf{Q}|>p_F;\ \boldsymbol{\alpha}<p_F\\ \Theta_{\alpha;\mathrm{Q}}{}^{n;i}|\boldsymbol{\alpha}+\mathbf{Q}|<p_F;\ \boldsymbol{\alpha}>p_F\\ \mathrm{X}_{\alpha;\mathrm{Q}}{}^{n;i}|\boldsymbol{\alpha}+\mathbf{Q}|>p_F;\ \boldsymbol{\alpha}>p_F\\ \Xi_{\alpha;\mathrm{Q}}{}^{n;i}|\boldsymbol{\alpha}+\mathbf{Q}|<p_F;\ \boldsymbol{\alpha}<p_F\end{bmatrix},\tag{9}$$

Then Eq. (7) becomes (writing $\omega_{p;Q}=\omega_{p+Q}-\omega_p$)

$$(\omega_n-\omega_{p;Q})\begin{bmatrix}\Phi_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \Theta_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \mathrm{X}_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \Xi_{\mathrm{p};\mathrm{Q}}{}^{n;i}\end{bmatrix}=n\bar{v}^i(\mathbf{Q})\sum_{\mathrm{p}'}(\Phi_{p';\mathrm{Q}}{}^{n;i}-\Theta_{p';\mathrm{Q}}{}^{n;i})\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\tag{10}$$

namely, Φ and Θ are coupled to produce eigenvalue equations:

$$(\omega_n-\omega_{p;\mathrm{Q}}\sigma_z)\boldsymbol{\Phi}_{\mathrm{p};\mathrm{Q}}{}^{n;i}=n\bar{v}^i(Q)(\sigma_z-i\sigma_y)\sum_{p'}\boldsymbol{\Phi}_{\mathrm{p}';\mathrm{Q}}{}^{n;i},$$
$$\boldsymbol{\Phi}_{p,\mathrm{Q}}{}^{n;i}=\begin{pmatrix}\Phi_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \Theta_{-\mathrm{p}-\mathrm{Q};\mathrm{Q}}{}^{n;i}\end{pmatrix},\tag{11}$$

(note that $|\mathbf{p}+\mathbf{Q}|>p_F$, $|\mathbf{p}|<p_F$), and X and Ξ can be solved immediately as functions of Φ and Θ.

The orthonormality relation for (11) is

$$\sum_{\mathrm{p}}\boldsymbol{\Phi}_{\mathrm{p};\mathrm{Q}}{}^{*m;i}\sigma_z\boldsymbol{\Phi}_{\mathrm{p};\mathrm{Q}}{}^{n;i}=\delta_{m;n}\omega_n/|\omega_n|,\quad \boldsymbol{\Phi}_{\mathrm{p};\mathrm{Q}}{}^{\pm,n;i}=\sigma_x(\boldsymbol{\Phi}^T)_{\mathrm{p};\mathrm{Q}}{}^{\mp,n;i*}.\tag{12}$$

(The $\pm$ sign represents the sign ω_n; the T represents its transposed matrix.) And the completeness relation is

$$\sigma_z\sum_n\boldsymbol{\Phi}_{\mathrm{p},\mathrm{Q}}{}^{n;i}\frac{\omega_n}{|\omega_n|}\boldsymbol{\Phi}_{\mathrm{p}',\mathrm{Q}}{}^{n;i*}=\delta_{\mathrm{p};\mathrm{p}'}\tag{13}$$

(this applies only for $|\mathbf{p}+\mathbf{Q}|>p_F$, $|\mathbf{p}|<p_F$).

The explicit solution of (10) is given ($\omega_n>0$) by

$$\begin{bmatrix}\Phi_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \Theta_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \mathrm{X}_{\mathrm{p};\mathrm{Q}}{}^{n;i}\\ \Xi_{\mathrm{p};\mathrm{Q}}{}^{n;i}\end{bmatrix}=\begin{bmatrix}\delta_{n;\mathrm{p}+\mathrm{Q},\mathrm{p}}(|\mathbf{p}+\mathbf{Q}|>p_F,\ |\mathbf{p}|<p_F)\\ 0\\ 0\\ 0\end{bmatrix}+\frac{1}{\omega_n\pm i\epsilon-\omega_{\mathrm{p};\mathrm{Q}}}N_{\mathrm{Q}}{}^{n;i},\tag{14}$$
$$N_{\mathrm{Q}}{}^{n;i}=n\bar{v}^i(Q)\sum_{\mathrm{p}'}(\Phi_{\mathrm{p}';\mathrm{Q}}{}^{n;i}-\Theta_{\mathrm{p}';\mathrm{Q}}{}^{n;i}),$$

where the first term in (14) represents the incident wave for a "pair-scattering" state ($\omega_{\mathrm{Q}}{}^{n;i}$ is in the continuum).

$$N_{\mathrm{Q}}{}^{n;i}=n\bar{v}^i(Q)/[1+f_{\mathrm{Q}}{}^i(\omega_{\mathrm{Q}}{}^{n;i})],$$
$$f_{\mathrm{Q}}{}^i(z)=n\bar{v}^i(Q)\sum_{\mathrm{p}}\frac{2\omega_{p;Q}n_{\mathrm{p}}(1-n_{\mathrm{p}+\mathrm{Q}})}{\omega_{p;Q}{}^2-(z\pm i\epsilon)^2},\tag{15}$$

with

$$\omega_Q{}^{n;i}=\omega_{p0;Q}$$

($|\mathbf{p}_0+\mathbf{Q}|>p_F$; $|\mathbf{p}_0|<p_F$ and $\omega_{p0;Q}$ is the energy of the incident wave.) For possible "bound" states (plasma states), where the first term on the right-hand side of (14) is zero, we have from the orthonormality relation (12) ($\omega_Q{}^{B;i}>0$):

$$|N_Q{}^{B;i}|^2=1\Big/\left[\sum_{\mathbf{p}}\left(\frac{1}{(\omega_Q{}^{B;i}-\omega_{p;Q})^2}-\frac{1}{(\omega_Q{}^{B;i}+\omega_{p;Q})^2}\right)(1-n_{\mathbf{p}+\mathbf{Q}})n_{\mathbf{p}}\right]$$

$$=n\bar{v}_j(Q)\Big/\left[\frac{\partial}{\partial z}f_Q{}^i(z)\right]_{z=\omega_Q{}^{B;i}}, \tag{16}$$

$$1+f_Q{}^i(\omega_Q{}^{B;i})=0.$$

Equation (16) determines both the energy of bound states and the normalization constant N.

In the approximation which neglects the dependence of $\omega_{p;Q}$ and $\bar{v}^j(Q)$ on the coupling parameter λ (we suppose v^j proportional to λ), we have

$$|N_Q{}^{B;i}|^2=\lambda\frac{\partial}{\partial\lambda}\omega_Q{}^{B;i}n\bar{v}^i(Q), \tag{17}$$

just as in the case of the electron gas treated before[6]; however, we will not need to use this kind of approximation to obtain Eq. (25) which appears in Sec. III.

The "creation" and "annihilation" operators (8) now become, by using (9) and X, Ξ given by (10):

$$\mathfrak{A}_Q{}^{*n;i}=\begin{pmatrix}A_Q{}^{*n;i}\\A_{-Q}{}^{n;i}\end{pmatrix}=\sum_p\Big\{(\mathbf{a}_{p+Q}{}^*\rho_i\mathbf{b}_p{}^*,\mathbf{b}_{-p}\rho_i\mathbf{a}_{p-Q})\Phi_{p;Q}{}^{n;i}$$
$$+\frac{1}{\omega_Q{}^{n;i}\pm i\epsilon-\omega_{p;Q}}N_Q{}^{n;i}(:\mathbf{a}_{p+Q}{}^*\rho_i\mathbf{a}_p:+:\mathbf{b}_{p+Q}\rho_i\mathbf{b}_p{}^*:)\Big\},\quad\begin{pmatrix}\omega_Q{}^{n;i}>0\\\omega_Q{}^{n;i}<0\end{pmatrix}. \tag{18}$$

We can see by using the relation (12) between $\Phi^{\pm,n}$ that $A_{-Q}{}^{n;i}$ is $(A_{-Q}{}^{*n;i})^\dagger$ (the dagger indicates Hermitian conjugate). By using the completeness relation (13), we get the inverse transform of (18):

$$\mathbf{a}_{p+Q}{}^*\rho_i\mathbf{b}_p{}^*=\sum_n\mathfrak{A}_Q{}^{*n;i}\frac{\omega_n}{|\omega_n|}\Phi_{p;Q}{}^{*n;i}-\sum_{p'}(\mathbf{a}_{p'+Q}{}^*\rho_i\mathbf{a}_{p'}+:\mathbf{b}_{p'+Q}\rho_i\mathbf{b}_{p'}{}^*:)\sum_n\frac{1}{\omega_n\pm i\epsilon-\omega_{p';Q}}N_Q{}^{n;i}\frac{\omega_n}{|\omega_n|}\Phi_{p;Q}{}^{*n;i},$$

$$\mathbf{b}_{-p}\rho_i\mathbf{a}_{-p-Q}=-\sum_n\mathfrak{A}_Q{}^{*n;i}\frac{\omega_n}{|\omega_n|}\Theta_{-p-Q;Q}{}^{*n;i}$$
$$+\sum_{p'}(\mathbf{a}_{p'+Q}{}^*\rho_i\mathbf{a}_{p'}+:\mathbf{b}_{p'+Q}\rho_i\mathbf{b}_{p'}{}^*:)\sum_n\frac{1}{\omega_n+i\epsilon-\omega_{p';Q}}N_Q{}^{n;i}\frac{\omega_n}{|\omega_n|}\Theta_{-p-Q;Q}{}^{*n;i}. \tag{19}$$

Hence for $:\sum_p\mathbf{c}_{p+Q}{}^*\rho_i\mathbf{c}_p:$, which we will write as $n_i(Q)$ by using the definition of $N_Q{}^{n;i}$ in (14), we have an expansion:

$$n_i(Q)=:\sum_{\mathbf{p}}\mathbf{c}_{p+Q}{}^*\rho_i\mathbf{c}_p:$$
$$=\sum_{(\omega_n>0)}\left(A_Q{}^{*n;i}\frac{N_Q{}^{*n;i}}{n\bar{v}_i(Q)}+A_{-Q}{}^{n;i}\frac{N_Q{}^{n;i}}{n\bar{v}_i(Q)}\right)+\sum_p:(\mathbf{a}_{p+Q}{}^*\rho_i\mathbf{a}_p+\mathbf{b}_{p+Q}\rho_i\mathbf{b}_p{}^*):$$
$$\times\left\{1-\sum_{n\,(\omega_n>0)}\left(\frac{1}{\omega_Q{}^{n;i}\pm i\epsilon-\omega_{p;Q}}-\frac{1}{\omega_Q{}^{n;i}\mp i\epsilon+\omega_{p;Q}}\right)\frac{|N_Q{}^{n;i}|^2}{n\bar{v}^i(Q)}\right\}. \tag{20}$$

(We have used the relation $|N_Q{}^{n;i}|^2=|N_{-Q}{}^{n;i}|^2$.)

At this point, one should note that $[A_Q{}^{n;i},A_Q{}^{*n;i}]=n$ (n is the number of the internal degrees of freedom) and so the A's are not normalized operators, but normalization is not essential to the following arguments.

Now, we want to make the "interaction" $\Delta_{\alpha;Q}$ in (5) as small as possible. By using expansion (20) for $n_j(q)$

[6] K. Sawada, K. A. Brueckner, N. Fukada, and R. Brout, Phys. Rev. **108**, 507 (1957).

$=\sum_p :\mathbf{c}_{\mathbf{p}+\mathbf{q}}^*\rho_j\mathbf{c}_\mathbf{p}:$ which appears in the 4-operator term in $\Delta_{\alpha;\mathbf{Q}}$, we get

$$\sum_{q;p}\{:(\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\rho_j\mathbf{c}_{\alpha+\mathbf{q}})(\mathbf{c}_{\mathbf{p}+\mathbf{q}}^*\rho_j\mathbf{c}_\mathbf{p}):-:(\mathbf{c}_{\mathbf{p}+\mathbf{q}}^*\rho_j\mathbf{c}_\mathbf{p})(\mathbf{c}_{\alpha+\mathbf{Q}-\mathbf{q}}^*\rho_j\sigma\mathbf{c}_\alpha):\}v^j(q)$$
$$=\sum_{qj}v^j(q)[\{\mathbf{a}_{\alpha+\mathbf{Q}}^*a\rho_j\mathbf{b}_{\alpha+\mathbf{q}}^*n_j(q)+\mathbf{a}_{\alpha+\mathbf{Q}}^*\sigma\rho_jn_j(q)\mathbf{a}_{\alpha+\mathbf{q}}+n_j(q)\mathbf{b}_{\alpha+\mathbf{Q}}\sigma\rho_j\mathbf{a}_{\alpha+\mathbf{q}}-(\mathbf{b}_{\alpha+\mathbf{q}})^{T*}(\rho_j)^T(\sigma)^Tn_j(q)(\mathbf{b}_{\alpha+\mathbf{Q}})^T\}$$
$$-\{\boldsymbol{\alpha}\rightarrow\boldsymbol{\alpha}-\mathbf{q};\sigma\rho\rightarrow\rho\sigma\}]$$

(where T is the transposed matrix), and by shifting the operators A^* and A to the left and the right, respectively, we reach the following expression for $\Delta_{\alpha;\mathbf{Q}}$:

$$\Delta_{\alpha,\mathbf{Q}}=\Delta_{\alpha,\mathbf{Q}}{}^{(1)}+\Delta_{\alpha;\mathbf{Q}}{}^{(2)}+\Delta_{\alpha;\mathbf{Q}}{}^{(3)},$$

$$\Delta_{\alpha;\mathbf{Q}}{}^{(1)}=\Big\{\Big\{\omega_{\alpha+Q}-\Big[\epsilon_{\alpha+Q}+nv^1(0)\sum_p n_p-\sum_{jq}v^j(q)n_{\alpha+Q+q}$$
$$+\sum_{jq}v^j(q)\sum_{(\omega_n>0)}\Big(\frac{n_{\alpha+Q-q}}{\omega_q{}^{n;i}\mp i\epsilon-\omega_{\alpha+Q;-q}}-\frac{1-n_{\alpha+Q-q}}{\omega_q{}^{n;i}\pm i\epsilon-\omega_{\alpha+Q-q;q}}\Big)\frac{|N_q{}^{n;i}|^2}{n\bar{v}^i(q)}\Big]\Big\}$$
$$-[\omega_\alpha-(\alpha+Q\rightarrow\alpha\text{ and complex conjugate in the above})]\Big\}\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\mathbf{c}_\alpha,\quad(21a)$$

$$\Delta_{\alpha;\mathbf{Q}}{}^{(2)}=\sum_j\sum_p\Big\{(\bar{v}^j(Q)-v^j(Q))(n_\alpha-n_{\alpha+Q})\,\mathrm{Tr}(\rho_j\sigma):\mathbf{c}_{\mathbf{p}+\mathbf{Q}}^*\rho_j\mathbf{c}_\mathbf{p}:$$
$$+v^j(\mathbf{p}-\mathbf{a})\Big[n_\alpha-\sum_{(\omega_n>0)}\Big(\frac{n_\alpha}{\omega_{p-\alpha}{}^{n;i}\mp i\epsilon-\omega_{p;\alpha-p}}-\frac{1-n_\alpha}{\omega_{p-\alpha}{}^{n;i}\pm i\epsilon-\omega_{\alpha;p-\alpha}}\Big)\frac{|N_{p-\alpha}{}^{n;i}|^2}{n\bar{v}_j(\mathbf{p}-\boldsymbol{\alpha})}\quad(21b)$$
$$-n_{\alpha+Q}+\sum_{(\omega_n>0)}\Big(\frac{n_{\alpha+Q}}{\omega_{p-\alpha}{}^{n;i}\pm i\epsilon-\omega_{p+Q;\alpha-p}}-\frac{1-n_{\alpha+Q}}{\omega_{p-\alpha}{}^{n;i}\mp i\epsilon-\omega_{\alpha+Q;p-\alpha}}\Big)\frac{|N_{p-\alpha}{}^{n;j}|^2}{n\bar{v}_j(\mathbf{p}-\boldsymbol{\alpha})}\Big]:\mathbf{c}_{\mathbf{p}+\mathbf{Q}}^*\rho_j\sigma\rho_j\mathbf{c}_\mathbf{p}:\Big\},$$

$$\Delta_{\alpha;\mathbf{Q}}{}^{(3)}=\sum_{qj}v^j(q)[:(\mathbf{c}_{\alpha+\mathbf{Q}}^*\sigma\rho_j\mathbf{c}_{\alpha+\mathbf{q}})(\mathbf{c}_{\mathbf{p}+\mathbf{q}}^*\rho_j\mathbf{c}_\mathbf{p}):-(\alpha\rightarrow\alpha-q;\sigma\rho\rightarrow\rho\sigma)]_{\text{re-ordered}}.\quad(21c)$$

$\Delta^{(1)}$ and $\Delta^{(2)}$ include $\Delta\omega$ and Δv, which arose from re-ordering of the 4-operator term in $\Delta_{\alpha;\mathbf{Q}}$.

In the next two chapters we shall discuss the conditions which minimize (21).

III. ONE-PARTICLE ENERGY: ELECTRON GAS

We can minimize $\Delta_{\alpha;\mathbf{Q}}{}^{(1)}$ as given by (21a) by taking ω_α as the one-particle energy determined by the condition: $\mathrm{Real}(\Delta_{\alpha;\mathbf{Q}}{}^{(1)})=0$:

$$\omega_\alpha=\text{Real part}\Big[\epsilon_\alpha+nv^1(0)\sum_\mathbf{p}n_\mathbf{p}-\sum_{jq}v^j(q)n_{\alpha+\mathbf{q}}$$
$$+\sum_{jq}v^j(q)\sum_{(\omega_n>0)}\Big(\frac{n_{\alpha-q}}{\omega_q{}^{n;i}\pm i\epsilon-\omega_{\alpha;-q}}-\frac{1-n_{\alpha-q}}{\omega_q{}^{n;i}\mp i\epsilon-\omega_{\alpha-q;q}}\Big)\frac{|N_q{}^{n;i}|^2}{n\bar{v}^j(q)}\Big].\quad(22)$$

This is the self-consistent equation for the one-particle energy. Because of the appearance of the zero-energy denominator in (22), $\Delta_{\alpha;\mathbf{Q}}{}^{(1)}$ contains an imaginary part. This corresponds to the possibility of decay of a single-particle excitation into one particle and one particle-hole pair.

In the case of an electron gas with Coulomb interaction, we reach the same energy expression as previously derived by Quinn and Ferrell, and by Rockmore,[4] if we use "bare" quantities on the right-hand side of (22). To show this, we omit $v(0)$ from (22) (because we assume a background of positive charges which neutralize the system). Also we set

$$\rho_j=1;\quad n=2,\quad v(q)=4\pi/\Omega q^2,\quad(23)$$

and for $N_q{}^n$ we use (14) with $v(q)$ for $\bar{v}(q)$ and ϵ_α for $\omega_\alpha(\epsilon_\alpha$ is kinetic energy). The sum $\sum_n(\omega_n>0)$ can be transformed into a contour integral by using (15) and (16).

If we use, as in the above, $\bar{v}=v$ then (22) gives a self-consistent equation for one-particle energy which contains all pair-pair scattering effects within the random-phase approximation. If we choose $\bar{v}$ so that $\Delta_{\alpha;\mathbf{Q}}{}^{(2)}$ vanishes, then (22) will provide us with an equation containing all pair-pair scattering effects. In the case of very-long-range forces, the dependence of $v^j(\mathbf{p}-\boldsymbol{\alpha})$ on the direction between $\mathbf{p}$ and $\boldsymbol{\alpha}$ is too strong to make "average" to get $\Delta_{\alpha;\mathbf{Q}}{}^{(2)}\simeq 0$ with our simple choice of momentum dependence for $\bar{v}^j$. But for short range forces this angle dependence is rather small and we expect

some "average" can be taken to make $\Delta_{\alpha j\mathbf{Q}}{}^{(2)}\simeq 0$; this is fully discussed in the next chapter for the case of nuclear matter.

Thus in the case of an electron gas we cannot construct a self-consistent equation for the effective pair-pair interaction with our simple assumption of momentum dependence of $\bar{v}^j$. But we can see explicitly in (21b) that the "exchange" pair interaction ($\mathbf{c}^*\rho_j\sigma\rho_j\mathbf{c}$ term) is "screened." For example, $n_\alpha=1$, $n_{\alpha+Q}=0(|\boldsymbol{\alpha}+\mathbf{Q}|>p_F, |\boldsymbol{\alpha}|<p_F)$, the coefficient 1 [multiplied by $:\mathbf{c}_{\mathbf{p}+\mathbf{Q}}{}^*\rho_j\sigma\rho_j\mathbf{c}_\mathbf{p}:$ $v^j(\mathbf{p}-\boldsymbol{\alpha})$ in (5)] is replaced in $\Delta_{\alpha;\mathbf{Q}}{}^{(2)}$ by

$$1\rightarrow 1-\sum_{(\omega_n>0)}{}_n\left(\frac{1}{\omega_{p-\alpha}{}^{n;j}\mp i\epsilon-\omega_{p;\alpha-p}}+\frac{1}{\omega_{p-\alpha}{}^{n;j}\pm i\epsilon-\omega_{\alpha+Q;p-\alpha}}\right)\frac{|N_{p-\alpha}{}^{n;j}|^2}{2\bar{v}^j(\mathbf{p}-\boldsymbol{\alpha})}. \quad (24)$$

Using (15), (16), and (17), and the same contour technique as in reference 6, we have

$$\sum_{(\omega_n>0)}{}_n\frac{1}{\omega_q{}^{n;j}\pm i\epsilon-\Delta}\frac{|N_q{}^{n;j}|^2}{n\bar{v}^j(q)}=\frac{f_q{}^j(\Delta)}{1+f_q{}^j(\Delta)}(\Delta>0)$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_q{}^j(iv)}{1+f_q{}^j(iv)}\frac{\Delta}{v^2+\Delta^2}dv, \quad (25)$$

where the f's are defined in (15). Hence for sufficiently small Q:

$$1\rightarrow 1-\frac{f_{|\mathbf{q}-\boldsymbol{\alpha}|}{}^j(|\omega_\alpha-\omega_p|)}{1+f_{|\mathbf{q}-\boldsymbol{\alpha}|}{}^j(|\omega_\alpha-\omega_p|)}+O(Q),$$

and in the case of Coulomb interaction $f_{|\mathbf{p}-\boldsymbol{\alpha}|}{}^j(|\omega_\alpha-\omega_p|)$, $\propto 1/|\mathbf{p}-\boldsymbol{\alpha}|^2$; hence, for small $|\mathbf{p}-\boldsymbol{\alpha}|$ the interaction is screened.

IV. EFFECTIVE PAIR INTERACTION: NUCLEAR MATTER

As we have mentioned briefly in the previous chapter, our simple assumption for momentum dependence of $\bar{v}^j$ restricts us to constructing $\bar{v}^j$ only for short-range forces. To see the effect of pair-pair scattering on $\bar{v}^j$, we treat the case of nuclear matter in this section regarding the v appearing in our original interaction (1) as a pseudopotential (or K matrix) which includes the effect of the particle-particle and hole-hole scattering. It was already shown by G-H-W[2] that Goldhaber-Teller modes exist very close to the energy qp_F/m (q: momentum of oscillation) if one uses feasible parameters for the pseudopotential and solves an eigenvalue equation corresponding to (16). We would like to examine how this conclusion can be changed if we include all pair-pair interactions self-consistently.[7]

Let us write down $\Delta_{\alpha;\mathbf{Q}}{}^{(2)}$ explicitly by using (25): ($\omega_p-\omega_\alpha>0$, etc. means the term vanishes for $\omega_p-\omega_\alpha<0$),

$$\Delta_{\alpha;\mathbf{Q}}{}^{(2)}=\sum_j\Big[(\bar{v}^j(Q)-v^j(Q))(n_\alpha-n_{\alpha+Q})\,\mathrm{Tr}(\rho_j\sigma):\mathbf{c}_{\mathbf{p}+\mathbf{Q}}{}^*\rho_j\mathbf{c}_\mathbf{p}:+v^j(p-\alpha)\Big\{n_\alpha\frac{1}{1+f_{p-\alpha}{}^j(|\omega_\alpha-\omega_p|)}$$
$$+\frac{f_{p-\alpha}{}^j(\omega_p-\omega_\alpha)}{1+f_{p-\alpha}{}^j(\omega_p-\omega_\alpha)}(\omega_p-\omega_\alpha>0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_{p-\alpha}{}^j(iv)}{1+f_{p-\alpha}{}^j(iv)}\frac{\omega_\alpha-\omega_p}{v^2+(\omega_\alpha-\omega_p)^2}dv$$
$$-n_{\alpha+Q}\frac{1}{1+f_{p-\alpha}{}^j(|\omega_{\alpha+Q}-\omega_{p+Q}|)}-\frac{f_{p-\alpha}{}^j(\omega_{p+Q}-\omega_{\alpha+Q})}{1+f_{p-\alpha}{}^j(\omega_{p+Q}-\omega_{\alpha+Q})}(\omega_{p+Q}-\omega_{\alpha+Q}>0)$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_{p-\alpha}{}^j(iv)}{1+f_{p-\alpha}{}^j(iv)}\frac{\omega_{\alpha+Q}-\omega_{p+Q}}{v^2+(\omega_{\alpha+Q}-\omega_{p+Q})^2}dv\Big\}:\mathbf{c}_{\mathbf{p}+\mathbf{Q}}{}^*\rho_j\sigma\rho_j\mathbf{c}_\mathbf{p}:\Big]. \quad (26)$$

Let us further confine ourselves to the case of small momentum transfer Q. Then, since the essential part of the eigenvalue equations comes from Φ and Θ [see (9), (10), and (11)], we may consider $|\mathbf{p}+\mathbf{Q}|>p_F$, $|\mathbf{p}|<p_F$ or $|\mathbf{p}+\mathbf{Q}|<p_F$, $|\mathbf{p}|>p_F$ in (26), and correspondingly $|\boldsymbol{\alpha}+\mathbf{Q}|>p_F$, $|\boldsymbol{\alpha}|<p_F$ or $|\boldsymbol{\alpha}+\mathbf{Q}|<p_F$, $|\boldsymbol{\alpha}|>p_F$ in (21b). For small Q, then $|\boldsymbol{\alpha}|\sim p_F$, $|\mathbf{p}|\sim p_F$, $|\boldsymbol{\alpha}+\mathbf{Q}|\sim p_F$ and $|\mathbf{p}+\mathbf{Q}|\sim p_F$ follows, and we may take $\omega_\alpha-\omega_p\sim O(Q)$ and the terms of (26) in curly brackets which are not multiplied by n_α or $n_{\alpha+Q}$ can be taken as $O(Q)$; then

$$\Delta_{\alpha;\mathbf{Q}}{}^{(2)}=(n_\alpha-n_{\alpha+\mathbf{Q}})\sum\mathbf{p}_j\Big[(\bar{v}^j(Q)-v^j(Q)):\mathbf{c}_{\mathbf{p}+\mathbf{Q}}{}^*\rho_j\mathbf{c}_\mathbf{p}:\,\mathrm{Tr}(\sigma\rho_j)+v^j(\mathbf{p}-\boldsymbol{\alpha})\frac{1}{1+f_{\mathbf{p}-\boldsymbol{\alpha}}{}^j(0)}:\mathbf{c}_{\mathbf{p}+\mathbf{Q}}{}^*\rho_j\sigma\rho_j\mathbf{c}_\mathbf{p}:\Big]+O(Q),$$

where from (15)

$$f_q{}^j(0)=\frac{\bar{v}^j(q)}{(p_F{}^2/2m)^{\frac{3}{2}}}\times\lambda(q),$$
$$\lambda(q)=\tfrac{1}{2}-\left(\frac{x}{8}-\frac{1}{2x}\right)\ln\left(\frac{2+x}{2-x}\right),\quad (x=|q|/p_F).$$

[7] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955). The pair interaction is small for the ground-state energy.

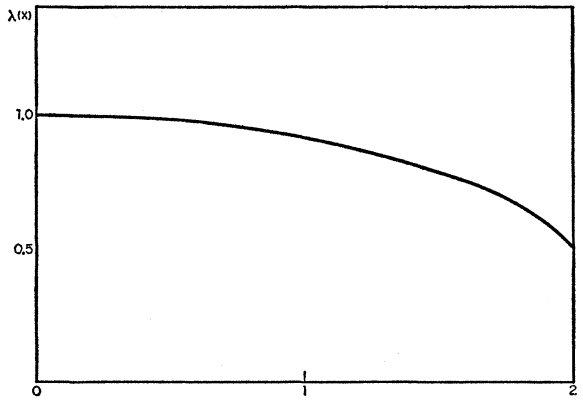

FIG. 4. A graph of the function
$\lambda(q)=\frac{1}{2}-(x/8-1/2x)\ln[(2+x)/(2-x)]$
against the momentum transfer $x=q/p_F$.

$|\mathbf{p}|\sim|\boldsymbol{\alpha}|\sim p_F$ but $|\mathbf{p}-\boldsymbol{\alpha}|$ goes from $0\rightarrow 2p_F$. Inside this range $\lambda(\mathbf{p}-\boldsymbol{\alpha})$ changes rather slowly from 1 to $\frac{1}{2}$ (Fig. 4). Further, we utilize here the short-range character of the pseudopotential and take v and $\bar{v}$ independent of momentum:

$$\bar{v}^i(Q)=\bar{v}^i,\quad v^i(Q)=v^i; \tag{27a}$$

we neglect the change of $\lambda(p-\alpha)$, and assume

$$f_{\mathrm{p}-\alpha}{}^i(0)\rightarrow f_{q\rightarrow 0}{}^i(0). \tag{27b}$$

The assumption of zero range for the pseudopotential and small Q leads to a simple equation for $\Delta_{\alpha;\mathrm{Q}}{}^{(2)}$. But for ranges of v (or $\bar{v}$) $\geq\frac{1}{3}(\hbar/\mu c)$ (μ: meson mass), Eq. (27b) becomes incorrect, because the angle dependence of $v(\mathbf{p}-\boldsymbol{\alpha})$ ($|\mathbf{p}|\sim|\boldsymbol{\alpha}|\sim p_F$) becomes large ($|\mathbf{p}-\boldsymbol{\alpha}|$ ranges from 0 to $2p_F$) since $2p_Fc\approx 540$ Mev and $\mu c^2\approx 140$ Mev.

Under these reservations, we get as part of the pair-pair interaction the term $\Delta_{\alpha;\mathrm{Q}}{}^{(2)}$ which should be equated to zero:

$$\Delta_{\alpha;\mathrm{Q}}{}^{(2)}=0=(n_\alpha-n_{\alpha+Q})\sum_{jp}\left[(\bar{v}^j-v^j)\,\mathrm{Tr}(\rho_j\sigma):\mathbf{c}_{\mathrm{p}+\mathrm{Q}}{}^*\rho_j\mathbf{c}_{\mathrm{p}}:+v^i\frac{1}{1+f_{q\rightarrow 0}{}^i(0)}:\mathbf{c}_{\mathrm{p}+\mathrm{Q}}{}^*\rho_j\sigma\rho_j\mathbf{c}_{\mathrm{p}}:\right]. \tag{28}$$

By the self-consistent equations (22) and (28), we have left only $\Delta_{\alpha;\mathrm{Q}}{}^{(3)}$ which contains A, A^*, already ordered. Hence the equation of motion for A only contains nonlinear interactions as was explained in Sec. II. Assuming two-body nuclear interactions in momentum space, we have

$$v^j(q)=v^1+v^\sigma(\sigma\cdot\sigma)+v^\tau(\tau\cdot\tau)+v^{\tau\sigma}(\tau\cdot\tau)(\sigma\cdot\sigma). \tag{29}$$

We get from (28); (taking $\sigma=1,\ \sigma_i,\ \tau_i,\ \sigma_i\tau_j$ respectively)

$$\begin{aligned}
4\bar{v}^1&=4v^1-\left(\frac{v_1}{1+f^\tau}+\frac{3v^\sigma}{1+f^\sigma}+\frac{3v^\tau}{1+f^\tau}+\frac{9v^{\tau\sigma}}{1+f^{\tau\sigma}}\right),\\
4\bar{v}^\sigma&=4v^\sigma-\left(\frac{v^1}{1+f^1}-\frac{v^\sigma}{1+f^\sigma}+\frac{3v^\tau}{1+f^\tau}-\frac{3v^{\tau\sigma}}{1+f^{\tau\sigma}}\right),\\
4\bar{v}^\tau&=4v^\tau-\left(\frac{v^1}{1+f^1}+\frac{3v^\sigma}{1+f^\sigma}-\frac{v^\sigma}{1+f^\tau}-\frac{3v^{\tau\sigma}}{1+f^{\tau\sigma}}\right),\\
4\bar{v}^{\tau\sigma}&=4v^{\tau\sigma}-\left(\frac{v^1}{1+f^\tau}-\frac{v^\sigma}{1+f^\sigma}-\frac{v^\tau}{1+f^\tau}+\frac{v^{\tau\sigma}}{1+f^{\tau\sigma}}\right),
\end{aligned} \tag{30}$$

where, from (27b) and (25),

$$\begin{aligned}
f^j&=f_{q\rightarrow 0}{}^j(0)=4\bar{v}_j\sum_{\mathrm{p}}\frac{2}{\omega_{p;\,q}}(1-n_{p+q})n_p\Big|_{q\rightarrow 0},\\
1+f^j&=1+\frac{3}{2}\frac{\bar{v}_jA}{\frac{1}{4}(p_F{}^2/2m)}.
\end{aligned} \tag{31}$$

Actually, we should use m^* (effective mass) instead of m in (31) because of the change of the one-particle energy (22), but for clarity we use m in the following.

As an actual potential for nuclear matter, we assume the form:

$$V(x)=[V_1+(\sigma\sigma)V_\sigma+(\tau\tau)V_\tau+(\tau\tau)(\sigma\sigma)V_{\tau\sigma}]g(x), \tag{32}$$

where $g(x)$ is a function of the space coordinate. The values V_1, etc., are, according to the shell model analysis by Thieberger,[8]

$$\begin{array}{ll} V_1= -40.3 \text{ Mev}, & V_1= -39.3 \text{ Mev}, \\ V_\sigma= 8.7 \text{ Mev}, & V_\sigma= 4.1 \text{ Mev}, \\ V_\tau=V_{\tau\sigma}=12.6 \text{ Mev}, & V_{\tau\sigma}=V_\tau= 9.4 \text{ Mev}. \\ \text{(with core included)} & \text{(without core)} \end{array} \tag{33a}$$

Transforming (33a) into momentum space $[v(q)=(1/\Omega)\int e^{iq\cdot x}V(x)dx^3]$, we have

$$\begin{aligned} v^jA=(v^j(0)A)&=V_j\left(\int\bar{v}(x)d^3x\Big/\frac{4\pi}{3}\gamma_0{}^3\right) \\ &=V^j\times\alpha, \end{aligned} \tag{33b}$$

where $\gamma_0A^{\frac{1}{3}}$ is the nuclear radius and (33b) defines α. The coefficient α can be determined by taking the Hartree-Fock ground-state energy of the system to be A times the mean binding energy per nucleon (-15 Mev).

$$A[\tfrac{3}{5}(p_F{}^2/2m)+\tfrac{3}{8}(v^1A-v^\sigma A-v^\tau A-3v^{\tau\sigma}A)]=-15 \text{ Mev } A, \tag{34}$$

and we get

$$\begin{array}{ll} \alpha=1.04, & \alpha=1.33, \\ \text{(with core)} & \text{(without core)} \end{array}$$

($p_F{}^2/2m\approx 40$ Mev). For the evaluation of $\bar{v}$ by (30), we take two sets of values which satisfy Eq. (34):

$$\begin{array}{ll} v^1A=-46 \text{ Mev}, & v^1A=-20.5 \text{ Mev}, \\ v^\sigma A=v^\tau A=v^{\tau\sigma}A=11.6 \text{ Mev}, & v^\sigma A=v^\tau A=v^{\tau\sigma}A=16.7 \text{ Mev}. \end{array} \tag{35}$$

and

One has a large attractive central force and the other has large σ, τ, and $\sigma\tau$ dependence. (33a) lies approximately in the range of (35).

The solution of (30) becomes:

$$\begin{array}{ll} \bar{v}^1A=-13.4 \text{ Mev} \quad (-78 \text{ Mev}), & \bar{v}^1A=-\ 8.23 \text{ Mev} \quad (-78 \text{ Mev}), \\ \left.\begin{matrix}\bar{v}^\sigma A\\ \bar{v}^\tau A\\ \bar{v}^{\tau\sigma}A\end{matrix}\right\}=+60 \quad \text{Mev} \quad (+26 \text{ Mev}), & \left.\begin{matrix}\bar{v}^\sigma A\\ \bar{v}^\tau A\\ \bar{v}^{\tau\sigma}A\end{matrix}\right\}=+50 \quad \text{Mev} \quad (+26 \text{ Mev}). \end{array} \tag{36}$$

and

[The quantities in the parentheses are obtained by taking all f's$=0$ in (30), that is the lowest order pair interaction corresponding to the one used by G-H-W.[2]] The remarkable point in (36) is that, since $v^1<0$, the term $-v^1/(1+f^1)$ in (30) is large and this has the effect of reducing $\bar{v}^1A$ and increasing $\bar{v}^\tau A$, $\bar{v}^\sigma A$, $\bar{v}^{\sigma\tau}A$ compared with the results obtained with $f=0$, though $\bar{v}^\tau A$, $\bar{v}^\sigma A$, $\bar{v}^{\tau\sigma}A$ are only about two times "enhanced" compared with $f=0$ value for the two different values (35) for v.

G-H-W[2] have shown that because of the negative sign for v^1A (and $\bar{v}^1A$ with $f=0$), there could exist solutions $(\omega_Q{}^{n;1})^2<0$ for the density fluctuation which leads to instability of the system, and in fact, if we take $\bar{v}^1A=-78$ Mev [obtained by taking $f=0$ in (30)] we get such a solution. For $\rho_j=1$, Eq. (16) gives an eigenvalue equation:

$$1+4\bar{v}^1A\sum\frac{2\omega_{p;Q}}{\omega_{p;Q}{}^2-(\omega^1)^2}(1-n_{p+Q})n_p=0=1+f_q{}^1(\omega^1), \tag{37}$$

[8] We are thankful to Dr. R. Thieberger for giving us his potential in a simple form for convenient use.

and

$$1+f_q{}^{\mathrm{I}}(i\Delta)>1+f_q{}^{1}(0)>1+f_{q\to 0}{}^{1}(0)=1+\frac{3}{2}\frac{\bar{v}^1 A}{(p_F{}^2/2m)}$$
$$=1-Q_1,$$

where $Q_1=-\frac{3}{2}\bar{v}^1 A/(p_F{}^2/2m)$ using the notation of G-H-W. With $\bar{v}^1 A=-78$ Mev, we get $1-Q_1<0$; hence there can exist $\omega^1=\pm i\Delta$ which satisfies the eigenvalue equation. But for $\bar{v}^1 A=-13.4$ or -8.23 Mev as we obtained in (36), $1-Q_1>0$ and no such solution exists; that is, the self-consistent solution satisfies the stability condition.[9] Writing the eigenvalue equation (16) in terms of G-H-W[2] notation:

$$\frac{1}{Q_j}=\left[-\frac{3}{2}\frac{\bar{v}^j A}{p_F{}^2/2m}\right]^{-1}=1-\frac{\bar{A}_j}{2}\ln\left|\frac{\bar{A}_j\pm 1}{\bar{A}_j-1}\right|,$$
$$\bar{A}_j=\omega_q{}^{B;\,i}\Big/\left(\frac{p_F}{m}\right), \tag{38}$$

we have

$$\frac{1}{Q_j}(j=\sigma,\ \tau,\ \sigma\tau)=-0.44 \quad \text{and} \quad -0.54, \tag{39}$$

($\bar{v}^\tau A$ obtained by taking $f=0$ in (30): $1/Q_\tau=-1.03$); hence, from the diagram given by G-H-W[2] (Fig. 3 of reference 2) we get about 15% increase for plasma frequency compared to $p_F q/m$ (i.e., $\bar{A}_j\fallingdotseq 1.15$).

It will be interesting to see how the ground-state energy and symmetry energies are changed as we include the pair-pair interaction in a self-consistent way. For this purpose, we evaluate the expectation value of the total Hamiltonian by starting from the distorted Fermi sea, indicating by $n_{P\uparrow}$, $n_{P\downarrow}$, $n_{N\uparrow}$, $n_{N\downarrow}$ the occupation numbers of states with proton, spin up, etc. After ordering the operators $a_{p;P\uparrow}{}^*$, $b_{p;P\uparrow}{}^*$, $a_{p;P\uparrow}$, $b_{p;P\uparrow}$, etc., we get as the expectation value of total Hamiltonian

$$\begin{aligned}(\Psi_0',H_T\Psi_0')=A\Big\{&\frac{3}{5}\frac{p_F{}^2}{2m}+\frac{p_F{}^2}{6m}\Big[\left(\frac{n_N-n_P}{A}\right)^2+\left(\frac{n_\uparrow-n_\downarrow}{A}\right)^2+\left(\frac{n_{N\uparrow}+n_{P\downarrow}-n_{N\downarrow}-n_{P\uparrow}}{A}\right)^2\Big]\\
&+\tfrac{1}{2}v^1 A+\tfrac{1}{2}v^\tau A\left(\frac{n_N-n_P}{A}\right)^2+\tfrac{1}{2}v^\sigma A\left(\frac{n_\uparrow-n_\downarrow}{A}\right)^2+\tfrac{1}{2}v^{\tau\sigma}A\left(\frac{n_{N\uparrow}+n_{P\downarrow}-n_{N\downarrow}-n_{P\uparrow}}{A}\right)^2\\
&-\frac{1}{8}\Big[A(v^1+3v^\sigma+3v^\tau+9v^{\tau\sigma})+A(v^1+3v^\sigma-v^\tau-3v^{\tau\sigma})\left(\frac{n_N-n_P}{A}\right)^2\\
&+A(v^1-v^\sigma+3v^\tau-3v^{\tau\sigma})\left(\frac{n_\uparrow-n_\downarrow}{A}\right)^2+A(v^1-v^\sigma-v^\tau+v^{\tau\sigma})\\
&\qquad\times\left(\frac{n_{N\uparrow}+n_{P\downarrow}-n_{N\downarrow}-n_{P\uparrow}}{A}\right)^2\Big]\Big\}+(\Psi_0':H_{\mathrm{int}}:\Psi_0'),\end{aligned} \tag{40}$$

where Ψ_0' is the wave function for the ground state of the system. We evaluate $(\Psi_0':H_{\mathrm{int}}:\Psi_0')$ by the same method we used to evaluate $\Delta_{\alpha;\mathrm{Q}}(5)$; namely, $:H_{\mathrm{int}}:$ is written explicitly as

$$:H_{\mathrm{int}}:=\sum_{ipq}v^i(q)\,(\mathbf{a}_{\mathrm{p+q}}{}^*\rho_i\mathbf{b}_{\mathrm{p}}{}^*n_i(q)+\mathbf{a}_{\mathrm{p+q}}{}^*\rho_i n_i(q)\,\mathbf{a}_{\mathrm{p}}+n_i(q)\,\mathbf{b}_{\mathrm{p+q}}\rho_i\mathbf{a}_{\mathrm{p}}-(\mathbf{b}_{\mathrm{p}})^{T*}(\rho_i)^T n_i(q)(\mathbf{b}_{\mathrm{p+q}})^T,$$

and we use the expansion (20) for $n_i(q)$. Shifting the operators A, A^* to the right and left, respectively, we have[10]

$$:H_{\mathrm{int}}:=-\sum_{pqj}v^j(q)\ \mathrm{Tr}[n_{p+q}\rho_j(1-n_p)\rho_j]\sum_{(\omega_n>0)}{}_n\ \frac{1}{2}\left(\frac{1}{\omega_q{}^{n;i}\pm i\epsilon-\omega_{p;q}}+\frac{1}{\omega_q{}^{n;i}\mp i\epsilon-\omega_{p;q}}\right)\frac{|N_q{}^{n;i}|^2}{n\bar{v}^j(q)}$$
$$+:H_{\mathrm{int}}:\ (\text{re-ordered about } A,\ A^*).$$

[9] K. Sawada and R. M. Rockmore, Phys. Rev. **116**, 1168 (1959).

[10] The method used here is slightly different from the method in K. Sawada, Phys. Rev. **106**, 372 (1957), where we took $:H_{\mathrm{int}}:$ of the form $:n(q)::n(q):$—diagonal form, expanded both $n(q)$'s into "pair" eigenmodes, and performed an "ordering." This is done in (A-13) and (A-14) of reference 6 [which is $n(q)\Psi_0$], and (14) and (15) of K. Sawada, Phys. Rev. **106**, 372 (1957), where we applied "pair" operators to the state vectors; but this is clearly equivalent to the ordering process. The technique of getting the total energy from the expectation values of the interaction energy becomes much too complicated if we determine ω and $\bar{v}$ self-consistently. So we simply took the expectation value of the total Hamiltonian.

The expectation value of the re-ordered term is approximately 0, since Ψ_0' is approximately Ψ_0, where Ψ_0 satisfies the relation $A\Psi_0=0$. In the above expression n_p is the matrix:

$$n_p=\tfrac{1}{4}(n_{p;P\uparrow}+n_{p;P\downarrow}+n_{p;N\uparrow}+n_{p;N\downarrow})+\tfrac{1}{4}(n_{p;P\uparrow}+n_{p;P\downarrow}-n_{p;N\uparrow}-n_{p;N\downarrow})\tau_3$$
$$+\tfrac{1}{4}(n_{p;P\uparrow}+n_{p;N\uparrow}-n_{p;P\downarrow}-n_{p;N\downarrow})\sigma_3+\tfrac{1}{4}(n_{p;P\uparrow}+n_{p;N\downarrow}-n_{p;P\downarrow}-n_{p;N\uparrow})\sigma_3\tau_3,$$

with $n_{p,ij}$ the occupation number (1 or 0) of states with momentum p, charge i, and spin j. By using (25), and evaluating the trace, we get:

$$(\Psi_0':H_{\rm int}:\Psi_0')=-4\sum_{\substack{\mathbf{p},\mathbf{q}j\\ |\mathbf{p}|>p_F\\ |\mathbf{p}+\mathbf{q}|<p_F}} v^j(q)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_q{}^j(iv)}{1+f_q{}^j(iv)}\frac{\omega_p-\omega_{p+q}}{v^2+(\omega_p-\omega_{p+q})^2}dv$$
$$+\tfrac{1}{8}\sum_{qj}\Bigg\{\left(\frac{f_q{}^1(|\omega_{p;q}|)v^1(q)}{1+f_q{}^1(|\omega_{p;q}|)}+\frac{3f_q{}^\sigma v^\sigma}{1+f_q{}^\sigma}-\frac{f_q{}^\tau v^\tau}{1+f_q{}^\tau}-\frac{3f_q{}^{\tau\sigma}v^{\tau\sigma}}{1+f_q{}^{\tau\sigma}}\right)$$
$$\times(n_{p;P\uparrow}+n_{p;P\downarrow}-n_{p;N\uparrow}-n_{p;N\downarrow})(n_{p+q;P\uparrow}+n_{p+q,P\downarrow}-n_{p+q;N\uparrow}-n_{p+q,N\downarrow})$$
$$+\left(\frac{f_q{}^1v^1(q)}{1+f_q{}^1}-\frac{f_q{}^\sigma v^\sigma(q)}{1+f_q{}^\sigma}+\frac{3f_q{}^\tau v^\tau(q)}{1+f_q{}^\tau}-\frac{3f_q{}^{\tau\sigma}v^{\tau\sigma}(q)}{1+f_q{}^{\tau\sigma}}\right)$$
$$\times(n_{p;P\uparrow}+n_{p;N\uparrow}-n_{p;P\downarrow}-n_{p;N\downarrow})(n_{p+q;P\downarrow}+n_{p+q;N\uparrow}-n_{p+q;P\downarrow}-n_{p+q;N\downarrow})$$
$$+\left(\frac{f_q{}^1v^1(q)}{1+f^1(q)}-\frac{f_q{}^\sigma v^\sigma(q)}{1+f_q{}^\sigma}-\frac{f_q{}^\tau v^\tau(q)}{1+f_q{}^\tau}+\frac{f_q{}^{\tau\sigma}v^{\tau\sigma}(q)}{1+f_q{}^{\tau\sigma}}\right)$$
$$\times(n_{p;P\uparrow}+n_{p;N\downarrow}-n_{p;P\downarrow}-n_{p;N\uparrow})(n_{p+q;P\uparrow}+n_{p+qN\downarrow}-n_{p+q;P\downarrow}-n_{p+q;N\uparrow})\Bigg\},\quad(41)$$

where $f_q{}^i$ is a shorthand notation for $f_q{}^j(|\omega_{p;q}|)$. The last three terms can also be reduced to a much simpler form. Let us consider the first of these terms, and suppose our distorted Fermi surface is obtained from the symmetric Fermi surface by changing the relative number densities of neutrons and protons (preserving total number A) by the amount

$$\left(\frac{L}{2\pi}\right)^2\int d\Omega_{\mathbf{p}F}(n_{pF;P\uparrow}+n_{pF;P\downarrow}-n_{pF;N\uparrow}-n_{pF;N\downarrow})=n_P-n_N,\quad(\delta n_P=-\delta n_N)$$

uniformly over all directions of momentum p_F (L is the length of quantization box; $L^3=\Omega$). Then the energy of this system is, from (41) (writing $\mathbf{p}+\mathbf{q}=\mathbf{p}'$),

$$\Delta E(n_P-n_N)=\frac{1}{8}\left(\frac{L}{2\pi}\right)^4\int d\Omega_{p'}\int d\Omega_p(n_{p,P\uparrow}+n_{p;P\downarrow}-n_{p;N\uparrow}-n_{p;N\downarrow})$$
$$\times(n_{p';P\uparrow}+n_{p';P\downarrow}-n_{p';N\uparrow}-n_{p';N\downarrow})\Bigg(\frac{f_{|\mathbf{p}-\mathbf{p}'|}{}^1(0)v^1(\mathbf{p}-\mathbf{p}')}{1+f_{|\mathbf{p}-\mathbf{p}'|}{}^1(0)}$$
$$+\frac{3v^\sigma(\mathbf{p}-\mathbf{p}')f^\sigma|\mathbf{p}-\mathbf{p}'|(0)}{1+f_{|\mathbf{p}-\mathbf{p}'|}{}^\sigma(0)}-\frac{v^\tau(p-p')f^\tau(\mathbf{p}-\mathbf{p}'|(0)}{1+f_{|\mathbf{p}-\mathbf{p}'|}{}^\tau(0)}-\frac{3v^{\tau\sigma}(\mathbf{p}-\mathbf{p}')f_{|p-p'|}{}^{\tau\sigma}(0)}{1+f_{|\mathbf{p}-\mathbf{p}'|}{}^{\tau\sigma}(0)}\Bigg)\Bigg|_{|\mathbf{p}|=|\mathbf{p}'|=p_F}.$$

In accordance with the short-range approximation in (27a) and (27b), we have

$$\Delta E(n_P-n_N)=\frac{A}{8}\left(\frac{f^1v^1A}{1+f^1}+\frac{3f^\sigma v^\sigma A}{1+f^\sigma}-\frac{f^\tau v^\tau A}{1+f^\tau}-\frac{3f^{\tau\sigma}v^{\tau\sigma}}{1+f^{\tau\sigma}}\right)\left(\frac{n_P-n_N}{A}\right)^2,$$

which represents a change in the symmetry energy. Similarly we can obtain $\Delta E(n_\uparrow-n_\downarrow)$ and $\Delta E(n_{p\uparrow}+n_{N\downarrow}$

$-n_{P\downarrow}-n_{N\uparrow}$). Thus we get, by using the relation (30) between $\bar{v}$ and v:[11]

$$(\Psi_0'H_T\Psi_0')=A\Big[\frac{3}{5}\frac{p_F^2}{2m}+\Big(\frac{p_F^2}{6m}+\frac{\bar{v}^\tau A}{2}\Big)\Big(\frac{n_P-n_N}{A}\Big)^2+\Big(\frac{p_F^2}{6m}+\frac{\bar{v}^\sigma A}{2}\Big)\Big(\frac{n_\uparrow-n_\downarrow}{A}\Big)^2$$
$$+\Big(\frac{p_F^2}{6m}+\frac{\bar{v}^{\tau\sigma}A}{2}\Big)\Big(\frac{n_{P\uparrow}+n_{N\downarrow}-n_{P\downarrow}-n_{N\uparrow}}{A}\Big)^2+\tfrac{3}{8}(v^\tau A-v^\sigma A-v^\tau A-3v^{\tau\sigma}A)\Big]$$
$$-4\sum_{\substack{\mathbf{p},\mathbf{q},j\\ |\mathbf{p}+\mathbf{q}|>p_F\\ |\mathbf{p}|<p_F}} v^j(q)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_q{}^j(iv)}{1+f_q{}^j(iv)}\frac{\omega_{p;q}}{v^2+\omega_{p;q}{}^2}dv. \quad (42)$$

The last term can be transformed by using the definition of $f_q{}^j(iv)$ from (15):

$$\tfrac{1}{2}\sum_j\sum_q\frac{v^j(q)}{\bar{v}^j(q)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{f_q{}^j(iv)}{1+f_q{}^j(iv)}-f_q{}^j(iv)\Big)dv,$$

which corresponds exactly to the expression for the expectation value of the interaction energy in the ground state for the electron gas [reference 6, the expression above (34) of this reference]. For nuclear matter, it was already shown that this contribution can be neglected.[7]

The symmetry energy for our model (35) is, in the case (i) where we take $f=0$ in (30): $\bar{v}^1A=-78$ Mev, $\bar{v}^\tau A=\bar{v}^\sigma A=\bar{v}^{\tau\sigma}A=+26$ Mev,

$$p_F{}^2/6m+13\text{ Mev}\simeq 26\text{ Mev}\ (\simeq 2p_F{}^2/6m);$$

in the case (ii) where we take the self-consistent solution (36)[12]:

$$43\text{ Mev}\ (\simeq 3.2p_F{}^2/6m),\quad\text{and}\quad 38\text{ Mev}\ (\simeq 2.8p_F{}^2/6m).$$

The experimental value is $\sim 2p_F{}^2/6m$. These figures can be improved very much if we use the technique to get the ground-state energy from the expectation value of the interaction energy. Let us suppose that the potential is multiplied with the parameter λ; then we can write (42) as follows:

$$E_0(\lambda)=(\Psi_0'(\lambda)H_T{}^{(\lambda)}\Psi_0'(\lambda))=(\Psi_0'(\lambda)H_0\Psi_0'(\lambda))+(\Psi_0'(\lambda)H_{\text{int}}{}^{(\lambda)}\Psi_0'(\lambda)). \quad (43)$$

Then we can use[6]

$$E_0(1)=\int_0^1 d\lambda\Big(\Psi_0'(\lambda)\frac{H_{\text{int}}{}^{(\lambda)}}{\lambda}\Psi_0'(\lambda)\Big)+E_0(0). \quad (44)$$

If we put the expression corresponding to $v\to\lambda v$ from (42), we have

$$(\Psi_0'(\lambda)H_{\text{int}}{}^{(\lambda)}\Psi_0'(\lambda))=A\Big[\frac{\bar{v}^\tau(\lambda)A}{2}\Big(\frac{n_P-n_N}{A}\Big)^2+\frac{\bar{v}^\sigma(\lambda)A}{2}\Big(\frac{n_\uparrow-n_\downarrow}{A}\Big)^2$$
$$+\frac{\bar{v}^{\tau\sigma}(\lambda)A}{2}\Big(\frac{n_{P\uparrow}+n_{N\downarrow}-n_{P\downarrow}-n_{N\uparrow}}{A}\Big)^2+\tfrac{3}{8}\lambda(v^1A-v^\sigma A-v^\tau A-3v^{\tau\sigma}A)\Big]$$
$$-4\sum_{\substack{\mathbf{p},q,j\\ |\mathbf{p}+\mathbf{q}|>p_F\\ |\mathbf{p}|<p_F}}\lambda v^j(q)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_q{}^j(\lambda;iv)}{1+f_q{}^j(\lambda;iv)}\frac{\omega_{p;q}}{v^2+\omega_{p;q}{}^2}dv, \quad (45)$$

where $\bar{v}^\tau(\lambda)$, etc., are given by Eq. (30) with $v^i\to\lambda v^i$, and $f_q{}^j(\lambda;iv)$ is given by the expression (15) with $\bar{v}^i(Q)\to\bar{v}^i(\lambda;Q)$.

By using our model potential (35) as $\lambda v^iA\,|_{\lambda=1}$, we have as the solution of self-consistent Eq. (30) the value given approximately in Fig. 5. Integration over λ as indicated in the equation (44) yields

$$\int_0^1 d\lambda\frac{\bar{v}^\tau(\lambda)A}{\lambda}=\alpha\bar{v}^\tau A\text{; same for }\bar{v}^\sigma(\lambda),\ \bar{v}^{\sigma\tau}(\lambda), \quad (46)$$

[11] Cf. K. A. Brueckner and R. Thieberger, Phys. Rev. Letters 4, 466 (1960); W. Brenig, Nuclear Phys. (to be published).

[12] Only by summing up the ordinary scattering matrices self-consistency (Brueckner method), Brueckner and Gammel (reference 1) obtain the value 2.69 $p_F{}^2/6m$. This is also bigger than the usually accepted phenomenological value $\sim 2p_F{}^2/6m$.

with $\alpha=0.70$ and 0.74, respectively. The energy (44) becomes

$$E_0(1)\equiv(\Psi_0',H_T\Psi_0')$$
$$=A\left[\frac{3}{5}\frac{p_F^2}{2m}+\left(\frac{p_F^2}{6m}+\frac{\alpha\bar{v}^\tau A}{2}\right)\left(\frac{n_P-n_N}{A}\right)^2+\left(\frac{p_F^2}{6m}+\frac{\alpha\bar{v}^\sigma A}{2}\right)\left(\frac{n_\uparrow-n_\downarrow}{A}\right)^2\right.$$
$$\left.+\left(\frac{p_F^2}{6m}+\frac{\alpha\bar{v}^{\tau\sigma}A}{2}\right)\left(\frac{n_{P\uparrow}+n_{N\downarrow}-n_{P\downarrow}-n_{N\uparrow}}{A}\right)^2+\tfrac{3}{8}(v^1A-v^\sigma A-v^\tau A-3v^{\tau\sigma}A)\right]$$
$$-4\int_0^1\sum_{\substack{pqj\\|\mathbf{p}+\mathbf{q}|>p_F\\p<p_F}}v^j(q)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_q{}^j(\lambda;iv)}{1+f_q{}^j(\lambda;iv)}\frac{\omega_{p;q}}{v^2+\omega_{p;q}{}^2}dvd\lambda. \quad (47)$$

Equation (47) gives as the symmetry energy

(iii) $p_F{}^2/6m+\frac{1}{2}\alpha\bar{v}^\tau A=34$ Mev ($\simeq 2.6p_F{}^2/6m$)
and 32 Mev ($\simeq 2.4p_F{}^2/6m$),

which is much better than (ii) and comparable to (i).

Summarizing, we can say that the higher order pair-pair interaction affects the ground-state energy rather slightly.

We can see the relation between $\bar{v}^iA$ and the symmetry energy, as suggested by Brenig[11] in (42) and its modification in (47).

V. CONCLUSION

By requiring self-consistency in constructing the approximate pair eigenmode, we have derived equations for the one-particle energy and pair-pair interaction strengths. We perform this program explicitly by using a simple "Ansatz" for the momentum dependence of the self-consistent interaction strengths, aiming only to see the effects of the pair (particle-hole) interaction. We have omitted the requirement of self-consistency for particle-particle and hole-hole scattering processes; supposing this to have been accomplished at a separate stage, and regarding our original interactions as a pseudopotential (or K matrix). But our model shows the importance of pair-pair interaction when the system has attractive forces. In particular, for Goldhaber-Teller modes, the resonant (plasma) energy lies at about $1.15(p_FQ/m)$ (Q: momentum of oscillation). It seems that this value will not depend critically on the magnitude of the original interaction though it does depend on the range of the force as we have mentioned concerning approximations (27a) and (27b). There exist no density plasma waves; the density wave has an energy corresponding to the individual particle-hole excitation.

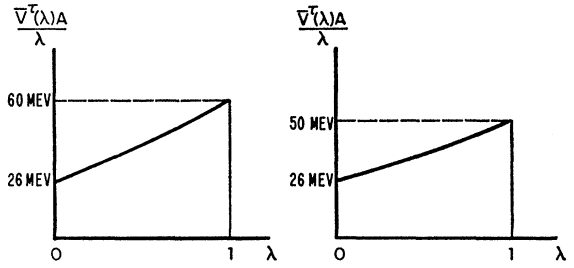

FIG. 5. Graphs for the solutions of the self-consistent equations (30) for the system with potential λv^i, where the v^i's are given in Eq. (35).

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Professor Keith A. Brueckner for suggesting this problem, and for his continuing help. The authors are also very grateful to Dr. Reuben Thieberger for giving them his results for shell-model potentials, and for many helpful discussions.