Energy Losses in a Many-Body System*†

STANLEY ENGELSBERG
Department of Physics, University of Illinois, Urbana, Illinois
(Received March 27, 1961)

The energy loss problem is formulated in such a way as to include all losses simultaneously. The lifetime and energy losses of a particle in a well-defined single-particle state with small transition probability are found to be related to the self-energy operator. As an illustration of the application of the relation obtained, a derivation of the Bethe sum rule and the Čerenkov losses is given for a particle incident on a many-body system.

INTRODUCTION

THE methods employed in calculating energy losses of a particle passing through matter usually vary according to the type of loss one is trying to describe. It is desirable to develop a general formulation of the energy loss problem, such that all manifestations of the interaction between the incident particle and the particles of the medium are included. In this paper we describe such a formulation and demonstrate the applicability of the technique involved.

The method to be proposed is one which employs the self-interactions of the incident particle as contained in the mass—or self-energy—operator for the single-particle Green's function introduced in the treatment[1] of relativistic field theory. It will be demonstrated that whenever a single particle state in the system is still meaningful, the self-energy operator contains the requisite information to predict both the lifetime for the incident particle leaving its initial state and, when a particular noncorrelation approximation is valid, the energy losses.

The applicability of the expression derived for the energy losses is illustrated by using it to obtain a proof of the Bethe sum rule. In the proof we reduce the relation for the energy losses to that obtained in the Born approximation. In this same approximation once we choose the system through which the particle passes to be described by a dielectric function, the formula for the Čerenkov energy losses appears directly.

SELF-ENERGY OPERATOR

We will first derive an expression for the self-energy of a particle or quasi-particle excitation when its interaction with the rest of the system is known.

The nonrelativistic field equation for a fermion interacting with a many-particle system is[2]

$$i\frac{\partial}{\partial t}\psi_\alpha(x)=-\frac{1}{2m}\nabla^2\psi_\alpha(x)+\sum_\beta (H_1(x))_{\alpha\beta}\psi_\beta(x). \quad (1)$$

The interactions are thereby chosen to be local. The Hermitian operator $(H_1(x))_{\alpha\beta}$, represents the interaction of the incident particle with the system. The interaction terms in the Hamiltonian, from which it derives, commute with the number operator of the fermion field, but may otherwise be completely general. The spin indices (α,β) will be suppressed in the discussion which follows.

The single-particle Green's function for the fermion field is defined by

$$G(x,x')=-i\frac{\langle\Phi_a,+\infty\,|\,(\psi(x)\psi^\dagger(x'))_+|\,\Phi_a,-\infty\rangle}{\langle\Phi_a,+\infty\,|\,\Phi_a,-\infty\rangle}\epsilon(t-t'), \quad (2)$$

where the time-ordered product

$$\begin{aligned}(A(x)B(x'))_+&=A(x)B(x') \quad \text{for} \quad t>t'\\ &=B(x')A(x) \quad \text{for} \quad t'>t,\end{aligned}$$

and the symbol

$$\begin{aligned}\epsilon(t)&=+1 \quad \text{for} \quad t>0,\\ &=-1 \quad \text{for} \quad t<0.\end{aligned}$$

The Heisenberg state vector $|\Phi_a,-\infty\rangle$ gives a description of the ground state of the many-particle system in terms of a complete set of observables at a given time. We adopt the notation

$$\langle F(x)\rangle=\frac{\langle\Phi_a,+\infty\,|\,F(x)\,|\,\Phi_a,-\infty\rangle}{\langle\Phi_a,+\infty\,|\,\Phi_a,-\infty\rangle}, \quad (3)$$

and Eq. (1) leads to the following equation for the Green's function:

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)G(x,x')+i\langle(H_1(x)\psi(x)\psi^\dagger(x'))_+\rangle\epsilon(t-t')=\delta(x-x'), \quad (4)$$

where we have made use of the equal-time anticommutation relations,

$$\{\psi(\mathbf{r},t),\psi^\dagger(\mathbf{r}',t)\}=\delta(\mathbf{r}-\mathbf{r}'). \quad (5)$$

The outgoing wave boundary condition is imposed on the Green's function, as a consequence of the expectation value considered.

We define the self-energy operator $\Sigma(x,x')$ by the

---

* Supported in part by the National Science Foundation.

† Based in part on a thesis submitted to Harvard University in partial fulfillment of the Ph.D. requirements, September, 1960.

[1] J. Schwinger, Proc. Nat. Acad. Sci. U. S. **37**, 452, 455 (1951).

[2] We have taken $\hbar=1$, and use the notation $x=(\mathbf{r},t)$, $p\cdot x=\mathbf{p}\cdot\mathbf{r}-p^0t$.

equation,

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2m}\nabla^2-\langle H_1(x)\rangle\right)G(x,x')+\int dx''\,\Sigma(x,x'')G(x'',x')=\delta(x-x'). \quad (6)$$

Equations (4) and (6) maintain that

$$\int dx''\,\Sigma(x,x'')G(x'',x')=i\langle(H_1(x)\psi(x)\psi^\dagger(x'))_+\rangle\epsilon(t-t')+\langle H_1(x)\rangle G(x,x'). \quad (7)$$

An expression for $\Sigma(x,x')$ may be obtained by using the technique of variational differentiation in conjunction with the action principle.[1] To do this, we add the term $J(x)H_1(x)$ to the original Lagrangian density which gave the equation of motion (1). The external current density $J(x)$ will be set equal to zero after it has been used in generating an expression for the self-energy operator. With this additional term in the Lagrangian, we may use the action principle to obtain

$$-i\frac{\delta}{\delta J(z)}\langle(\psi(x)\psi^\dagger(x'))_+\rangle\epsilon(t-t')=\langle(H_1(z)\psi(x)\psi^\dagger(x'))_+\rangle\epsilon(t-t')-i\langle H_1(z)\rangle G(x,x'). \quad (8)$$

The combination of Eqs. (7) and (8) results in

$$\int d^4x''\,\Sigma(x,x'')G(x'',x')=-i\int d^4x_1d^4x_2d^4x_3\frac{\delta\langle H_1(x_1)\rangle}{\delta J(x)}G(x,x_2)\times\frac{\delta G^{-1}(x_2,x_3)}{\delta\langle H_1(x_1)\rangle}G(x_3,x'). \quad (9)$$

The self-energy operator may then be written in standard form:

$$\Sigma(x,x')=i\int d^4x_1d^4x_2\,D(x,x_1)G(x,x_2)\Gamma(x_2,x';x_1), \quad (10)$$

where the Green's function D is

$$D(x,x')=\delta\langle H_1(x')\rangle/\delta J(x)=i\{\langle(H_1(x)H_1(x'))_+\rangle-\langle H_1(x)\rangle\langle H_1(x')\rangle\}, \quad (11)$$

and the vertex function is

$$\Gamma(x,x';y)=-\delta G^{-1}(x,x')/\delta\langle H_1(y)\rangle. \quad (12)$$

For a translationally invariant system, with the Fourier transforms defined by

$$\Sigma(x,x')=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}\Sigma(p), \quad (13)$$

and

$$\Gamma(x,x';y)=\int\frac{d^4pd^4q}{(2\pi)^8}e^{ip\cdot(x-x')}e^{iq\cdot(y-x)}\Gamma(p,q), \quad (14)$$

the Fourier transform of Eq. (10) is written as

$$\Sigma(p)=i\int\frac{d^4q}{(2\pi)^4}D(q)G(p-q)\Gamma(p,q). \quad (15)$$

TOTAL TRANSITION RATE AND ENERGY LOSSES

For a particle initially in a wave-packet state $\varphi_{\mathbf{p}'}(\mathbf{r},t')$ and incident on a many-particle system in its ground state $|\Phi_a,-\infty\rangle$, and finally in the wave-packet state $\varphi_{\mathbf{p}}^*(\mathbf{r},t)$, with the system excited to the state $|\Phi_b,+\infty\rangle$, the transition probability is given by

$$(\text{T.P.})_{ab}=\lim_{\substack{t\to+\infty\\ t'\to-\infty}}\frac{|\langle\Phi_b,+\infty|\psi(\mathbf{p},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle|^2}{\sum_{b,\mathbf{p}}|\langle\Phi_b,+\infty|\psi(\mathbf{p},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle|^2}, \quad (16)$$

where

$$\psi(\mathbf{p},t)=\int d^3r\;\varphi_{\mathbf{p}}^*(\mathbf{r},t)\psi(\mathbf{r},t)$$
$$\psi^\dagger(\mathbf{p}',t')=\int d^3r'\;\varphi_{\mathbf{p}'}(\mathbf{r}',t')\psi^\dagger(\mathbf{r}',t'). \quad (17)$$

We will denote the denominator of Eq. (16) by

$$A=\sum_{b,\mathbf{p}}|\langle\Phi_b,+\infty|\psi(\mathbf{p},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle|^2, \quad (18)$$

which normalizes the transition probability, so that $\sum_{b,\mathbf{p}}(\text{T.P.})_{ab}=1$. The transition rate from state Φ_a to state Φ_b is

$$W_{ab}=\frac{\partial}{\partial t}(\text{T.P.})_{ab}, \quad (19)$$

since the transition rate is a function of $t-t'$ only. Now,

$$\sum_{b,\mathbf{p}}W_{ab}=\frac{\partial}{\partial t}\sum_{b,\mathbf{p}}(\text{T.P.})_{ab}=0. \quad (20)$$

Therefore, the total transition rate out of the initial state is given by

$$\frac{1}{\tau}=\sum_{b\neq a,\mathbf{p}\neq\mathbf{p}'}W_{ab}=-W_{aa}. \quad (21)$$

Equations (1), (16), (19), and (21) lead to

$$\frac{1}{\tau}=-i\int d^3rd^3r_1\, A^{-1}\varphi_{\mathbf{p}'}(\mathbf{r},t)\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{r},t)|\,\Phi_a,+\infty\rangle$$

$$\times\langle\Phi_a,+\infty\,|[(1/2m)\nabla_1^2\psi(\mathbf{r}_1,t)-H_1(\mathbf{r}_1,t)\psi(\mathbf{r}_1,t)]\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle+\text{complex conjugate}$$

$$=2\,\mathrm{Im}\int d^3r\, A^{-1}\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p}',t)|\,\Phi_a,+\infty\rangle\varphi_{\mathbf{p}'}{}^*(\mathbf{r},t)$$

$$\times\langle\Phi_a,+\infty\,|[(1/2m)\nabla^2\psi(\mathbf{r},t)-H_1(\mathbf{r},t)\psi(\mathbf{r},t)]\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle. \quad (22)$$

The kinetic energy term in the expectation value does not contribute to the lifetime since the expression which results is real. Since we are interested in the limit $t\to+\infty$, $t'\to-\infty$, we may use the notation introduced before to write

$$\langle\Phi_a,+\infty\,|H_1(\mathbf{r},t)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle$$

$$=\int d^3r'\langle(H_1(\mathbf{r},t)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t'))_+\rangle$$

$$\times\epsilon(t-t')\varphi_{\mathbf{p}'}(\mathbf{r}',t')\langle\Phi_a,+\infty\,|\,\Phi_a,-\infty\rangle$$

$$=-\int d^3r'd^4x''\,\Sigma(x,x'')$$

$$\times\langle\Phi_a,+\infty\,|\psi(x'')\psi^\dagger(x')|\,\Phi_a,-\infty\rangle\varphi_{\mathbf{p}'}(\mathbf{r}',t')$$

$$+\int d^3r'\langle H_1(x)\rangle\langle\Phi_a,+\infty\,|\psi(x)\psi^\dagger(x')|\,\Phi_a,-\infty\rangle$$

$$\times\varphi_{\mathbf{p}'}(\mathbf{r}',t'). \quad (23)$$

The lifetime then becomes

$$\frac{1}{\tau}=2\,\mathrm{Im}\int d^3rd^4x''A^{-1}\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p}',t)|\,\Phi_a,+\infty\rangle$$

$$\times\langle\Phi_a,+\infty\,|\psi(x'')\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle$$

$$\times\varphi_{\mathbf{p}'}{}^*(\mathbf{r},t)\,\Sigma(x,x''). \quad (24)$$

We use the four-momentum operator to refer the coordinate of ψ to the point x

$$\psi(x'')=e^{iP\cdot(x-x'')}\psi(x)e^{-iP\cdot(x-x'')}. \quad (25)$$

If the incident particle is in a wave-packet state which is predominantly characterized by the four-momentum p' then the state $\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle$ has four-momentum eigenvalues greater than the ground state of the system $|\Phi_a,-\infty\rangle$ by the value p'. Thus

$$\frac{1}{\tau}=2\,\mathrm{Im}\int d^3rd^4x''\,\Sigma(x,x'')e^{-ip'\cdot(x-x'')}$$

$$\times\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p}',t)|\,\Phi_a,+\infty\rangle$$

$$\times\langle\Phi_a,+\infty\,|\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle A^{-1}\varphi_{\mathbf{p}'}{}^*(\mathbf{r},t)$$

$$=2\,\mathrm{Im}A^{-1}\,\Sigma(p')\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p}',t)|\,\Phi_a,+\infty\rangle$$

$$\times\langle\Phi_a,+\infty\,|\psi(\mathbf{p}',t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle. \quad (26)$$

At this stage we make the approximation which, if made at the outset, would yield an infinite lifetime. We assume that the single-particle state is well defined, that is, that the width of the single-particle energy state is small, and that the total transition probability for leaving the initial state is much less than one.

$$A\approx\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p}',t)|\,\Phi_a,+\infty\rangle$$

$$\times\langle\Phi_a,+\infty\,|\psi(\mathbf{p}',t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle. \quad (27)$$

We now have the desired result for the lifetime.

$$1/\tau=2\,\mathrm{Im}\Sigma(p'). \quad (28)$$

Other derivations of this result require the same approximation; however, the proof is usually shown by illustrating the function $|G(\mathbf{p}',t-t')|^2$ which describes the propagation of single-particle excitations as a decaying exponential in time.[3]

We now consider the evaluation of the total energy lost per unit time by the incident particle. The transition probability for energy loss ω is

$$\mathrm{T.P.}(\omega)=\lim_{\substack{t\to+\infty\\ t'\to-\infty}}\sum_{b,\mathbf{p}}\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{p},t)|\,\Phi_b,+\infty\rangle A^{-1}\langle\Phi_b,+\infty\,|\delta(H-E_a-\omega)\psi(\mathbf{p},t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle. \quad (29)$$

The delta function will pick out only those terms in the sum over b and $\mathbf{p}$ such that the system finally has an energy $E_a+\omega$, as compared with the initial energy E_a. We sum over the complete set of intermediate states

$$\sum\nolimits_b|\Phi_b,+\infty\rangle\langle\Phi_b,+\infty\,|=1, \quad (30)$$

and choose the exponential representation of the delta function

$$\delta(H-E_a-\omega)=\int_{-\infty}^{\infty}\frac{dt_1}{2\pi}\exp[-i(H-E_a-\omega)t_1]. \quad (31)$$

The completeness of wave-packet states,

$$\sum\nolimits_{\mathbf{p}}\varphi_{\mathbf{p}}(\mathbf{r},t)\varphi_{\mathbf{p}}{}^*(\mathbf{r}',t)=\delta(\mathbf{r}-\mathbf{r}'), \quad (32)$$

[3] V. M. Galitskii and A. M. Migdal, Soviet Phys.—JETP **34**(7), 1 (1958).

may be utilized in the expression for the transition rate for energy loss ω:

$$\frac{1}{\tau(\omega)}=\frac{\partial}{\partial t}\int d^3r\int_{-\infty}^{\infty}\frac{dt_1}{2\pi}e^{i\omega t_1}A^{-1}\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\times\psi^\dagger(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle. \quad (33)$$

The factor e^{-iHt_1} has been employed to translate the time coordinate of the field operators. The field equation (1) is used to obtain

$$\frac{1}{\tau(\omega)}=-2\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{dt_1}{2\pi}d^3r\times A^{-1}\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)\times H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle. \quad (34)$$

The kinetic energy term which gives no contribution has been omitted. To obtain the total energy loss per unit time we consider

$$\frac{dE}{d\tau}=\int_{-\infty}^{\infty}d\omega\frac{1}{\tau(\omega)}\omega=-2\,\mathrm{Im}\int_{-\infty}^{\infty}d\omega\frac{dt_1}{2\pi}d^3r\,A^{-1}\times\left\{\frac{1}{i}\frac{\partial}{\partial t_1}e^{i\omega t_1}\right\}\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)\times H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle. \quad (35)$$

If we integrate by parts over t_1, and perform the ω integration we find

$$\frac{dE}{d\tau}=2\,\mathrm{Im}\int d^3r\,A^{-1}\frac{1}{i}\frac{\partial}{\partial t_1}\times\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)\times H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle|_{t_1=0}. \quad (36)$$

At this stage the result is still exact and we see that the energy losses actually depend upon the correlations of a two-particle function with the interacting field. We will make the approximation that it is only the single-particle correlations with the interacting field which are dominant. Thus, we choose the self-interactions of the particle to be the most important in the noncorrelation approximation,

$$\begin{aligned}&\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle\\&\approx\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)H_1(\mathbf{r},t+t_1)|\Phi_a,+\infty\rangle\langle\Phi_a,+\infty|\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle\\&+\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)|\Phi_a,+\infty\rangle\langle\Phi_a,+\infty|H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle\\&-\langle\Phi_a,-\infty|\psi(\mathbf{p}',t'+t_1)\psi^\dagger(\mathbf{r},t+t_1)|\Phi_a,+\infty\rangle\langle\Phi_a,+\infty|H_1(\mathbf{r},t+t_1)|\Phi_a,+\infty\rangle\\&\times\langle\Phi_a,+\infty|\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle.\end{aligned} \quad (37)$$

This approximation, when made in the expression for $1/\tau(\omega)$, maintains the desired property:

$$\int_{-\infty}^{\infty}d\omega\frac{1}{\tau(\omega)}=0. \quad (38)$$

This result comes about through a cancellation of the integral of the first two terms on the right-hand side of Eq. (37). The integral of the first term of Eq. (37) represents the transition rate for no energy loss, and that of the second, the transition rate to all states with non-zero energy loss. This equivalence is a restatement of the equality given in Eq. (21). Of course, Eq. (37) is exact for a noninteracting system in which the correlations are zero. When the width of the single particle energy state is small as assumed, the corrections to Eq. (37) calculated using intermediate states which differ from the ground state will be negligible.

If we maintain the dominance of self-interactions, Eq. (35) becomes

$$\frac{dE}{d\tau}=2\,\mathrm{Im}\int d^3r\langle\Phi_a,-\infty|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{r},t)|\Phi_a,+\infty\rangle A^{-1}\times\frac{1}{i}\frac{\partial}{\partial t_1}\langle\Phi_a,+\infty|H_1(\mathbf{r},t+t_1)\times\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle|_{t_1=0}. \quad (39)$$

Now since we are interested in $t\to\infty$, $t'\to-\infty$, $t_1\to0^+$ we may use Eqs. (8), (11), and (12) to obtain

$$\begin{aligned}&\langle\Phi_a,+\infty|H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\Phi_a,-\infty\rangle\\&=\int d^3r'\langle\Phi_a,+\infty|\Phi_a,-\infty\rangle\times\langle(H_1(\mathbf{r},t+t_1)\psi(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t'))_+\rangle\epsilon(t-t')\varphi_{\mathbf{p}'}(\mathbf{r}')\\&=\int d^3r'\langle\Phi_a,+\infty|\Phi_a,-\infty\rangle\varphi_{\mathbf{p}'}(\mathbf{r}')\times\Big\{\langle H_1(\mathbf{r},t+t_1)\rangle\langle\psi(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t')\rangle\\&+\int d^4x''\frac{d^4p\,d^4q}{(2\pi)^8}e^{ip\cdot(x-x'')}e^{-iq^0t_1}D(q)\times G(p-q)\Gamma(p,q)G(x'',x')\Big\}.\end{aligned} \quad (40)$$

When combined with Eqs. (25) and (39) the above

equation leads to

$$\frac{dE}{d\tau}=2\,\mathrm{Im}i\int d^3r\,A^{-1}\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{r},t)|\,\Phi_a,+\infty\rangle \times\langle\Phi_a,+\infty\,|\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,-\infty\rangle \times\int\frac{d^4q}{(2\pi)^4}q^0D(q)G(p'-q)\Gamma(p',q). \quad (41)$$

We must again make the approximation equivalent to Eq. (27) which requires that the single-particle state be well defined and the total transition probability for leaving the initial state be much less than one.

$$A\approx\int d^3r\,\langle\Phi_a,-\infty\,|\psi(\mathbf{p}',t')\psi^\dagger(\mathbf{r},t)|\,\Phi_a,+\infty\rangle \times\langle\Phi_a,+\infty\,|\psi(\mathbf{r},t)\psi^\dagger(\mathbf{p}',t')|\,\Phi_a,+\infty\rangle. \quad (42)$$

With this requirement, we obtain the total energy loss per unit time.

$$\frac{dE}{d\tau}=2\,\mathrm{Im}i\int\frac{d^4q}{(2\pi)^4}q^0D(q)G(p'-q)\Gamma(p',q). \quad (43)$$

We may note that this is related to the result obtained for the lifetime except that the integrand is multiplied by the energy associated with the interacting field Green's function. When we consider systems initially at nonzero temperatures we must take a statistical average over the possible ensemble of states in the expression for the transition probability. The result for the stopping is again given by Eq. (43) after we replace the ground state Green's functions by thermodynamic Green's function.[4]

We next consider two examples to which Eq. (43) may be applied. In these examples it is sufficient to consider a lowest order Born approximation in order to obtain the already well-known results. However, there are many problems in which the single-particle state has a sharp energy but yet the Born approximation neglects the major effects. It is in these problems that the preceding formulation will prove most useful.

BETHE SUM RULE

The electromagnetic interactions between charged particles afford one example to which Eq. (43) may be applied. We will first consider the Coulomb interactions and give a demonstration of the Bethe sum rule. The field equation obeyed by a charged fermion is

$$i\frac{\partial}{\partial t}\psi(x)=\frac{1}{2m}\left(\frac{1}{i}\nabla-\frac{e}{c}\mathbf{A}(x)\right)^2\psi(x) +e^2\int dx'\,\mathfrak{D}(x-x')\psi^\dagger(x')\psi(x')\psi(x), \quad (44)$$

where

$$\mathfrak{D}(x)=\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}\frac{1}{\mathbf{k}^2} \quad (45)$$

and $\mathbf{A}$ is chosen to be transverse

$$\nabla\cdot\mathbf{A}=0. \quad (46)$$

We have taken the Coulomb interaction to act between identical particles. Were we to calculate the energy losses of a particle not identical to the other particles of the system, the equations would be simplified slightly. Our system is chosen to have a fixed homogeneous charged background which neutralizes the systems charge. Since the fixed background has no dynamical effects we have not included it in Eq. (44). We have also excluded the spin interactions of the particles in the system.

We introduce the source terms

$$\mathbf{J}(x)\cdot\mathbf{A}(x)-e\psi^\dagger(x)\psi(x)U(x)$$

into the Lagrangian density. After the equations for the Green's functions are generated, the external forcing terms $\mathbf{J}$ and U will be set equal to zero. The equation for the fermion Green's function is then written as

$$\left\{i\frac{\partial}{\partial t}-U_{\mathrm{eff}}(x)-\frac{1}{2m}\left(\frac{1}{i}\nabla-\frac{e}{c}\langle\mathbf{A}(x)\rangle\right)^2\right\}G(x,x') +\int d^4x''\{\Sigma_C(x,x'')+\Sigma_T(x,x'')\}G(x'',x') =\delta(x-x'). \quad (47)$$

We have separated the self-energy effects due to Coulomb interactions from those due to the transverse electromagnetic field. In Eq. (47) the Hartree potential,

$$U_{\mathrm{eff}}(x)=U(x)+e\int dx'\,\mathfrak{D}(x-x')\langle\rho(x')\rangle, \quad (48)$$

is introduced. The expectation value $\langle\rho\rangle$ is of the total charge density, including the nondynamical positive background. The methods outlined before are used to evaluate

$$\Sigma_C(x,y)=-ie^2\int d^4x'd^4x''\,D(x,x') \times G(x,x'')\Gamma_C(x'',y,x'), \quad (49)$$

where the interacting-field Green's function is

$$D(x,y)=\mathfrak{D}(x,y)-i\int d^4x'd^4x''\,\mathfrak{D}(x-x')\mathfrak{D}(y-x'') \times\{\langle(\rho(x')\rho(x''))_+\rangle-\langle\rho(x')\rangle\langle\rho(x'')\rangle\}, \quad (50)$$

and the vertex operator is

$$\Gamma_C(x,y,z)=-\delta G^{-1}(x,y)/\delta U_{\mathrm{eff}}(z). \quad (51)$$

[4] Such functions have been introduced and studied previously, e.g., P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

Let us denote the Fourier transform of the charge density commutator by

$$C(\mathbf{k},\omega)=\int d^4x\, e^{-ik\cdot(x-x')}\langle[\rho(x),\rho(x')]\rangle. \tag{52}$$

Then the Fourier transform of the Coulomb Green's function is

$$D(\mathbf{k},\omega)=\frac{1}{\mathbf{k}^2}+\frac{1}{\mathbf{k}^4}\int_0^\infty \frac{d\omega'}{\pi}\frac{\omega' C(\mathbf{k},\omega')}{\omega^2-\omega'^2+i\epsilon}, \quad \epsilon\to 0^+, \tag{53}$$

where the terms involving the expectation value of the total charge density are set equal to zero, positive and negative charges canceling.

The self-energy operator is evaluated by treating the incident particle separately from the rest of the system. It is considered as an external agent which produces an electromagnetic field. The field produced is altered by the system and it is the reaction of the altered field back on the particle which is calculated in the self-energy. For an energetic incident particle the real part of the self-energy will be a small correction to the total energy and we will choose the uncorrected connection between momentum and energy to hold for the particle in the system. The vertex operator is taken in its lowest order approximation

$$\Gamma_C(x,y;z)=\delta(x-y)\delta(x-z). \tag{54}$$

The Coulomb self-energy is then

$$\Sigma_C(\mathbf{p},\mathbf{p}^2/2m) =-ie^2\int\frac{d^3k d\omega}{(2\pi)^4}D(\mathbf{k},\omega)G\left(\mathbf{p}-\mathbf{k},\frac{\mathbf{p}^2}{2m}-\omega\right). \tag{55}$$

As indicated above, we choose a free-particle approximation for the incident particle:

$$G(\mathbf{p},p^0)=\frac{1}{p^0-p^2/2m+(p^2/2m-\mu)i\delta}, \quad \delta\to 0^+, \tag{56}$$

where μ is the Fermi energy of the system. If the system is composed of particles which are not identical to the incident particle, all states are accessible, and $\mu=0$.

Equation (43) for the total energy loss per unit time gives

$$\frac{dE_C}{d\tau}=-2\,\mathrm{Im}\, ie^2\int\frac{d^4k}{(2\pi)^4}\omega D(k)G\left(\mathbf{p}-\mathbf{k},\frac{\mathbf{p}^2}{2m}-\omega\right). \tag{57}$$

The Born approximation has been chosen as a starting point in several works and used to derive expressions identical with Eq. (57).[5] The most generally chosen notation in these works for the interacting-field Green's function $D(\mathbf{k},\omega)$ is $\mathbf{k}^{-2}\epsilon_L^{-1}(\mathbf{k},\omega)$. This choice is natural in this example since $D(\mathbf{k},\omega)$ satisfies Maxwell's equation for the scalar potential in a medium with a single external delta function charge distribution.

Performing the ω integration in Eq. (57), there results

$$\begin{aligned}\frac{dE_C}{d\tau}=2e^2\,\mathrm{Im}\Bigg\{&\int_{(\mathbf{p}-\mathbf{k})^2>2m\mu}\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^4}\times\int_0^\infty\frac{d\omega'}{2\pi}\frac{\omega' C(k,\omega')}{\omega'+\mathbf{k}^2/2m-\mathbf{p}\cdot\mathbf{k}/m-i\delta}\\&+\int_{(\mathbf{p}-\mathbf{k})^2<2m\mu}\frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^4}\times\int_0^\infty\frac{d\omega'}{2\pi}\frac{\omega' C(\mathbf{k},\omega')}{\omega'-\mathbf{k}^2/2m+\mathbf{p}\cdot\mathbf{k}/m-i\delta}\Bigg\}.\end{aligned} \tag{58}$$

We can obtain an imaginary part to the integrals only from the delta function terms is the integration over angle

$$\begin{aligned}\frac{dE_C}{d\tau}=\frac{2\pi e^2}{v}\Bigg\{&\int_{0\atop (\mathbf{p}-\mathbf{k})^2>2m\mu}^\infty\frac{dk}{(2\pi)^2}\frac{1}{k^3}\times\int_{0\atop \omega'+\mathbf{k}^2/2m<vk}^\infty\frac{d\omega'}{2\pi}\omega' C(k,\omega')\\&+\int_{0\atop (\mathbf{p}-\mathbf{k})^2<2m\mu}^\infty\frac{dk}{(2\pi)^2}\frac{1}{k^3}\times\int_{0\atop |\omega'-\mathbf{k}^2/2m|<vk}^\infty\frac{d\omega'}{2\pi}\omega' C(k,\omega')\Bigg\}.\end{aligned} \tag{59}$$

where $v=|\mathbf{p}|/m$ is the magnitude of the velocity of the incident particles, and k is now used to denote the magnitude of the momentum transfer. The second term in Eq. (59) would not contribute for a fast incident particle since the conditions on the integral demand $\omega'+p^2/2m<\mu$. Thus, for $p^2/2m>\mu$ the energy loss becomes

$$\frac{dE_C}{d\tau}=\frac{e^2}{(2\pi)^2 v}\int_{\omega'+\mathbf{k}^2/2m<vk} dk\frac{1}{k^3}\int_0^\infty d\omega'\,\omega' C(k,\omega'). \tag{60}$$

The condition placed on the integral incorporates

[5] J. Lindhardt, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **28**, No. 8 (1954). L. Van Hove, Phys. Rev. **95**, 249 (1954). U. Fano, *ibid.* **103**, 1202 (1956). P. Nozières and D. Pines, Nuovo cimento **9**, 470 (1958).

energy and momentum conservation

$$mv-(m^2v^2-2m\omega')^{\frac{1}{2}}<k<mv+(m^2v^2-2m\omega')^{\frac{1}{2}},\quad \omega'<mv^2/2. \tag{61}$$

Current conservation is now used to determine the value of the integral over ω'.

$$\frac{\partial}{\partial t}\langle[\rho(x),\rho(x')]\rangle|_{t'=t}=-\nabla\cdot\langle[\mathbf{j}(x),\rho(x')]\rangle|_{t'=t}=i\nabla^2\delta(\mathbf{r}-\mathbf{r}')ne^2/m, \tag{62}$$

where n is the particle density. When Eq. (62) is written in terms of the Fourier transformed function

$$\int_0^\infty \frac{d\omega}{\pi}\omega C(\mathbf{k},\omega)=\mathbf{k}^2ne^2/m. \tag{63}$$

When the incident energy $mv^2/2$ is large enough, we extend the integral in the energy loss to infinity and neglect any possible small positive contribution which may occur.

$$\frac{dE_C}{d\tau}=\frac{ne^4}{4\pi mv}\int^{2mv}\frac{dk}{k}. \tag{64}$$

Thus we see there are two additional assumptions inherent to the Born approximation. One is the neglect of vertex corrections in the self-energy operator. The second is the choice of the free-particle Green's function to describe the incident particle.

ČERENKOV LOSSES

We may develop an expression for the energy losses which take place in the form of Čerenkov radiation by considering the term of Eq. (44) linear in the electromagnetic field. The transverse self-energy is

$$\Sigma_T\left(\mathbf{p},\frac{\mathbf{p}^2}{2m}\right)=\frac{ie^2}{c^2}\int\frac{d^3kd\omega}{(2\pi)^4}G\left(\mathbf{p}-\mathbf{k},\frac{p^2}{2m}-\omega\right)\times v_mD_{ml}(\mathbf{k},\omega)v_l, \tag{65}$$

when we make the lowest order approximation for the vertex function and denote the velocity components of the incident particle by v_m. In this case the interacting-field Green's function is

$$D_{ml}(x,x')=\frac{\delta\langle A_m(x)\rangle}{\delta J_l(x')}=i\langle(A_m(x)A_l(x'))_+\rangle. \tag{66}$$

We will derive an expression for the photon Green's function when the system is described classically by a frequency-dependent transverse dielectric function. Maxwell's equation for the transverse field is

$$\nabla^2\mathbf{A}(x)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(x)=-\mathbf{J}(x)-\mathbf{j}(x), \tag{67}$$

where the external current source $\mathbf{J}$ acts to induce currents $\mathbf{j}$ in the system. To derive an equation for the photon Green's function, we take the variational derivation of Eq. (67) with respect to J.

$$\left(-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)D_{ml}{}^T(x,x')=(\delta_{ml}\delta(x-x'))^T+\int d^4x''\frac{\delta A_\mu(x'')}{\delta J_l(x')}\frac{\delta j_m(x)}{\delta A_\mu(x'')}. \tag{68}$$

If we define the polarization tensor

$$P_{m\mu}(x,x')=\delta j_m(x)/\delta A_\mu(x'), \tag{69}$$

in the case of a homogeneous isotropic system the Fourier transform of Eq. (68) is

$$\left(\mathbf{k}^2-\frac{\omega^2}{c^2}\right)D_{ml}{}^T(\mathbf{k},\omega)=\delta_{ml}{}^T+P_{m\mu}(\mathbf{k},\omega)D_{\mu l}(\mathbf{k},\omega). \tag{70}$$

We have written

$$\delta_{ml}{}^T=\delta_{ml}-k_mk_l/\mathbf{k}^2. \tag{71}$$

For a classical isotropic system there is no coupling between the transverse and Coulomb field:

$$P_{m0}{}^T(\mathbf{k},\omega)=P_{0m}{}^T(\mathbf{k},\omega)=0. \tag{72}$$

The question we are asking when we evaluate P is: When the field measured in a medium is $\mathbf{A}$, what are the polarization currents induced, given in terms of $\epsilon_T(\omega)$? In the medium we will need the current $\mathbf{J}'$ to cause a field $\mathbf{A}$; however, to create the same field strength in a vacuum, we need a different current, $\mathbf{J}$. The polarization current is then the difference

$$\mathbf{j}(x)=\mathbf{J}(x)-\mathbf{J}'(x). \tag{73}$$

In the medium we write the Fourier transform of Maxwell's equation

$$\left(\mathbf{k}^2-\frac{\epsilon_T(\omega)}{c^2}\omega^2\right)\mathbf{A}(\mathbf{k},\omega)=\mathbf{J}'(\mathbf{k},\omega). \tag{74}$$

Equations (69), (73), and (74) yield the evaluation

$$P_{ml}{}^T(\mathbf{k},\omega)=\{\epsilon_T(\omega)-1\}\frac{\omega^2}{c^2}\delta_{ml}{}^T. \tag{75}$$

Thus, the photon Green's function satisfying outgoing wave boundary conditions is

$$D_{ml}{}^T(\mathbf{k},\omega)=\delta_{ml}{}^T\left(\mathbf{k}^2-\epsilon_T(\omega)\frac{\omega^2}{c^2}-i\gamma\right)^{-1},\quad \gamma\to0^+ \tag{76}$$

when $\epsilon_T(\omega)$ is real and positive. The energy losses are then obtained using Eq. (65) with the approximation of

Eq. (56) for the incident-particle Green's function. We need only set $\mu=0$ for the classical system.

$$\frac{dE_T}{d\tau}=2\,\mathrm{Im}\,i\frac{e^2v^2}{c^2}\int\frac{d^4k}{(2\pi)^4}\frac{\omega}{\mathbf{p}\cdot\mathbf{k}/m-\omega-\mathbf{k}^2/2m+i\lambda}\times\frac{1-x^2}{\mathbf{k}^2-\epsilon_T\omega^2/c^2-i\gamma}. \quad (77)$$

x is used to denote the cosine of the angle between $\mathbf{p}$ and $\mathbf{k}$, and $\lambda\to 0^+$. We assume that ϵ_T is such that there is only one singularity of the integrand in the lower half plane at a frequency $\omega=kc/[\epsilon_T(\omega)]^{\frac{1}{2}}-i\gamma$. The integral over frequency in Eq. (77) is performed to give

$$\frac{dE_T}{d\tau}=\mathrm{Im}\,e^2v^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{\epsilon_T}\times\left\{\frac{1-x^2}{ck\epsilon_T^{-\frac{1}{2}}+\mathbf{k}^2/2m-\mathbf{v}\cdot\mathbf{k}-i\lambda}\right\}. \quad (78)$$

When ϵ_T has the properties stated above, the integral will have an imaginary part resulting from the delta function contribution in the integration over x.[6]

$$\frac{dE_T}{d\tau}=\frac{e^2v}{4\pi}\int_{(c/v\epsilon_T^{\frac{1}{2}}+k/2p)<1}dk\frac{k}{\epsilon_T}\left[1-\left(\frac{c}{v\epsilon_T^{\frac{1}{2}}}+\frac{k}{2p}\right)^2\right]. \quad (79)$$

The result obtained classically for the emission of Čerenkov radiation[7] may be obtained from Eq. (79) by neglecting the quantum-mechanical correction $k/2p$.

In the preceding examples we have separated two particular forms of energy loss and shown they are both contained in the expression given for the stopping. However, it should be noted that a major advantage of Eq. (43) is that it contains all the possible forms of energy loss provided one choses the complete interacting-field Green's function and vertex function.

These techniques will be used in a later paper to describe the collective energy losses in crystals.

ACKNOWLEDGMENTS

I should like to thank Professor J. Schwinger for suggesting the problem considered in this paper. Several discussions of the problem with Dr. G. Baym were helpful in clarifying it. I am also grateful to Professor R. Haag, Professor D. Pines, and Professor J. Schrieffer for reading the manuscript and offering useful comments.

[6] When the relativistic Green's function for the incident particle is used, the term $k/2p$ in Eq. (79) representing the quantum-mechanical correction to the classical result is replaced by $(k/2p)(1-1/\epsilon_T)$.

[7] I. Tamm, J. Phys. (U.S.S.R.) **1**, 439 (1939).