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Effect of Periodic Adiabatic Time Variations on Interacting Systems

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It is shown that a many-particle system subject to periodic adiabatic variation of certain of its parameters is to a certain extent equivalent to a non-time-varying system with a radically modified interaction between the particles. The particular case of an electron gas in a metal is discussed in some detail.

It is well known in classical mechanics that small periodic forces applied to a system often transfer it from one kind of motion to a completely different kind. Over time intervals long compared with the period of the forces the system follows an almost smooth orbit which differs qualitatively from the orbit in the absence of the periodic forces. Two well-known examples are the electromagnetic suspension of bodies against the action of gravity, and the alternating-gradient forcussing of charged beams.

In this paper we show that similar effects should be obtainable in a quantum-mechanical system, provided it is possible to vary some parameter of the system in an adiabatic manner. In particular, under certain circumstances it should prove possible to cause a vanishing, or a reversal in sign of the interaction between the constituents of a many-particle system, at least if the interaction is weak. As a special case we shall discuss artificial enhancement of the transition temperature of superconductors by reversal of the sign of the electron-electron interaction in a region of momentum space where it is normally repulsive. It must be pointed out at the outset that the convergence of certain procedures used in this paper is assumed without being proved, and that in fact it might be possible to disprove convergence, thereby casting doubt on the validity of the conclusions. It should be noted, however, that one case of "decoupling of the interaction" by modulation techniques has already been verified in the laboratory: In ferromagnetic resonance certain phenomena caused by interaction terms in the spin-wave Hamiltonian have been quenched in this manner.[1] Though the theory of this particular process was given in terms of equations of motion,[2] it could equally well have been presented in a Hamiltonian formulation such as is used in this paper.

GENERAL THEORY

Consider a system whose Hamiltonian can be written

$$H=H_0+H_1(t)+V, \tag{1}$$

where H_1 and H_0 commute with each other but not with V. The time variation of H_1 is assumed to be explicitly contained in a *numerical parameter:* H_0 and V are time-independent. Though V may be quite general, for the purpose of this paper it will denote the interaction between the constituents of the system. We first transform the Hamiltonian (1) into a partial interaction representation by means of the transformation $\exp(iS)H\exp(-iS)$, where

$$S=\int H_1(t)dt.$$

In this representation, Schrödinger's equation reads

$$i\partial\phi/\partial t=[H_0+V(t)]\phi, \tag{2}$$

where $V(t)=\exp(iS)V\exp(-iS)$, and where $\hbar$ has been equated to unity. In Eq. (2), H_0 has remained untransformed, inasmuch as H_1 commutes with H_0 and the time integration only involves the varying parameter, not the operator constituents of H_1. We now assume further that the time variation is sinusoidal. Then it is obvious that $V(t)$ may be expanded in a Fourier series with operator coefficients V_n:

$$V(t)=\sum_{n=-\infty}^{+\infty} V_n\exp(inft),$$

where f is the frequency of the sinusoidal variation. The main point of this paper is that by suitable choice of the depth and the frequency of the modulation any particular coefficient, and more especially V_0 may be given a wide range of either positive or negative values, or may be made to vanish if desired. The conditions under which such manipulations may be of interest will now be examined.

We define a quasi-stationary solution of Eq. (2) as one which in the limit $f=0$ reverts to the usual exponential time dependence characteristic of the wave function of a time-independent Hamiltonian. Such a solution must have the form

$$\phi=\exp(-iEt)\sum_{n=-\infty}^{+\infty}\phi_n\exp(inft).$$

From (2) and from the Fourier series for V it follows that the functions ϕ_n satisfy the recurrence relations,

$$(E-nf)\phi_n=H_0\phi_n+\sum_{m=-\infty}^{m=+\infty}V_{n-m}\phi_m. \tag{3}$$

[1] T. S. Hartwick, E. R. Peressini, and M. T. Weiss, Phys. Rev. Letters **6**, 176 (1961).
[2] H. Suhl, Phys. Rev. Letters **6**, 174 (1961).

We now present plausible arguments that in the limit of small V, a particular ϕ_n, say ϕ_0, is the dominant part of the wave function, and that to order V, the equation for ϕ_0 is

$$(H_0+V_0)\phi_0=E\phi_0. \tag{4}$$

From (3), we have

$$E\phi_0=(H_0+V_0)\phi_0+\sum_{m\neq 0} V_{-m}\phi_m. \tag{5}$$

Assuming that ϕ_0 is large compared with the other harmonics, we may solve for these approximately:

$$(E-mf-H_0-V_0)\phi_m=V_m\phi_0,$$

which, together with (5), gives

$$(E-H_0-V_0)\phi_0 = \sum_{m\neq 0} V_{-m}(E-mf-H_0-V_0)^{-1}V_m\phi_0. \tag{5a}$$

If the sum on the right-hand side converges, it will be of second order in the interaction. More accurate solutions for ϕ_m will lead to still higher correction terms to Eq. (4), which we henceforth regard as a satisfactory zero-order approximation to our problem. The reason for giving V_0 preferential treatment vis-à-vis the other coefficients and not relegating it to perturbation theory is that in some cases one will wish to arrange the modulation in such a way that V_0 is an attractive interaction where the original V was repulsive. Perturbation theory starting with H_0 as the unperturbed Hamiltonian is then not usually justified, since the ground-state energy will then not in general be an analytic function of V_0 for small V_0. In fact, it is only under these circumstances that the present theory corresponds to more than a mere rearrangement of ordinary time-dependent theory.

It is to be noted that no special significance attaches to the choice of ϕ_0 as the "large" harmonic. Had we picked some other ϕ_n, this would still have led to an equation of the form (5a), with a slight relabeling of the various quantities.

EXAMPLE. ARTIFICIAL ENHANCEMENT OF SUPERCONDUCTIVITY

We consider an electron gas in a metal, in the limit of high density. In the absence of interaction, the electrons are described by Bloch functions, which we approximate by plane waves $\exp(i\mathbf{kr})$. The corresponding single-particle energies are denoted by ϵ_k. It is assumed that these may be varied adiabatically. For the present we assume that the interaction among the electrons is purely repulsive; the Fourier transform of the interaction being denoted by v_q. In second quantized notation, with c_{ks}^*, c_{ks} denoting creation and annihilation operators in a state with momentum k and spin orientation s, the Hamiltonian is

$$H=\sum_{ks} \epsilon_k c_{ks}^* c_{ks}+\sum_{ks} \eta_k(t) c_{ks}^* c_{ks} - \sum_{qkk'ss'} v_q c_{k+qs}^* c_{k'-qs'}^* c_{ks} c_{k's'}.$$

Here $\eta_k(t)$ is the time-varying part of ϵ_k. For the sake of definiteness we assume that the parameter that is being varied is the effective electron mass. In that case

$$\eta_k(t)=-\epsilon_k\delta m^* \cos ft/m^*,$$

where δm^* is the maximum excursion of the effective mass from its average value m^*. In the interaction representation the Hamiltonian becomes

$$H=\sum_{ks} \epsilon_k c_{ks}^* c_{ks} -\sum v_q c_{k+qs}^* c_{k'-qs'}^* c_{ks} c_{k's'} \exp[-iF(k,k',q,t)],$$

where

$$F(k,k',q,t)=(\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_{k'}-\epsilon_k)\delta m^* \sin ft/fm^*.$$

The time-independent part of the potential has thus changed from v_q to

$$v_q J_0[(\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_k-\epsilon_{k'})\delta m^*/fm^*],$$

and its sign will depend on the value of the argument of the Bessel function. Bearing in mind that under suitable circumstances $v_q J_0$ may be negative, we now attempt the pairing of electrons with opposite momenta and spin orientations characteristic of the Bardeen-Cooper-Schrieffer theory of superconductivity,[3] neglecting the part of the interaction that cannot be expressed in terms of such pairs. The time independent part of the Hamiltonian truncated in this manner is

$$H=\sum_{ks} \epsilon_k c_{ks}^* c_{ks}+\sum_{kk'} v_{k-k'} c_{k\uparrow}^* c_{-k\downarrow}^* c_{-k'\downarrow} c_{k'\uparrow} \times J_0[2(\epsilon_k-\epsilon_{k'})\delta m^*/m^* f], \tag{6}$$

where the kinetic energies are confined such that

$$z_{2n-1}<2(\epsilon_k-\epsilon_{k'})\delta m^*/m^* f<z_{2n}, \quad (n=1,2\cdots) \tag{7}$$

where z_n is the nth zero of the zero-order Bessel function. It is to be expected that the attractive interaction becomes successively less effective as n increases, since the corresponding kinetic energies, on the whole, will move further and further away from the Fermi surface. We therefore restrict the argument to the case $n=1$, in which case the attractive region consists of the interior of the two strips S, S' in the ϵ_k, $\epsilon_{k'}$ plane shown in Fig. 1. These should be compared with the single strip B shown in Fig. 2 in which the interaction is attractive in the BCS theory, due to excitation and de-excitation of virtual phonons. We note that B contains the Fermi energy, which is taken as the

[3] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

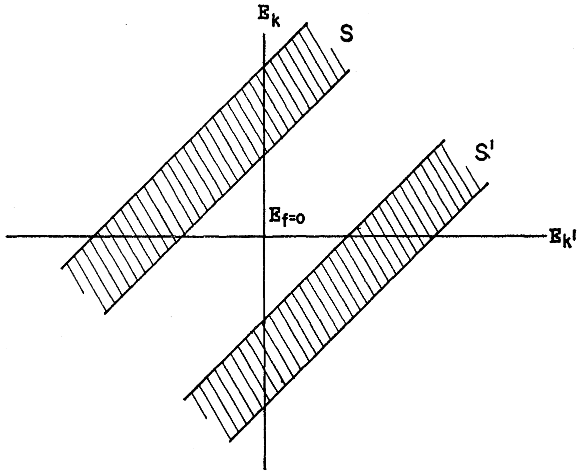

FIG. 1. The interaction is attractive in the cross-hatched strips S, S' symmetrically disposed about the Fermi energy.

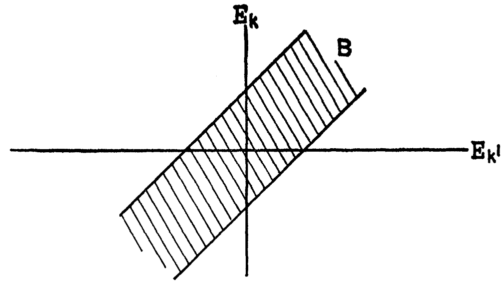

FIG. 2. In the BCS theory, the interaction is attractive in the strip B, containing the Fermi energy.

reference zero in both Figs. 1 and 2. In order to make further progress with the BCS theory, one replaces B by a square, centered at the origin, whose sides equal the width of the strip, in that case equal to the maximum available phonon energy. Similarly, to make further progress in the present case it is necessary to replace the strips S, S' by two equivalent squares delimited by $a_1<|\epsilon_k|<a_2$ and $a_1<|\epsilon_{k'}|<a_2$, where $a_{1,2}=fm^*z_{1,2}/\delta m^*$. The determining equation for the energy gap now takes the form

$$1=2V\sum_{a_1}^{a_2}\frac{1}{(\epsilon_k^2+\epsilon_0^2)^{\frac{1}{2}}},$$

assuming that the actual interaction may be replaced by an effective average V. The density of states may still be regarded as substantially constant, equal to $N(0)$, over the range of summation. The equation for the gap then becomes

$$\sinh^{-1}(a_2/\epsilon_0)-\sinh^{-1}(a_1/\epsilon_0)=1/VN(0).$$

Taking the sinh of both sides it is readily found that a solution will exist only if

$$z_2/z_1>\exp[1/VN(0)].$$

This condition is inconsistent with the present theory, which is valid only in the limit of small $VN(0)$. Therefore even if V were large enough to satisfy the last inequality, this would be no guarantee that the originally normal metal becomes a superconductor, though the possibility cannot be excluded.

The less ambitious project of raising the transition temperature of an existing superconductor holds more promise, however, even in the context of the present theory. Outside the strip B in Fig. 2 the interaction in the superconductor is repulsive. If we now adjust the modulation depth and frequency in such a way that the strips S, S' abut the upper and lower lines bounding the strip B, the width of the region in which the interaction is attractive will have been increased in the ratio z_2/z_1. The condition on δm^* and f is simply

$$fm^*z_1=\omega_d\delta m^*.$$

The effective mass modulation of a superconducting electron gas then effectively multiplies the Debye frequency ω_d, and therefore the transition temperature by the ratio z_2/z_1.

SOME LIMITS OF VALIDITY AND CONCLUSIONS

From the preceding example one might gain the impression that the interaction strength, and its sign is controlled by the *ratio* of frequency to depth of modulation, and that the absolute values of these quantities separately are not critical. However, referring to the correction to the quasi-stationary Schrödinger Eq. (5a), we see immediately that the frequency of modulation (times $\hbar$) must be sufficiently greater than the interaction if the correction terms are to have any chance of converging. Assuming V to be off-diagonal entirely, $(E_0-H_0-V_0-mf)^{-1}$ acts on an excited state, and is, therefore, of order $1/(V+mf)$. Hence, for $m=1$, we require $f\gg V$; then, for cancellation or sign-reversal of the interaction, the depth of the modulation must also exceed V, since by the condition for a zero of the Bessel function, the frequency and depth of modulation must be of the same orders.

To produce modulation depths in excess of the shielded coulomb repulsion in a metal may prove excessively difficult. For this reason, it is desirable to first try the principles described here on genuinely weakly interacting systems.

Finally, it must be pointed out that where a rigorous theory is possible, it may transcend the limits of validity described here. Such is the case in reference 2, where it turned out that with square wave modulation one could decouple the spin wave interaction with very *low* modulation frequencies also.

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