Theory of $\pi-N$ Scattering in the Strip Approximation to the Mandelstam Representation*

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The strip approximation to the Mandelstam representation is applied to the $\pi-N$ problem, and the basic equations given. The asymptotic behavior of the invariant amplitudes in the physical regions is discussed in terms of the unitarity condition on partial-wave amplitudes, the constancy of high-energy scattering cross sections, and the Pomeranchuk theorem, and it is shown to imply that no subtractions should be necessary except in the $J=\frac{1}{2}$ wave of the $\pi-N$ channel and the $J=0$ wave of the $\pi+\pi\rightarrow N+\bar{N}$ channel. This obviates the difficulties encountered by earlier workers when they subtracted higher waves.

I. INTRODUCTION

IN the preceding paper Chew and Frautschi (CF)[1] have discussed the pion-pion scattering problem in the Mandelstam representation,[2] taking into account the regions of the double-spectral functions (hereafter abbreviated as dsf's) nearest the physical regions. The purpose of this paper is to initiate a similar program for the pion-nucleon problem.

In the next section we describe the kinematics and the location of the singularities of the invariant amplitudes for the problem in terms of the Mandelstam diagram. The notation for the dsf's in the different regions is also fixed. The unitarity condition on the elements of the scattering matrix is the used in Sec. III to put limits on the possible asymptotic high-energy behaviors of the invariant amplitudes in all three channels. The implications of the constancy of high-energy pion-nucleon cross sections and of the Pomeranchuk theorem have also been analyzed in this section. This knowledge of asymptotic behavior is used in Sec. IV to argue that the only independent subtractions that can be carried out are the subtraction of the $J=\frac{1}{2}$ part of the amplitude in the $\pi-N$ channel and of the $J=0$ part in the $\pi+\pi\rightarrow N+\bar{N}$ channel. We are thus spared the necessity of making subtractions of the $J=\frac{3}{2}$ part in the $\pi-N$ channel and of the $J=1$ part in the $\pi+\pi\rightarrow N+\bar{N}$ channel, which have given rise to difficulties in previous work using partial-wave dispersion relations.[3] We then give the subtracted dispersion relations. In Sec. V we give expressions for the double-spectral functions in the strip approximation. These expressions, together with the subtracted dispersion relations given in Sec. IV and the already known partial-wave dispersion relations, are the basic equations for the $\pi-N$ problem in this approach. Solution of these equations will be considered in a subsequent communication.

II. MANDELSTAM DIAGRAM

The present approach is best described in terms of the Mandelstam diagram. We shall use the usual invariant variables s, $\bar{s}$, t defined by

$$\begin{aligned} s&=-(P_1+P_2)^2,\\ \bar{s}&=-(P_1+P_4)^2, \qquad (2.1)\\ t&=-(P_1+P_3)^2, \end{aligned}$$

where P_1, P_3 are the four-momenta of the pions, and P_2, P_4 of the nucleons, all in-going (Fig. 1). They satisfy

$$s+\bar{s}+t=2m^2+2\mu^2\equiv\Sigma. \qquad (2.2)$$

We shall use these variables also as labels for the channels for which they are the squares of the energy in the barycentric system.

The physical regions of the three channels are bounded by the curves

$$t=0, \quad s\bar{s}=(m^2-\mu^2)^2.$$

The boundary curves for the regions in which the double spectral functions are nonzero were calculated by Mandelstam.[2,4] The Mandelstam diagram (Fig. 2) shows, in terms of s, $\bar{s}$, t as triangular coordinates, the physical regions of the three channels of the four-line diagram of Fig. 1, as well as the regions where the double spectral functions fail to vanish.

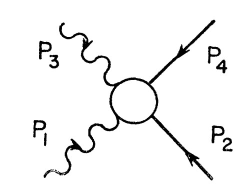

FIG. 1. The four-line diagram for the $\pi-N$ problem.

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[1] G. F. Chew and S. Frautschi, Phys. Rev. **123**, 1478 (1961), herafter referred to as CF.

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[3] W. Frazer, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), p. 282. See also G. F. Chew and S. Mandelstam, Lawrence Radiation Laboratory Report UCRL-9126, March, 1960 (unpublished). See also G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289, June, 1960 (unpublished), Sec. XI.

[4] The equations for the boundary curves for A_{12} given in reference 2 contain an error. The correct equation is given in W. Frazer and J. Fulco, Phys. Rev. **117**, 1603 (1960).

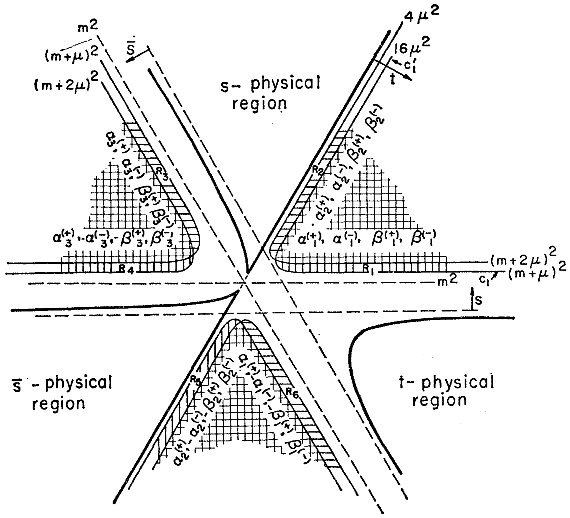

FIG. 2. The Mandelstam diagram for the $\pi-N$ problem.

According to Cutkosky,[5] the dsf can be expressed as a sum of contributions of all possible four-vertex diagrams. The only diagrams contributing to the dsf's[6] $A_{13}^{(\pm)}(s,t)$, and $B_{13}^{(\pm)}(s,t)$ in the strip region R_1 of Fig. 2 are the ones shown in Fig. 3(a) and (b). They are the diagrams in which there are only two particles in the intermediate state in the s direction, while an arbitrary number of particles is exchanged in the t direction. These diagrams in fact contribute to the entire region bounded by the curve C_1C_1' with asymptotes $s=(m+\mu)^2$ and $t=16\mu^2$. But although they give the dsf exactly in the strip region R_1, they give only the "elastic" (in the s channel) part thereof outside the strip, where there are also contributions from further Cutkosky diagrams which have at least three particles in each direction in the intermediate state. We have at present no way of calculating these latter diagrams, and hence we shall neglect them here. The "strip approximation" of Chew and Frautschi[1] consists in calculating the part of the dsf given by the Cutkosky diagrams shown in Fig. 3, and further assuming that the scattering amplitudes in the physical regions are dominated by the adjacent strips of the dsf's. Henceforth all our statements and equations giving relationships between absorptive parts and the dsf's will be in the "strip approximation," unless a statement is made to the contrary.

We denote the parts of the dsf's $A_{13}^{(\pm)}(s,t)$, $B_{13}^{(\pm)}(s,t)$ given by the Cutkosky diagrams of Fig. 3(a) and (b) by $\alpha_1^{(\pm)}(s,t)$ and $\beta_1^{(\pm)}(s,t)$, respectively. Similarly, the parts of the dsf's $A_{13}^{(\pm)}(s,t)$, $B_{13}^{(\pm)}(s,t)$ given by the diagram of Fig. 3(c) (which are the exact dsf's as far as the strip R_2 is concerned) will be denoted by $\alpha_2^{(\pm)}(t,s)$, $\beta_2^{(\pm)}(t,s)$, respectively. The $\alpha_3^{(\pm)}(\bar{s},s)$, $\beta_3^{(\pm)}(\bar{s},s)$ are also defined in a similar manner. These strip functions α and β have been indicated in Fig. 2 alongside the corresponding strips where they are the exact dsf's. In labeling the arguments of the strip functions α and β we adopt the convention that the first variable increases in a direction perpendicular to the strip while the second increases parallel to it.

We will make a few remarks about certain qualitative features of $\pi-N$ scattering which may be expected to follow from the strip approximation. First, as we have already seen, the contribution to the dsf in strip R_2 comes from Fig. 3(c), in which arbitrary inelastic processes are allowed to give rise to the intermediate state in the s channel, but only two pions are exchanged. The inelastic character (for the s channel) of the strip function in R_2 may be expected to introduce a substantial imaginary part into those phase shifts of $\pi-N$ scattering which are dominated by the 2π-exchange process, namely the high-angular-momentum phase shifts for $s\geq(m+2\mu)^2$. It is well known that the phase shifts start becoming complex at $s=(m+2\mu)^2$ because of opening up of inelastic channels, but it is usually assumed that the imaginary parts are small up to considerably higher energies. The effect we are discussing should make the imaginary part of d-, f-, $\cdots$, and higher phase shifts comparable to the real part

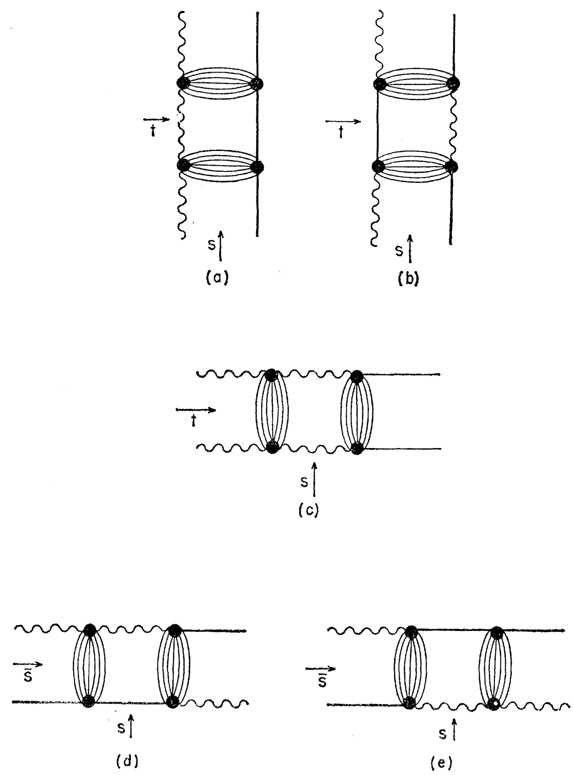

FIG. 3. The Cutkosky diagrams contributing to the dsf's in the strip regions $R_1(a/b)$, $R_2(c)$, and $R_3(d/e)$.

[5] R. E. Cutkosky, Phys. Rev. Letters 4, 624 (1960); J. Math. Phys. 1, 429 (1960).

[6] For the definition of the scattering amplitudes $A^{(\pm)}$, $B^{(\pm)}$ and the dsf's $A_{13}^{(\pm)}(s,t)$ etc., see any earlier work on pion-nucleon scattering using Mandelstam representation, e.g., S. Frautschi and D. Walecka, Phys. Rev. 120, 1486 (1960); W. Frazer and J. Fulco, Phys. Rev. 119, 1420 (1960).

very soon after they start showing up at all. This fact will have to be taken into account in the phase-shift analysis of $\pi-N$ scattering above approx 300 Mev.

Secondly, we notice that diagram 3(c) has a $\pi-\pi$ scattering part. We may therefore expect substantial direct contributions of the $\pi-\pi$ interaction to the forward amplitude in the $\pi-N$ scattering, and therefore to the total cross section, again in the region above about 300 Mev, where strip R_2 is assumed to dominate. This direct contribution will, however, vanish in the low-energy elastic region. Finally, if the concentration of the dsf's in the strips is responsible for the characteristic features of high-energy diffraction scattering in the forward direction, as discussed in CF, we would expect the backward peak in $\pi-N$ scattering, if it exists at all, to be much broader than the forward peak. In fact, since the nearest singularity (pole) in the backward direction is about four times as distant as the mean distance of R_2 from the forward direction (Fig. 2), we may expect the backward peak to be about four times as broad (measured in terms of t or $\bar{s}$) as the forward peak. The present experimental data, while definitely indicating a broad backward maximum in $\pi^- - p$ scattering at 2 Bev/c,[7] need to be extended to more backward directions and also to higher energies in order to confirm this point.

III. ASYMPTOTIC BEHAVIOR OF AMPLITUDES IN THE PHYSICAL REGION

We now study some of the restrictions imposed by the unitarity condition on the asymptotic behaviors of our invariant amplitudes. It is most convenient to do so in terms of partial wave amplitudes, since the unitarity condition assumes a very simple form in terms of these.

A. Unitarity Limitations in the s Channel ($\pi+N\rightarrow\pi+N$)

In this channel we have the following partial-wave expansions for the invariant amplitudes[8]:

$$A^{(\pm)}=4\pi\left[\left(\frac{W+m}{E+m}\right)f_1^{(\pm)}-\left(\frac{W-m}{E-m}\right)f_2^{(\pm)}\right] \tag{3.1a}$$

$$\underset{s\to\infty}{\longrightarrow} 8\pi(f_1^{(\pm)}-f_2^{(\pm)})\,; \tag{3.1b}$$

$$B^{(\pm)}=4\pi\left[\frac{1}{E+m}f_1^{(\pm)}+\frac{1}{E-m}f_2^{(\pm)}\right] \tag{3.2a}$$

$$\underset{s\to\infty}{\longrightarrow}\frac{8\pi}{s^{\frac{1}{2}}}(f_1^{(\pm)}+f_2^{(\pm)})\,; \tag{3.2b}$$

where

$$f_1^{(\pm)}(s,\cos\theta)=\sum_{l=0}f_{l,+}^{(\pm)}(s)P_{l+1}'(\cos\theta)-\sum_{l=2}f_{l,-}^{(\pm)}(s)P_{l-1}'(\cos\theta), \tag{3.3}$$

$$f_2^{(\pm)}(s,\cos\theta)=\sum_{l=1}[f_{l,-}^{(\pm)}(s)-f_{l,+}^{(\pm)}(s)]P_l'(\cos\theta), \tag{3.4}$$

and

$$f_{l,\pm}^{(\pm)}=\exp(i\delta_{l,\pm}^{(\pm)})\sin\delta_{l,\pm}^{(\pm)}/k. \tag{3.5}$$

One then readily sees that

$$A^{(\pm)}\underset{s\to\infty}{\longrightarrow}8\pi\sum_l(f_{l,+}^{(\pm)}-f_{l+1,-}^{(\pm)})(P_{l+1}'+P_l'), \tag{3.6}$$

$$B^{(\pm)}\underset{s\to\infty}{\longrightarrow}\frac{8\pi}{s^{\frac{1}{2}}}\sum_l(f_{l,+}^{(\pm)}+f_{l+1,-}^{(\pm)})(P_{l+1}'-P_l'). \tag{3.7}$$

Now we note

$$|P_l'(z)|\leq\tfrac{1}{2}l(l+1)\quad\text{for}\quad -1\leq z\leq 1, \tag{3.8}$$

where the equality holds for $z=\pm1$, and owing to unitarity,

$$|f_{l,\pm}^{(\pm)}|\leq 1/k. \tag{3.9}$$

Equations (3.8) and (3.9) enable us to put upper bounds on $A^{(\pm)}$, $B^{(\pm)}$. Thus, in the forward direction,

$$A^{(\pm)}(s,\theta=0)\underset{s\to\infty}{=}8\pi\sum_l(f_{l,+}^{(\pm)}-f_{l+1,-}^{(\pm)})(l+1)^2\lesssim\frac{16\pi}{k}\sum_{l=0}^{L=kR}(l+1)^2\sim\frac{1}{k}k^3R^3\sim s,$$

$$B^{(\pm)}(s,\theta=0)\underset{s\to\infty}{=}\frac{8\pi}{s^{\frac{1}{2}}}\sum_l(f_{l,+}^{(\pm)}+f_{l+1,-}^{(\pm)})(l+1)\lesssim\frac{16\pi}{s^{\frac{1}{2}}}\frac{1}{k}\sum_l(l+1)\sim\frac{1}{s^{\frac{1}{2}}}\frac{1}{k}k^2R^2\sim\text{const.}$$

Here R is the range of interaction, assumed constant. Thus we get,[9] in the forward direction,

$$\begin{aligned}A^{(\pm)}(s,\theta=0)&=O(s),\\ B^{(\pm)}(s,\theta=0)&=O(1).\end{aligned} \tag{3.10}$$

Similarly in the backward direction ($\theta=\pi$) we get

$$\begin{aligned}A^{(\pm)}(s,\theta=\pi)&=O(s^{\frac{1}{2}}),\\ B^{(\pm)}(s,\theta=\pi)&=O(s^{\frac{1}{2}}).\end{aligned} \tag{3.11}$$

Incidentally, we note that for $0\leq\theta\leq\pi$,

$$\begin{aligned}A^{(\pm)}(s,\cos\theta)&=O(s),\\ B^{(\pm)}(s,\cos\theta)&=O(s^{\frac{1}{2}}).\end{aligned} \tag{3.12}$$

[7] M. Perle (private communication).

[8] G. F. Chew, F. E. Low, M. L. Goldberger, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[9] We shall use the equation $f(x)=O(x^\alpha)$ to denote that $f(x)$ increases *at most* like x^α as $x\rightarrow\infty$, while by $f(x)=o(x^\alpha)$ we shall understand $\lim_{x\to\infty}[f(x)/x^\alpha]=0$.

It is of interest, from the point of view of the discussion in the next section, to put more stringent conditions than Eq. (3.12) on the asymptotic behavior of $A^{(-)}$, namely

$$A^{(-)}(s,t)=o(s). \tag{3.12a}$$

The argument proceeds as follows—one has

$$\begin{aligned}\mathrm{Re}A^{(-)}(s,t)&=\frac{P}{\pi}\int ds'\,A_s^{(-)}(s',t)\\&\quad\times\{(s'-s)^{-1}-[s'-\Sigma+s-2k^2(1-\cos\theta)]^{-1}\}\\&\underset{s\to\infty}{\sim}\frac{P}{\pi}\int ds'\,A_s^{(-)}(s',t)\\&\quad\times\{(s'-s)^{-1}-[s'+\tfrac{1}{2}s(1+\cos\theta)]^{-1}\}\\&\sim s\ln s,\end{aligned}$$

if $A_s^{(-)}(s,t)\sim s$. This, however, is impossible, since $\mathrm{Re}A^{(-)}$ is bounded by s [Eq. (3.12)]. Hence $A_s^{(-)}$ must go as $o(s)$, and then $\mathrm{Re}A^{(-)}$ also goes as $o(s)$, thus giving (3.12a).

B. Unitarity Limitations in the t Channel

The partial-wave expansion of the invariant amplitudes in this channel is given by[10]

$$\begin{aligned}A^{(\pm)}=-\frac{8\pi i}{p^2}\left(\frac{p}{q}\right)^{\frac{1}{2}}\sum_J(J+\tfrac{1}{2})&\\\times\left\{\frac{m\cos\theta_3}{[J(J+1)]^{\frac{1}{2}}}P_J'(\cos\theta_3)S_{-J}^{(\pm)}\right.&\\\left.-\frac{t^{\frac{1}{2}}}{2}P_J(\cos\theta_3)S_{+J}^{(\pm)}\right\},&\end{aligned} \tag{3.13}$$

$$\begin{aligned}B^{(\pm)}=-\frac{8\pi i}{pq}\left(\frac{p}{q}\right)^{\frac{1}{2}}\sum_J\frac{(J+\frac{1}{2})}{[J(J+1)]^{\frac{1}{2}}}&\\\times P_J'(\cos\theta_3)S_{-J}^{(\pm)},&\end{aligned} \tag{3.14}$$

where

$$t=4(q^2+\mu^2)=4(p^2+m^2). \tag{3.15}$$

The unitarity requirement

$$|S_{\pm J}^{(\pm)}|\leq 1, \tag{3.16}$$

combined with Eq. (3.8) then gives us the result that for $-1<\cos\theta_3<1$,

$$A^{(\pm)}(t,\cos\theta_3)=O(t^{\frac{1}{2}}), \tag{3.17}$$

$$B^{(\pm)}(t,\cos\theta_3)=O(t^{\frac{1}{2}}). \tag{3.18}$$

[10] W. Frazer and J. Fulco, Phys. Rev. 117, 1603 (1960).

C. Limitations Imposed by Constancy of High-Energy $\pi-N$ Cross Sections and Pomeranchuk Theorem

The total cross sections for $\pi^{\pm}-p$ reactions are given by

$$\begin{aligned}\sigma(\pi^+p)&=(\omega^2-1)^{-\frac{1}{2}}\\&\quad\times\mathrm{Im}[A^{(+)}-A^{(-)}+\omega(B^{(+)}-B^{(-)})]_{\theta=0},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\sigma(\pi^-p)&=(\omega^2-1)^{-\frac{1}{2}}\\&\quad\times\mathrm{Im}[A^{(+)}+A^{(-)}+\omega(B^{(+)}+B^{(-)})]_{\theta=0},\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\omega&=(s-m^2-\mu^2)/2m\\&=\text{energy of pion in the lab system.}\end{aligned} \tag{3.21}$$

Now the Pomeranchuk theorem states,[11] and this is in agreement with present experimental data, that at high energies

$$\sigma(\pi^+p)=\sigma(\pi^-p)=\sigma, \tag{3.22}$$

where σ is a constant. Hence from Eqs. (3.19) and (3.20) we should have

$$\mathrm{Im}\left(\frac{A^{(+)}}{s/2m}+B^{(+)}\right)_{\theta=0}\underset{s\to\infty}{\longrightarrow}\sigma, \tag{3.23}$$

$$\mathrm{Im}\left(\frac{A^{(-)}}{s/2m}+B^{(-)}\right)_{\theta=0}\underset{s\to\infty}{\longrightarrow}0, \tag{3.24}$$

The condition (3.23) implies that at least one amplitude out of $A^{(+)}$ and $B^{(+)}$ must assume the maximal behavior allowed by the unitarity requirements (3.10). No such stringent requirement is implied by (3.22) on the amplitudes $A^{(-)}$ and $B^{(-)}$.

IV. SUBTRACTIONS

A. Independent Subtractions

We can now discuss the number of subtractions to be carried out in the double spectral representation in order to make it meaningful. Some subtractions must in general be carried out for this purpose, and give rise to single spectral integrals. These may be called independent subtractions, since the corresponding subtraction terms are not expressible in terms of the dsf's. Over and above, one may carry out further subtractions if one so desires, but these will give rise to single spectral integrals which are determined in terms of the dsf's. We shall determine the number of independent subtractions for our problem on the basis of the asymptotic behaviors of the amplitudes in the physical regions, discussed in the preceding section, following an argument due to Froissart.[12]

[11] I. Ya. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 725 (1958) [translation: Soviet Phys.—JETP 7, 499 (1958)].
[12] M. Froissart (private communication).

The essence of the argument is that the subtraction of arbitrarily high partial waves in one channel is not consistent with the unitarity requirements on the asymptotic behavior in the crossed channels. Thus the subtraction of the $J=\frac{3}{2}$ part in the s channel would introduce a term of the form

$$t\int[\sigma(s')ds'/(s'-s)],$$

into the spectral representation of the amplitude, which, if independent, would contradict the maximal $t^{\frac{1}{2}}$ behavior in the t channel allowed by unitarity (see Eqs. 3.17, 3.18). Hence the only independent subtraction that could be tolerated in the s channel is that of the $J=\frac{1}{2}$ part. If one subtracts out the $J=\frac{3}{2}$ part also, one has to remember at every stage of approximation that it is not independent, but determined by the dsf's, otherwise one meets with the difficulty of spurious divergences of the kind encountered by previous workers.[3]

A similar argument shows that only a $J=0$ part can be subtracted out in the t channel, and since only $A^{(+)}$ has a $J=0$ part, the only independent subtraction in the t channel is that of the $J=0$ part of the $A^{(+)}$ amplitude.

B. Subtracted Expressions for the Absorptive Parts

The subtracted representations for absorptive parts in the s channel are now given by

$$\begin{aligned}
A_s^{(\pm)}(s,\bar{s},t)=&\frac{W+m}{E+m}4\pi\,\mathrm{Im}f_{S\frac{1}{2}}^{(\pm)}-\frac{W-m}{E-m}4\pi\,\mathrm{Im}f_{P\frac{1}{2}}^{(\pm)}\\
&+\frac{1}{\pi}\int dt'[\alpha_1^{(\pm)}(s,t')+\alpha_2^{(\pm)}(t',s)]\left\{\frac{1}{t'-t}-\frac{1}{2k^2}Q_0\left(1+\frac{t'}{2k^2}\right)+\frac{a_1(s)}{2k^2}Q_1\left(1+\frac{t'}{2k^2}\right)\right\}\\
&+\frac{a_2(s)}{\pi}\frac{1}{2k^2}\int dt'\{\beta_1^{(\pm)}(s,t')+\beta_2^{(\pm)}(t',s)\}Q_1\left(1+\frac{t'}{2k^2}\right)+\frac{1}{\pi}\int d\bar{s}'[\alpha_3^{(\pm)}(\bar{s}',s)\pm\alpha_3^{(\pm)}(s,\bar{s}')]\\
&\times\left\{\frac{1}{\bar{s}'-\bar{s}}+\frac{1}{2k^2}Q_0\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right)-\frac{a_1(s)}{2k^2}Q_1\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right)\right\}\\
&-\frac{1}{\pi}\frac{a_2(s)}{2k^2}\int d\bar{s}'[\beta_3^{(\pm)}(\bar{s}',s)\mp\beta_3^{(\pm)}(s,\bar{s}')]\left\{Q_1\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right)\right\};
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
B_s^{(\pm)}(s,\bar{s},t)=&\frac{1}{E+m}4\pi\,\mathrm{Im}f_{S\frac{1}{2}}^{(\pm)}+\frac{1}{E-m}4\pi\,\mathrm{Im}f_{P\frac{1}{2}}^{(\pm)}\\
&+\frac{1}{\pi}\int dt'[\beta_1^{(\pm)}(s,t')+\beta_2^{(\pm)}(t',s)]\left\{\frac{1}{t'-t}-\frac{1}{2k^2}Q_0\left(1+\frac{t'}{2k^2}\right)+\frac{b_1(s)}{2k^2}Q_1\left(1+\frac{t'}{2k^2}\right)\right\}\\
&+\frac{1}{\pi}\frac{b_2(s)}{2k^2}\int dt'[\alpha_1^{(\pm)}(s,t')+\alpha_2^{(\pm)}(t',s)]Q_1\left(1+\frac{t'}{2k^2}\right)+\frac{1}{\pi}\int d\bar{s}'[\beta_3^{(\pm)}(\bar{s}',s)\mp\beta_3^{(\pm)}(s,\bar{s}')]\\
&\times\left\{\frac{1}{\bar{s}'-\bar{s}}+\frac{1}{2k^2}Q_0\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right)-\frac{b_1(s)}{2k^2}Q_1\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right)\right\}\\
&-\frac{1}{\pi}\frac{b_2(s)}{2k^2}\int d\bar{s}'[\alpha_3^{(\pm)}(\bar{s}',s)\pm\alpha_3^{(\pm)}(s,\bar{s}')]Q_1\left(1+\frac{\Sigma-s-\bar{s}'}{2k^2}\right),
\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}
a_1(s)&=1+\frac{4m^2(s-m^2+\mu^2)}{[s-(m-\mu)^2][s-(m+\mu)^2]},\\
a_2(s)&=\frac{2m[(s-m^2)^2-\mu^4]}{[s-(m-\mu)^2][s-(m+\mu)^2]},\\
b_1(s)&=-a_1(s),\\
b_2(s)&=-\frac{4m(s+m^2-\mu^2)}{[s-(m-\mu)^2][s-(m+\mu)^2]},
\end{aligned}\tag{4.3}$$

and

$$Q_0(x)=\tfrac{1}{2}\ln[(x+1)/(x-1)],$$
$$Q_1(x)=\tfrac{1}{2}x\ln[(x+1)/(x-1)]-1. \tag{4.4}$$

The expressions for the subtracted representations for the t-absorptive parts are

$$A_t^{(+)}(s,\bar{s},t)=-\frac{4\pi}{p^2}\operatorname{Im}f_+^{(+)0}(t)+\frac{1}{\pi}\int ds'\left\{\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}-\frac{1}{pq}Q_0\left(\frac{s'+p^2+q^2}{2pq}\right)\right\}$$
$$\times[\alpha_1^{(+)}(s',t)+\alpha_2^{(+)}(t,s')]+\frac{m}{p^2}\frac{1}{\pi}\int ds'[\beta_1^{(+)}(\bar{s},t)+\beta_2^{(+)}(t,s')]Q_1\left(\frac{s'+p^2+q^2}{2pq}\right), \tag{4.5}$$

$$A_t^{(-)}(s,\bar{s},t)=\pi^{-1}\int ds'[\alpha_1^{(-)}(s',t)+\alpha_2^{(-)}(t,s')]\{(s'-s)^{-1}-(s'-\bar{s})^{-1}\}, \tag{4.6}$$

$$B_t^{(\pm)}(s,\bar{s},t)=\pi^{-1}\int ds'[\beta_1^{(\pm)}(s',t)+\beta_2^{(\pm)}(t,s')]\{(s'-s)^{-1}\mp(s'-\bar{s})^{-1}\}. \tag{4.7}$$

The partial-wave dispersion relations for the subtracted quantities $f_{S\frac{1}{2}}^{(\pm)}$, $f_{P\frac{1}{2}}^{(\pm)}$ in the s channel, and $f_+^{(+)0}$ in the t channel, which occur above have been given already by Frazer and Fulco,[6,10] and by Frautschi and Walecka.[6]

V. EXPRESSIONS FOR THE STRIP FUNCTIONS IN TERMS OF ABSORPTIVE PARTS

The contributions to the dsf's of the Cutkosky diagrams of the type shown in Fig. 3 will now be calculated by using the generalized unitarity condition for each channel, wherein only the lowest-mass two-particle states are retained in the intermediate-state summation.

A. Channel $\pi+N\rightarrow\pi+N$

We will describe the $\pi-N$ scattering channel by k, the magnitude of the barycentric momentum of the pion, and θ, the angle through which it is scattered. Then

$$s=[(k^2+\mu^2)^{\frac{1}{2}}+(k^2+m^2)^{\frac{1}{2}}]^2\equiv W^2,$$
$$t=-2k^2(1-\cos\theta)\equiv-2k^2(1-z), \tag{5.1}$$

and

$$\bar{s}=2m^2+2\mu^2-s-t.$$

The generalized unitarity condition applied to this channel gives the strip functions $\alpha_1^{(\pm)}$, $\beta_1^{(\pm)}$, $\alpha_3^{(\pm)}$, $\beta_3^{(\pm)}$. The expressions for these were essentially given by Mandelstam,[2] but since they contained a few algebraic errors, we shall give the correct expressions in our notation.

$$\alpha_1^{(\pm)}(s,t)=\sum_{i=1}^{4}\frac{m}{8\pi^2kW}\left[\int dt'dt''\,K_s(s;t,t',t'')l_i(s;t,t',t'')G_{i;tt}^{(\pm)}(s;t',t'')\right.$$
$$\left.+\int d\bar{s}'d\bar{s}''\,K_s(s;t,\Sigma-s-\bar{s}',\Sigma-s-\bar{s}'')l_i(s;t,\Sigma-s-\bar{s}',\Sigma-s-\bar{s}'')G_{i;\bar{s}\bar{s}}^{(\pm)}(s;\bar{s}',\bar{s}'')\right], \tag{5.2}$$

$$\alpha_3^{(\pm)}(\bar{s},s)=\sum_{i=1}^{4}\frac{m}{8\pi^2kW}\left[\int dt'd\bar{s}''\,K_{\bar{s}}(s;\Sigma-s-\bar{s},t',\Sigma-s-\bar{s}'')\right.$$
$$\left.\times l_i(s;\Sigma-s-\bar{s},t',\Sigma-s-\bar{s}'')\{G_{i;t\bar{s}}^{(\pm)}(s;t',\bar{s}'')+\text{H.c.}\}\right]; \tag{5.3}$$

where

$$K_s(s;x_1,x_2,x_3)=[x_1^2+x_2^2+x_3^2-2(x_1x_2+x_2x_3+x_3x_1)-(x_1x_2x_3/k^2)]^{-\frac{1}{2}}\theta(x_1-x_{1+}), \tag{5.4}$$

and

$$K_{\bar{s}}(s;x_1,x_2,x_3)=-[x_1^2+x_2^2+x_3^2-2(x_1x_2+x_2x_3+x_3x_1)-(x_1x_2x_3/k^2)]^{-\frac{1}{2}}\theta(x_{1-}-x_1), \tag{5.5}$$

with

$$x_{1\pm}=\{[x_2(1+x_3/4k^2)]^{\frac{1}{2}}\pm[x_3(1+x_2/4k^2)]^{\frac{1}{2}}\}^2, \tag{5.6}$$

and $G_{i;\lambda\mu}^{(\pm)}(s;x,y)$ are bilinear combinations of absorptive parts[13] defined by

$$G_{1;\lambda\mu}^{(+)}(s;x,y)=A_\lambda^{*(+)}(s,x)A_\mu^{(+)}(s,y)+2A_\lambda^{*(-)}(s,x)A_\mu^{(-)}(s,y), \tag{5.7}$$

$$G_{2;\lambda\mu}^{(+)}(s;x,y)=A_\lambda^{*(+)}(s,x)B_\mu^{(+)}(s,y)+2A_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y)$$
$$=G_{3;\lambda\mu}^{*(+)}(s;y,x), \tag{5.8}$$

$$G_{4;\lambda\mu}^{(+)}(s;x,y)=B_\lambda^{*(+)}(s,x)B_\mu^{(+)}(s,y)+2B_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y), \tag{5.9}$$

$$G_{1;\lambda\mu}^{(-)}(s;x,y)=A_\lambda^{*(-)}(s,x)A_\mu^{(+)}(s,y)+A_\lambda^{*(+)}(s,x)A_\mu^{(-)}(s,y)+A_\lambda^{*(-)}(s,x)A_\mu^{(-)}(s,y), \tag{5.10}$$

$$G_{2;\lambda\mu}^{(-)}(s;x,y)=A_\lambda^{*(-)}(s,x)B_\mu^{(+)}(s,y)+A_\lambda^{*(+)}(s,x)B_\mu^{(-)}(s,y)+A_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y),$$
$$=G_{3;\lambda\mu}^{*(-)}(s;y,x); \tag{5.11}$$

$$G_{4;\lambda\mu}^{(-)}(s;x,y)=B_\lambda^{*(-)}(s,x)B_\mu^{(+)}(s,y)+B_\lambda^{*(+)}(s,x)B_\mu^{(-)}(s,y)+B_\lambda^{*(-)}(s,x)B_\mu^{(-)}(s,y); \tag{5.12}$$

and the l_i's are kinematical factors given by

$$l_1(s;t,t',t'')=1+\frac{(t'+t''-t)(s+\mu^2-m^2)}{4[(m^2-\mu^2)^2-s\bar{s}]}, \tag{5.13}$$

$$l_2(s;t,t',t'')=l_3(s;t,t'',t')=\frac{(s-m^2-\mu^2)(t'-t''+t)}{4mt}+\frac{m(t-t'-t'')(s+\mu^2-m^2)}{4[(m^2-\mu^2)^2-s\bar{s}]}, \tag{5.14}$$

and

$$l_4(s;t,t',t'')=\frac{(t-t'-t'')(s-m^2)(s+\mu^2-m^2)}{4[(m^2-\mu^2)^2-s\bar{s}]}. \tag{5.15}$$

The corresponding expressions for the strip functions $\beta_1^\pm(s,t)$ and $\beta_3^\pm(\bar{s},s)$ are obtained from Eqs. (5.2) and (5.3) by replacing the kinematical factors l_i therein by m_i, defined by

$$m_1(s;t,t',t'')=\frac{(s+m^2-\mu^2)(t-t'-t'')}{4m[(m^2-\mu^2)^2-s\bar{s}]}, \tag{5.16}$$

$$m_2(s;t,t',t'')=m_3(s;t,t'',t')=\frac{t-t'+t''}{2t}-\frac{(t-t'-t'')(s+m^2-\mu^2)}{4[(m^2-\mu^2)^2-s\bar{s}]}, \tag{5.17}$$

$$m_4(s;t,t',t'')=\frac{s-m^2-\mu^2}{2m}-\frac{(s-m^2)(s+m^2-\mu^2)(t-t'-t'')}{4m[(m^2-\mu^2)^2-s\bar{s}]}. \tag{5.18}$$

B. Channel $\pi+\pi\rightarrow N+\bar{N}$

We have the following expressions for s, $\bar{s}$, t in this channel:

$$\begin{aligned} s&=-q^2-p^2+2pq\zeta,\\ \bar{s}&=-q^2-p^2-2pq\zeta,\\ t&=4(q^2+\mu^2)=4(p^2+m^2)\equiv W_t^2, \end{aligned} \tag{5.19}$$

where $q=|\mathbf{q}_1|=|\mathbf{q}_2|$ is the magnitude of the momentum of the pions, $p=|\mathbf{p}_1|=|\mathbf{p}_2|$ is the magnitude of the momentum of the nucleons, and $\zeta=(\hat{q}_1\cdot\hat{p}_1)$.

In writing the generalized unitarity condition for this channel we shall need the S-matrix element for $\pi-\pi$ scattering. We shall define this with the same normalization as in Chew and Mandelstam,[3] viz.,

$$\langle q_1'q_2'|S|q_1q_2\rangle=\langle q_1'q_2'|q_1q_2\rangle+\frac{2i(2\pi)^5\delta^{(4)}(q_1'+q_2'-q_1-q_2)}{(q_{10}'q_{20}'q_{10}q_{20})^{\frac{1}{2}}}\langle q_1'q_2'|\mathcal{A}|q_1q_2\rangle. \tag{5.20}$$

With this normalization, $\mathcal{A}$ has the partial-wave expansion

$$\mathcal{A}=\frac{(q^2+\mu^2)^{\frac{1}{2}}}{q}\sum_l(2l+1)e^{i\delta_l}\sin\delta_l P_l(\hat{q}_1\cdot\hat{q}_1'). \tag{5.21}$$

[13] Note that our notation for the absorptive parts differs somewhat from that in CF.

The generalized unitarity condition with only 2π intermediate states then gives,

$$\mathrm{Im}A(t,\zeta)=\frac{q}{2\pi W_t}\int d\Omega'\left[A^*(t,\zeta'')-\frac{mq}{p}\frac{\zeta''-\zeta\zeta'}{1-\zeta^2}B^*(t,\zeta'')\right]\mathcal{A}(t,\zeta'), \tag{5.22}$$

and

$$\mathrm{Im}B(t,\zeta)=\frac{q}{2\pi W_t}\int d\Omega'\left[\frac{\zeta'-\zeta\zeta''}{1-\zeta^2}B^*(t,\zeta'')\mathcal{A}(t,\zeta')\right], \tag{5.23}$$

where (see Fig. 4)

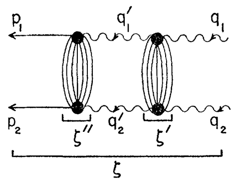

FIG. 4. The two-particle intermediate state for the $\pi+\pi\rightarrow N+\bar{N}$ channel.

$$\zeta=(\hat{q}_1\cdot\hat{p}_1),\quad \zeta'=(\hat{q}_1\cdot\hat{q}_1'),\quad \zeta''=(\hat{q}_1'\cdot\hat{p}_1),$$

and

$$d^3q_1'=q_1'^2dq_1'd\Omega'.$$

These equations hold separately for the $T=0$ and $T=1$ amplitudes and lead to the following expressions for the strip functions $\alpha_2^{(\pm)}$, $\beta_2^{(\pm)}$:

$$\begin{aligned}\alpha_2^{(\pm)}(t,s)=\frac{2q}{\pi W_t}\Bigg[\int\frac{ds'}{2q^2}\frac{ds''}{2pq}K_t(t;s,s',s'')\{A_s^{*(\pm)}(t,s'')\mathcal{A}_s^{(\epsilon)}(t,s')-n_\alpha(t;s,s',s'')B_s^{*(\pm)}(t,s'')\mathcal{A}_s^{(\epsilon)}(t,s')\}\\ +\int\frac{d\bar{s}'}{2q^2}\frac{d\bar{s}''}{2pq}K_t(t;s,4\mu^2-t-\bar{s}',\Sigma-t-\bar{s}'')\\ \times\{A_{\bar{s}}^{*(\pm)}(t,\bar{s}'')\mathcal{A}_{\bar{s}}^{(\epsilon)}(t,\bar{s}')-n_\alpha(t;s,4\mu^2-t-\bar{s}',\Sigma-t-\bar{s}'')B_{\bar{s}}^{*(\pm)}(t,\bar{s}'')\mathcal{A}_{\bar{s}}^{(\epsilon)}(t,\bar{s}')\}\Bigg],\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\beta_2^{(\pm)}(t,s)=\frac{2q}{\pi W_t}\Bigg[\int\frac{ds'}{2q^2}\frac{ds''}{2pq}K_t(t;s,s',s'')n_\beta(t;s,s',s'')B_s^{*(\pm)}(t,s'')\mathcal{A}_s^{(\epsilon)}(t,s')\\ +\int\frac{d\bar{s}'}{2q^2}\frac{d\bar{s}''}{2pq}K_t(t;s,4\mu^2-t-\bar{s}',\Sigma-t-\bar{s}'')n_\beta(t;s,4\mu^2-t-\bar{s}',\Sigma-t-\bar{s}'')B_{\bar{s}}^{*(\pm)}(t,\bar{s}'')\mathcal{A}_{\bar{s}}^{(\epsilon)}(t,\bar{s}')\Bigg],\end{aligned} \tag{5.25}$$

where $(\epsilon)=(0)$ for $\alpha_2^{(+)}$ and $\beta_2^{(+)}$; $(\epsilon)=(1)$ for $\alpha_2^{(-)}$ and β_2^{-}; and

$$K_t(t;x,y,z)=\left\{\left(\frac{x+p^2+q^2}{2pq}\right)^2+\left(1+\frac{y}{2q^2}\right)^2+\left(\frac{z+p^2+q^2}{2pq}\right)^2-1-2\left(\frac{x+p^2+q^2}{2pq}\right)\left(1+\frac{y}{2q^2}\right)\left(\frac{z+p^2+q^2}{2pq}\right)\right\}^{-\frac{1}{2}} \tag{5.26}$$

if the quantity under the square root is positive, and zero otherwise, and n_α, n_β are kinematical factors given by

$$n_\alpha(t;s,s',s'')=\frac{m[2q^2(s''-s)-s'(s+p^2+q^2)]}{4p^2q^2-(s+p^2+q^2)^2}, \tag{5.27}$$

$$n_\beta(t;s,s',s'')=\frac{4p^2q^2[1+s'/2q^2]-[ss''+(p^2+q^2)(s+s'')+(p^2+q^2)^2]}{4p^2q^2-(s+p^2+q^2)^2}. \tag{5.28}$$

Equations (4.1) through (4.7) for the absorptive parts, the crossing relations, and Eqs. (5.2) through (5.18) and (5.24) through (5.28) for the strip functions, together with the known dispersion relations for the low partial-wave absorptive parts, constitute the basic equations for the $\pi-N$ problem in this approach.

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Hypervirial Theorems for Variational Wave Functions*

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It is shown that a sufficient condition for an optimal energy variational wave function ψ_0 to satisfy the hypervirial relation $(\psi_0,[H,W]\psi_0)=0$ is for the trial function ψ to admit variations of the form $\partial\psi/\partial a = (i/\hbar)W\psi$. Here H is the Hamiltonian, W is a Hermitian operator, and a is a variational parameter. Explicit forms of such trial functions are exhibited for several W's. The case in which W generates a point transformation of the coordinates is discussed in detail. Conditions are given for the existence of simultaneous hypervirial theorems.

I. INTRODUCTION

THE diagonal elements (in the energy representation) of the Heisenberg equations of motion[1] are called the hypervirial relations.[2] If χ is a (bound state) eigenfunction of a Hamiltonian H and if W (which henceforth is assumed to be Hermitian) is a time-independent operator, the hypervirial theorem for W states that

$$(\chi,[H,W]\chi)=0, \tag{1}$$

where $[H,W]\equiv HW-WH$ is the commutator of H and W. Physically, this is, of course, just the statement that, for a stationary state, the expectation value of W is independent of time.[3] For a particular choice of W, Eq. (1) yields the familiar virial theorem.[4] For other choices of W, the hypervirial relations lead to generalizations of the virial theorem.

It is well known that if a parameter is introduced into an *approximate*[5] wave function ψ in such a manner that all distances are scaled, and if the parameter is varied so as to obtain the optimum energy, then the corresponding optimal function ψ_0 satisfies the virial theorem.[6] Analogously, we show that, under certain conditions, it is possible to introduce a parameter into a trial function ψ so that the variationally determined approximate wave function ψ_0 satisfies the hypervirial theorem

$$(\psi_0,[H,W]\psi_0)=0. \tag{2}$$

The general plan of this paper is as follows: In Sec. II the conditions are derived in a formal manner. In Secs. III and IV these conditions are put into explicit form for certain special W's. In Sec. V the satisfaction of simultaneous hypervirial relations is discussed, and in Sec. VI possible applications and extensions of our results are considered. For simplicity of presentation, Cartesian coordinates are used throughout the main body

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[1] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 140.

[2] J. O. Hirschfelder, J. Chem. Phys. **33**, 1762 (1960).

[3] More formally, Eq. (1) is also a partial expression of the fact that the eigenvalues of H are invariant to unitary transformation. Namely, if we subject H to the unitary transformation generated by W, then the first-order change in H is proportional to $i[H,W]$. Equation (1) then correctly tells us that the first-order energy shift vanishes.

[4] In reference 2 it is shown that if $W=\frac{1}{2}\sum_i(x_ip_i+p_ix_i)$, where the x_i are the Cartesian coordinates of the system and the p_i are the corresponding momentum operators, Eq. (1) is a statement of the quantum mechanical virial theorem, originally derived by M. Born, W. Heisenberg, and F. Jordan [Z. Physik **35**, 557 (1925)] and again by J. C. Slater [J. Chem. Phys. **1**, 687 (1933)].

[5] Throughout this paper, all approximate wave functions are assumed to satisfy the continuity-boundary conditions required of physically acceptable bound stationary state wave functions: (1) The function must be single-valued and analytic in all of its variables at every point in configuration space where the potential energy is analytic. (2) The function and its first derivatives must be absolutely and quadratically integrable over the whole of configuration space. (3) The function must vanish at infinity faster than any negative power of the Cartesian coordinates. See E. C. Kemble, *Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), Sec. 32.

[6] E. A. Hylleraas, Z. Physik **54**, 347 (1929); V. Fock, *ibid.* **63**, 855 (1930); J. O. Hirschfelder and J. F. Kincaid, Phys. Rev. **32**, 658 (1937); and P. O. Löwdin, *Advances in Chemical Physics*, edited by I. Prigogine (Interscience Publishers, Inc., New York, 1959), Vol. II. p. 219.