Transfer of Helicity in Radiation and Absorption of High-Energy Photons*

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Nearly complete transfer of momentum between a high-energy electron (or positron) and a photon in a Coulomb field implies that helicity is also transferred. This is not a consequence of conservation of total angular momentum but, rather, of spin angular momentum, and follows from a demonstration that it is possible to use free-particle spinors (though not free wave functions) for the high-energy particles. Polarization correlations of the lower energy particle in such a process are discussed. Applications are made to bremsstrahlung, pair production, photoeffect, and one-photon pair annihilation.

WE wish to show that, in a Coulomb field, almost-complete transfer of momentum between a high-energy electron (or positron) and a photon implies that helicity is also transferred. By this, we mean complete correlation between right (or left) circular polarization of the photon and forward (or backward) longitudinal polarization of the electron. Although the high-energy particles in such a process are nearly collinear, the result does *not* follow from conservation of total angular momentum J_z along that direction. The z component of total angular momentum for the lower energy electron or positron is *not* completely correlated. Rather, the result can be interpreted as a consequence of conservation of spin angular momentum σ_z. We shall give a general proof of these statements and later indicate where they extend the results obtained for specific processes, such as bremsstrahlung at the "tip",[1,2] pair production in which one member of the pair takes all the energy,[1] and atomic photoeffect.[2–6]

The argument consists in (1) noting that use of a free-particle spinor for the high-energy electron will result in conservation of helicity and (2) showing that for these processes the spinor (though not the complete wave function) may indeed be approximated by the free-particle form. When this approximation is made, the spinor-dependence of the matrix element (or its complex conjugate) is contained in one of the forms

$$u^*(\mathbf{p})\boldsymbol{\alpha}\cdot\mathbf{e}\psi(r), \qquad \psi^*(r)\boldsymbol{\alpha}\cdot\mathbf{e}v(\mathbf{p}), \tag{1}$$

where u and v are free electron and positron spinors, and ψ represents the wave function of the lower energy particle (bound or continuum). For photon states of right (left) circular polarization, $\boldsymbol{\alpha}\cdot\mathbf{e}$ simply applies raising (lowering) operators $\sigma_\pm=\sigma_x\pm i\sigma_y$ to the free spinors, and it is easy to see that the matrix elements vanish unless the photon and the high-energy electron (or positron) have the same helicity, regardless of the structure of ψ. (With a slight generalization of this argument one can understand the transfer of helicity to the higher energy photon in two-photon annihilation of a fast positron.) This conclusion has used the near collinearity of the particles. If the incident energy is E,[7] the relevant deviations from collinearity are $O(1/E)$. The deviations from complete persistence of helicity will be of the same order, unless the matrix elements are suppressed owing to the near collinearity, and we shall later see that this does not happen.

The solution of the Dirac equation for an electron in the potential V may be approximated by a Sommerfeld-Maue (SM) type wave function[8] explicitly displaying the spinor dependence which is our present concern:

$$\psi_{\mathrm{SM}}=e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2E}\right)Fu(\mathbf{p}), \tag{2}$$

where F is a solution of

$$(\nabla^2+2i\mathbf{p}\cdot\boldsymbol{\nabla}-2EV)F=0, \tag{3}$$

chosen to satisfy the desired boundary conditions. For a $1/r$ potential, ψ_{SM} may be obtained from the exact solution with the neglect of $O(1/l)$ in each partial wave.[9,10] The important values of l are expected to be $O(ER)$ and the error in ψ_{SM}, $O(1/ER)$, where R represents the region of r space important for the process. A similar conclusion can be obtained for a screened potential.[11] To determine R, note $\boldsymbol{\Delta}\cdot\mathbf{r}=O(1)$, where $\boldsymbol{\Delta}$ is the momentum transfer to the nucleus, if there is to be an important contribution to the matrix element. One may show (from conservation of energy) that these high-energy processes are not possible without momentum transfer; $\Delta_z=O(1/\epsilon)$, where ϵ is the energy of the slower particle in the process. Thus the important r's are $O(\epsilon)$ and the error in using ψ_{SM} is $O(1/\epsilon E)$.

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[1] H. Olsen and L. C. Maximon, Phys. Rev. **114**, 887 (1959), which contains references to earlier work.

[2] U. Fano, K. W. McVoy, and J. R. Albers, Phys. Rev. **116**, 1159 (1959). This paper gives a detailed discussion, in Born approximation, of mechanisms of radiation.

[3] H. Olsen, Kgl. Norske Videnskab. Selskabs. Forh. **31**, No. 11 (1958).

[4] H. Banerjee, Nuovo cimento **11**, 220 (1959).

[5] U. Fano, K. W. McVoy, and J. R. Albers, Phys. Rev. **116**, 1147 (1959).

[6] B. Nagel, Arkiv Fysik **18**, 1 (1960).

[7] We use unrationalized units and set $\hbar=c=m_e=1$; $O(x)$ shall mean "of order x."

[8] H. Olsen, L. C. Maximon, and H. Wergeland, Phys. Rev. **106**, 27 (1957).

[9] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954). The error is actually $O(a^2/l)$, where $a\equiv Ze^2$.

[10] R. H. Boyer, Phys. Rev. **117**, 475 (1960), also neglecting $O(1/Er)$.

[11] R. H. Pratt, Phys. Rev. **117**, 1017 (1960), hereafter designated as I.

An electron wave function in the potential V hence has the spinor $u(\mathbf{p})$ of a free electron, except for corrections of $O(1/E)$ [the $\boldsymbol{\alpha}\cdot\nabla/2E$ term of Eq. (2)] and $O(1/\epsilon E)$. However, the matrix element with the free spinor will be suppressed: Whenever all three particles have relativistic momenta in similar directions, the spinor factor $u_f^*\boldsymbol{\alpha}\cdot\mathbf{e}u_i$ is reduced to $O(1/\epsilon)$.[9] Hence, in general, the free-spinor term of ψ_{SM} contributes $O(1/\epsilon)$, the second term $O(1/E)$, and the neglected remainder $O(1/\epsilon E)$. When $\epsilon \ll E$ the electron spinor may be approximated by $u(\mathbf{p})$ and helicity persists, neglecting $O(\epsilon/E)$. This completes the proof.

Conservation of angular momentum provides an obvious, if incorrect, explanation for persistence of helicity. The high-energy particles are nearly collinear, and so to conserve J_z an electron with $j_z=\pm\frac{1}{2}$ must be correlated with a photon of $j_z=\pm 1$ (assuming the lower energy particle is also in a $j=\frac{1}{2}$ state). This argument, exact when the high-energy particles are exactly collinear, also predicts complete correlation with the j_z of the lower energy particle. However, we shall make an explicit calculation and find only partial correlations. How do deviations from collinearity $O(1/E)$ cause deviations $O(1)$ in correlations of low-energy particles? Apply the electron operator j_z^{op} to ψ_i in the typical matrix element $\psi_f^*\boldsymbol{\alpha}\cdot\mathbf{e}e^{i\mathbf{k}\cdot\mathbf{r}}\psi_i$ and then bring it through to act on ψ_f. For photon states $\mathbf{e}$ of opposite helicities, electron states j_z and $j_z\pm 1$ are connected, thus completely correlating all three particles. This assumes j_z^{op} (taken along the high-energy electron) commutes with $e^{i\mathbf{k}\cdot\mathbf{r}}$, which is true only if photon and electron are precisely collinear. Otherwise j_z^{op} will fail to commute with a term $e^{i\mathbf{k}_\perp\cdot\mathbf{r}}$, which can be $O(1)$, and so the angular momentum argument fails.

A correct explanation for persistence of helicity is provided by conservation of spin angular momentum. Apply the electron spin operator σ_z^{op} as above. This operator does commute with $e^{i\mathbf{k}\cdot\mathbf{r}}$, so it is true that for opposite photon helicites σ_z is connected with $\sigma_z\pm 1$. The high-energy electron wave function may, according to our earlier discussion, be considered an eigenstate of σ_z^{op}. The low-energy electron is not an eigenstate of σ_z^{op}, but is some combination of $\sigma_z\pm\frac{1}{2}$ (even if $j>\frac{1}{2}$). Hence there is complete correlation of electron σ_z's with photon helicity, but only for the high-energy electron does this mean a complete correlation of j_z. In summary: Spin angular momentum (unlike total angular momentum) predicts complete transfer of helicity between high-energy particles, regardless of the low-energy state (not just for $j=\frac{1}{2}$). Low-energy σ_z (rather than j_z) states are also completely correlated. In the precise forward direction, where conservation of J_z is also valid, the low-energy particle is completely correlated with $j_z=\pm\frac{1}{2}$ even if $j>\frac{1}{2}$, and hence other substates cannot contribute[12].

We now make an explicit calculation of polarization correlations. First we obtain all possible correlations of the high-energy particles, including the helicity correlations as a special case. Then we introduce specific low-energy states and determine correlations with the angular momentum j_z of the low-energy particle.

The complete polarization dependence of the cross section for a process described by the first term of Eq. (1) is contained in

$$\psi^*(r')\Omega\psi(r),\quad \Omega=\boldsymbol{\alpha}\cdot\mathbf{e}^*u(\mathbf{p})u^*(\mathbf{p})\boldsymbol{\alpha}\cdot\mathbf{e},\tag{4}$$

and the polarization of the high-energy particles is in the operator Ω. We characterize these polarizations in a standard way. In two-component notation

$$u(\mathbf{p})=\begin{pmatrix} u_A \\ \sigma_3 u_A\end{pmatrix},\quad u_A^*u_A=1,\quad \boldsymbol{\alpha}\cdot\mathbf{e}=\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\mathbf{e} \\ \boldsymbol{\sigma}\cdot\mathbf{e} & 0\end{pmatrix},\tag{5}$$

where the z axis has been taken along the photon, and the $O(1/E)$ deviations from collinearity are neglected. Then

$$u_Au_A^*=\tfrac{1}{2}(1+\boldsymbol{\zeta}\cdot\boldsymbol{\sigma}),\tag{6}$$

where $\boldsymbol{\zeta}$ is the spin vector of the electron in its rest system. For the photon write

$$\boldsymbol{\sigma}\cdot\mathbf{e}=\sigma_1e_1+\sigma_2e_2,\quad e_1^*e_1+e_2^*e_2=1,\tag{7}$$

and define

$$\xi_1=e_1^*e_1-e_2^*e_2,\quad \xi_2=e_1e_2^*+e_2e_1^*,\quad \xi_3=i(e_1e_2^*-e_2e_1^*).\tag{8}$$

The operator Ω of Eq. (4) may then be written

$$\Omega=\begin{pmatrix} 1 & -\sigma_3 \\ -\sigma_3 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2}(A-B) & 0 \\ 0 & \frac{1}{2}(A+B)\end{pmatrix},\tag{9}$$

where

$$\begin{aligned} A&=(1+\zeta_3\xi_3)-(\zeta_3+\xi_3)\sigma_3,\\ B&=(\zeta_1\xi_1+\zeta_2\xi_2)\sigma_1+(\zeta_1\xi_2-\zeta_2\xi_1)\sigma_2.\end{aligned}\tag{10}$$

For helicity states, only ζ_3 and ξ_3 are nonzero, $B=0$, and the coefficients $(1+\zeta_3\xi_3)$ and $(\zeta_3+\xi_3)$ of A also vanish unless electron and photon have the same helicity. No other correlations allowed by (10) are complete. The same analysis may be made for a process described by the second term of Eq. (1). In Eq. (4) $r\leftrightarrow r'$, $\mathbf{e}\leftrightarrow\mathbf{e}^*$, and $u\rightarrow v$. For the operator Ω these have the consequences $\xi_3\rightarrow-\xi_3$, $\boldsymbol{\zeta}\rightarrow-\boldsymbol{\zeta}$.

Further results depend on the choice of low-energy wave function, but they are still quite general. In this paper we shall calculate K-shell photoeffect through $O(a\equiv Ze^2)$, but the results are also valid for L_{I} photoeffect,[13] for one-photon annihilation of fast positrons,[11] and for high-energy bremsstrahlung and pair production

[12] These forward cross sections are finite, even for processes, such as K-shell photoeffect, for which they vanish to lowest order in $a\equiv Ze^2$. See the recent work of Nagel (reference 6), and of K. Mork and H. Olsen, Proceedings of the Physics Seminar in Trondheim, No. 5, 1960.

[13] R. H. Pratt, Phys. Rev. 119, 1619 (1960).

involving a low-energy electron with $\epsilon-1=O(a^2)$.[14] We obtain the polarization correlations of the total cross section, for a pure Coulomb potential, using methods previously developed.[11] Then in Eq. (I-19),

$$F(r,r')=\psi^*(r)\Omega\psi(r'). \tag{11}$$

Portions of F now depend on the direction $\hat{p}$, but these vanish in the integration. Eq. (I-22) is (except for normalization) to be replaced by

$$(c+d)\left\{1+\left(\frac{1-\epsilon}{1+\epsilon}\right)\left[\cos\theta'\cos\theta+\left(\frac{c-d}{c+d}\right)\sin\theta'\sin\theta\right]\right.$$
$$\left.+i\left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{1}{2}}(\cos\theta-\cos\theta')\right\}, \tag{12}$$

where

$$c=\tfrac{1}{2}(1+\zeta_3\xi_3),\quad d=\pm\tfrac{1}{2}(\zeta_3+\xi_3)=j_z(\zeta_3+\xi_3). \tag{13}$$

The sign of d corresponds to the sign of angular momentum j_z in the bound state. There is no dependence on transverse electron polarization or linear photon polarization. Integrating, the cross section is proportional to

$$[\tfrac{2}{3}-(4\pi a/15)](c+d)+\tfrac{1}{3}(c-d). \tag{14}$$

If the high-energy particles are in the same pure helicity states, then $c=1$ and $d=\pm1$ according as j_z for the bound state is correlated (conservation of J_z) or anticorrelated. For small Z the electron spin will flip only two-thirds of the time, and in heavier elements the non-spin-flip term may even predominate.

[14] R. H. Pratt, Phys. Rev. **120**, 1717 (1960).

We now briefly summarize previous results for specific processes and indicate where they have been extended. Perhaps the most general discussion of polarization correlations is that given by Olsen and Maximon[1] for bremsstrahlung and pair production. Sommerfeld-Maue type wave functions are used for both high- and low-energy particles, so the calculations are valid to all orders in a general screened potential, but require that the lower energy particle be extremely relativistic. This restriction is removed in the present work. The careful discussion of Fano, McVoy, and Albers,[2] valid to lowest order in a pure Coulomb potential, should also be noted, as it is also applicable to the high-frequency limit of the spectrum and to K-shell photoeffect. In another paper[5] the same authors explicitly give the correlations, to lowest order, of the high-energy particles in the photoeffect; these were also obtained in the work of Olsen.[3] Relevant calculations of K-shell photoeffect, including $O(a)$, have been made by Nagel[6] and Banerjee,[4] the latter author also discussing one-photon positron annihilation. The present paper extends these results to arbitrary shells and to all orders in a general screened potential. Correlations with the low-energy particle in the process do not appear to have previously been considered.

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