Analyticity of Amplitudes and Separable Potentials

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Expressions for the partial scattering amplitudes from nonlocal separable potentials are written down in the form of dispersion relations. These relations are automatically expressible in the N/D form discussed by Chew and Mandelstam. Criteria for "acceptable" separable potential shapes are discussed. The relation between "local" and separable potentials is clarified with the help of a concrete potential shape which conforms to the above criteria. With such a potential shape the physical meaning of the "range of the interaction" in terms of separable potentials becomes clearer.

As an elementary application of such "analytic" potentials, the low-energy 2-body parameters are evaluated.

1. INTRODUCTION

WHILE the dispersion relations approach to physical problems was originally confined almost entirely to the framework of quantum field theory, it has in more recent times proved equally popular in the theory of potential scattering. The essential simplicity of potential scattering compared with field theory has enabled some authors[1,2] to give fairly rigorous demonstrations of the validity of the Mandelstam representation for the former, though its validity for the corresponding field-theoretical problem has still remained largely at the conjectural stage at which its author left it.[3]

Specifically, the "proofs" of double dispersion relations for potential scattering have so far been confined to local Yukawa-type potentials.[1,2] Since, however, there is no occasion in these "proofs" to make explicit use of the causality condition, it leaves unanswered the question as to whether the "locality" of the potential is connected with causality in such an essential manner as to appear as a *necessary* condition for the validity of dispersion relations. On the other hand, it has been suggested[4] that amplitudes arising from interactions of finite range in field theory have properties partly analogous to those of *single partial wave* amplitudes with a *nonlocal* potential. It should therefore be of some interest to study the analyticity properties of scattering amplitudes deduced directly from nonlocal potentials of various shapes and in particular see if and for what types of such potentials, they satisfy the Mandelstam representation. Now, from a practical point of view, the most useful manifestation of the Mandelstam representation is provided by the dispersion relations for the partial waves amplitudes (A_l). Thus the nonlocal potential which naturally suggests itself for earliest study is the separable variety. Though this model is rather trivial, being exactly soluble, its very solubility can be exploited to examine all aspects of the problem of analyticity of amplitudes in the simplest possible manner, and deduce the possible potential shapes for which the individual A_l's satisfy dispersion relations.

This note was motivated partly by these considerations and partly by a desire on our part to conform to the discipline of dispersion relations within the framework of the separable potential approach to problems in the low-energy region which we have been pursuing for some time.[5,6]

In Sec. 2 we collect the results for partial wave amplitudes in the formal shape of dispersion relations and, through the latter, examine the restrictions that must be imposed on possible potential shapes. The resemblance of the results obtained with this separable potential model to the N/D representation of Chew and Mandelstam[7] is noted.

In Sec. 3, we discuss the idea of an "equivalent separable potential" corresponding to a local potential, with the help of a special separable potential form which satisfies the necessary analyticity conditions deduced from Sec. 2. Some practical aspects of working with such "analytic" potentials are also discussed, and as an elementary application, the low-energy two-body parameters are evaluated.

2. DISPERSION REPRESENTATION FOR $A_l(s)$

In the notation of an earlier paper,[5] consider a central separable potential of the form

$$(\mathbf{p}|V|\mathbf{p}')=\sum_l -\frac{\lambda_l}{M}(2l+1)v_l(p)v_l(p')P_l(\hat{p}\cdot\hat{p}'). \quad (1)$$

The corresponding T matrix which satisfies the integral equation

$$(\mathbf{p}|T|\mathbf{k})=(\mathbf{p}|V|\mathbf{k})+M\int\frac{(\mathbf{p}|V|\mathbf{q})(\mathbf{q}|V|\mathbf{k})}{k^2+i\epsilon-q^2}d\mathbf{q}, \quad (2)$$

has the solution

$$(\mathbf{p}|T|\mathbf{k})=\sum_l -\frac{\lambda_l}{M}(2l+1)D_l^{-1}(k^2+i\epsilon)\times v_l(p)v_l(k)P_l(\hat{p}\cdot\hat{k}), \quad (3)$$

[1] R. Blankenbecler *et al.*, Ann. Phys. **10**, 62 (1960).
[2] A. Klein, J. Math. Phys. **1**, 41 (1960).
[3] See, however, R. J. Eden, Phys. Rev. **120**, 1514 (1960).
[4] M. A. Ruderman and S. Gasiorowicz, Nuovo cimento **8**, 861 (1958).
[5] A. N. Mitra and V. L. Narasimham, Nuclear Phys. **14**, 407 (1960).
[6] A. N. Mitra and S. P. Pandya, Nuclear Phys. **20**, 455 (1960).
[7] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

where

$$D_l(s)=1-\lambda_l J_l(s). \quad (4)$$

and

$$J_l(s)=\int d\mathbf{q}\, v_l^2(q)(q^2-s)^{-1}. \quad (5)$$

Here s will be regarded as a complex variable which, in the standard manner, takes the value $k^2+i\epsilon$ on the energy shell. The scattering amplitude $f(k^2, \cos\theta)$ is expressed by the normalization

$$f(k^2,\cos\theta)=-2\pi^2 M(\mathbf{p}|T|\mathbf{k})_{(p=k)} \quad (6)$$

$$=\sum_l (2l+1)A_l(k^2)P_l(\cos\theta), \quad (7)$$

where the partial scattering amplitude,

$$A_l(s)=s^{-\frac{1}{2}}e^{+i\delta_l(s)}\sin\delta_l(s), \quad (8)$$

is given from comparison with (3) by

$$A_l(s)=N_l(s)/D_l(s), \quad (9)$$

$$N_l(s)=2\pi^2\lambda_l v_l^2(s^{\frac{1}{2}}). \quad (10)$$

Before considering the properties of $A_l(s)$ defined by (9) and (10), we recall the analytic structure of $A_l(s)$ obtained with a local potential approach,[2] viz., (a) $A_l(s)$ has a branch cut from $s=0$ to ∞ (physical region); (b) it has a finite number of poles for real $s<0$, corresponding to one or more bound states; (c) it has a series of branch cuts to the left of these poles, represented by the points $-\frac{1}{4}n^2\mu^2$, $n=1, 2, 3$, where μ^{-1} is the range of the corresponding Yukawa potential; (d) it has no other singularities and tends to zero as $s\rightarrow\infty$.

The position of the nearest left-hand branch cut, $s_0=-\frac{1}{4}\mu^2$, allows a direct interpretation of the inverse range parameter μ of the potential as the value of $(-4s_0)^{\frac{1}{2}}$.

It is now seen that property (a) of our $A_l(s)$ derives from the representation (5) for $D_l(s)$. Property (b) is also satisfied since that only zero of $D_l(s)$ is given by the solution of the eigenvalue equation

$$\lambda_l^{-1}=\int d\mathbf{q}\, v_l^2(q)(\alpha_l^2+q^2)^{-1}, \quad (11)$$

corresponding to the point $s=-\alpha_l^2$. The corresponding bound state with binding energy α_l^2/M exists only if λ_l is positive and large enough to allow for a real solution α_l from Eq. (11).

As for properties (c) and (d), these are dependent on the explicit shape of $v_l(p)$. The potential shapes considered by Yamaguchi[8] or in our previous work[5,6,9] were

$$v_l(p)=p^l(\beta_l^2+p^2)^{-(l+2)/2} \quad\text{and}\quad p^l(\beta_l^2+p^2)^{-(l+1)/2}. \quad (12)$$

[8] Y. Yamaguchi, Phys. Rev. **95**, 1628, 1635 (1954).
[9] A. N. Mitra and J. H. Naqvi, Nuclear Phys. (to be published).

Unfortunately these shapes, while maintaing correct behavior for $p^2=s>0$, give, according to (10), functions $N_l(s)$ which are too strongly singular for branch cut behavior at the left-hand singularity $s=-\beta_l^2$. Rather these singularities are poles of higher orders increasing with l. Thus a heuristic interpretation of β_l as the "inverse range" of the interaction, (e.g., by observing that for $l=0$ the wave function has a Hulthén type structure[8] $r^{-1}[\exp(-\alpha_0 r)-(\exp-\beta_0 r)]$ does not fit in with the analytic behavior of the corresponding $A_l(s)$ implied in property (c). On the other hand, we can now make use of property (c) to have a concrete criterion for the choice of separable potential shapes which lead to the correct analytic behavior of $A_l(s)$. Indeed, what we need of $v_l(p)$ are the following.

(A) $v_l^2(s^{\frac{1}{2}})$ is a positive definite analytic function of s, vanishing as s^l for $s\rightarrow 0$.

(B) It has a branch cut behavior from $s=-\infty$ to $s=-\frac{1}{4}\beta_l^2$, where $\frac{1}{4}\beta_l^2>\alpha_l^2$.

(C) It tends to zero as $s\rightarrow\infty$, preferably as fast as the work with local potentials suggest,[2] viz., $\sim s^{-1}\ln s$ as $s\rightarrow\infty$.

These conditions are sufficient to ensure the correct analytic behavior of $A_l(s)$. We can largely incorporate these conditions in the "spectral representation,"

$$v_l^2(s^{\frac{1}{2}})=f_l(s)=\int_{-\infty}^{-\frac{1}{4}\beta_l^2} ds'\frac{\rho_l(s')}{s'(s'-s)}, \quad (13)$$

where the "spectral function" $\rho_l(s')$ should be an analytic function bounded at ∞. It need not be positive definite by itself but must be so chosen as to lead to condition (A) for $f_l(s)$.

We are unable to find the most general class of functions $\rho_l(s')$ of Eq. (13), but certain particular functions are easily constructed. For example, for $l=0$, $\rho_0(s)=1$ is the simplest choice which satisfies all the requirements (A)–(C), leading to

$$f_0(s)=\frac{1}{s}\ln(\beta_0^2+4s/\beta_0^2). \quad (14)$$

Similar expressions for general l are discussed in Sec. 3.

Assuming now that v_l has the representation (13) we see that all the conditions for a dispersion representation for $A_l(s)$ given by (9) and (10) are satisfied, so that in the usual way

$$A_l(s)=\frac{R_l(-\alpha_l^2)}{s+\alpha_l^2}+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}A_l(s')ds'}{s'-s}$$
$$+\frac{1}{\pi}\int_{-\infty}^{-\frac{1}{4}\beta_l^2}\frac{\mathrm{Im}A_l(s')ds'}{s'-s}. \quad (15)$$

Here $R_l(-\alpha_l^2)$ is the residue of $A_l(s)$ at the pole

$s=-\alpha_l^2$, which works out to be

$$R_l^{-2}(-\alpha_l^2)=-(2\pi^2)^{-1}v_l^{-2}(i\alpha_l)\times\int d\mathbf{q}\, v_l^2(q)(\alpha_l^2+q^2)^{-2}. \quad (16)$$

The magnitude of this residue would in turn govern the normalization of the corresponding bound state wave function in a manner discussed in reference 4. The unitarity condition, whose content (for the neglect of inelastic processes) is [see Eq. (8)]

$$\mathrm{Im}A_l^{-1}(s)=-s^{\frac{1}{2}}, \quad (\text{real } s>0), \quad (17)$$

is seen to be trivially satisfied by writing Eq. (9) in *reciprocal* form and making use of (4) and (5).

Finally, it is tempting to recognize the strong resemblance of the results obtained for $A_l(s)$ with the so-called N/D representation of Chew and Mandelstam.[7] It may be recalled that in the N/D method, the "numerator" is assigned the role of representing the entire effect of the left-hand branch cut, and the "denominator" the corresponding role for the right-hand branch cut. Here it appears from Eq. (9) that the N/D representation is already built into the structure of $A_l(s)$, with the necessary analyticity conditions on $N_l(s)$ being satisfied through the conditions (A)–(C) imposed on the potential shape. Such a model thus provides perhaps one of the simplest backgrounds against which one could, e.g., test the Chew-Mandelstam approximation scheme for solving their $\pi-\pi$ integral equations.

3. EQUIVALENT SEPARABLE POTENTIAL

A possible generalization of Eq. (14) for arbitrary l suggests itself in a natural way by expressing (14) in terms of a Legendre function of the second kind, viz.,

$$f_0(s)=(2/s)Q_0(1+\beta_0^2/2s). \quad (18)$$

It is now not difficult to guess that the desired generalization consists in writing

$$f_l(s)=(2/s)Q_l(1+\beta_l^2/2s), \quad (19)$$

which has the "spectral shape" (13) with

$$\rho_l(s')=P_l(1+\beta_l^2/2s'). \quad (20)$$

By making use of the known properties of $Q_l(x)$ it is easily seen that the form (19) satisfies all the conditions (A)–(C) imposed on the potential. Thus a possible "analytic structure" of $v_l(p)$ is represented by the shape

$$v_l(p)=[(2/p^2)Q_l(1+(\beta_l^2/2p^2)]^{\frac{1}{2}}, \quad (21)$$

where β_l now has the direct significance of an inverse range for the interaction in the state of angular momentum l.

It is possible to derive an approximate connection between the set of separable potentials (21) and a local Yukawa potential of range β^{-1} for the special case when all the λ_l's and β_l's are equal. Indeed, for this special case, the *Born approximation* for the scattering amplitude $f(k^2,\cos\theta)$ defined by (7) is computed to be

$$\begin{aligned}f_B(k^2,\cos\theta)&=(4\pi^2\lambda/k^2)\textstyle\sum_l (2l+1)P_l(\cos\theta)\\&\qquad\times Q_l(1+\tfrac{1}{2}\beta^2k^{-2})\\&=8\pi^2\lambda/[\beta^2+2k^2(1-\cos\theta)],\end{aligned} \quad (22)$$

which is just the Born approximation amplitude for scattering from the Yukawa potential,

$$V(r)=-(4\pi^2\lambda/Mr)e^{-\beta r}. \quad (23)$$

In a sense, therefore, one could speak of the set of separable potentials defined by (21) as an "equivalent separable potential" corresponding to (23). The interpretation can be generalized. For any other local potential $-(\lambda/M)V(r)$ which satisfies the Mandelstam representation, one calculates the Born approximation amplitude

$$f_B(k^2,\cos\theta)=(\lambda/2\pi)\int d\mathbf{r}\; V(r)\exp[i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')], \quad (24)$$

and deduces $v_l^2(p)$ as

$$\begin{aligned}v_l^2(p)&=(4\pi^2\lambda)^{-1}\int_{-1}^{+1} f_B(p^2,x)P_l(x)dx\\&=\pi^{-2}\int_0^\infty r^2dr\; V(r)[j_l(pr)]^2.\end{aligned} \quad (25)$$

Conversely, if one derives an analytic expression for $v_l^2(p)$ satisfying the conditions (A)–(C), it should be possible to deduce the corresponding local potential through equations like (22) and (24).

Though such a deduction makes explicit use of the Born approximation, this limitation need not be serious since it appears from contemporary work that the Born series generally conforms to the usual analyticity conditions term by term. Of course, in the separable-potential approach there is no scope for resorting to conventional "higher Born approximations," since for each partial wave these are effectively incorporated in the function $D_l^{-1}(s)$ of (9).

From a practical point of view, while working with separable potentials, it would be too restrictive to make all the β_l's equal and correspondingly the λ_l's, though it is only under such circumstances that one can derive an equivalent local potential. Further, since a separable potential approach is useful only for fairly low energies, one needs in practice only a few partial waves. One could still use "analytic forms" like (21) where β_l would represent the inverse range operative in the state l.

It is useful to note that potentials like (21) need not pose any algebraic problem of evaluation of integrals since the relevant integrals always involve $v_l^2(p)$ [see,

[10] Making use of the relation $(y-x)^{-1}=\sum_l (2l+1)P_l(x)Q_l(y)$.

e.g., (5) or (16)]. These integrals, in turn, can be evaluated in an elementary manner by using the integral representation

$$Q_l(y)=\frac{1}{2}\int_{-1}^{+1} dx\, P_l(x)(y-x)^{-1}. \tag{26}$$

Thus, with an s-state potential of the type (21) (with $l=0$) operative between 2 nucleons, the deuteron binding energy α^2/M and the s-phase shift δ_0 are given by

$$(\alpha/4\pi^2\lambda_0)=\ln(2\alpha+\beta_0/\beta_0), \tag{27}$$

and

$$k\cot\delta_0=\left(1-\frac{4\pi^2\lambda_0}{k}\tan^{-1}\frac{2k}{\beta_0}\right)\left(\frac{2\pi^2\lambda_0}{k^2}\ln\frac{\beta_0{}^2+4k^2}{\beta_0{}^2}\right)^{-1}, \tag{28}$$

corresponding to the triplet effective-range parameters

$$0.8\approx(a\alpha)^{-1}=2x^{-2}[x-\ln(1+x)], \tag{29}$$

and

$$0.2\approx\tfrac{1}{2}r_0\alpha=\ln(1+x)-\tfrac{1}{3}x, \tag{30}$$

where $x=2\alpha/\beta_0$. These relations are satisfied with $\beta_0\approx5\alpha$ corresponding to a range of 0.9×10^{-13} cm. While this value is somewhat smaller than the meson Compton wavelength, viz. $\mu^{-1}\approx1.4\times10^{-13}$ cm, it is reasonable enough to warrant more detailed calculations with such potential shapes, and such calculations are in progress.

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Subtractions in Dispersion Relations*

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The following theorem is proved: If an analytic function $f(z)$ has singularities only on the real axis and is bounded in magnitude at infinity by a finite but arbitrary power of z, then $f(z)$ has essentially the same limits everywhere at infinity. This theorem enables one to express the contribution from the infinite circle of the Cauchy contour integral in terms of the boundary values of $f(z)$ at infinity along only one of the cuts extending to infinity. The exact dispersion relation is thus determined. As examples, we derive the forward and double pion-nucleon dispersion relations, assuming that the total cross section approaches a finite limit at infinite energy. We see how the subtractions are determined completely by the theorem.

I. INTRODUCTION

In order to determine the number of subtractions in dispersion relations, we usually introduce subtractions until the dispersion integrals appear to be convergent on the basis of conjectured asymptotic behaviors of integrands along the cuts. It could, however, be that the contribution from the infinite circle of the Cauchy contour integral we started with does not yet vanish, which implies that the subtracted dispersion relation has to be supplemented by some finite terms. It could also be that the last subtraction was unnecessary since the dispersion relation prior to the last subtraction was already finite because of the cancellation of divergences among dispersion integrals (in the case when there are two cuts extending to infinity).

Obviously the best way to find out the exact dispersion relation is to deal directly with the integral over the infinite circle in the original Cauchy integral. The question then arises how we know the behavior of the function at arbitrary infinite points in the complex plane. This is exactly why we wish to prove the theorem (stated in Sec. II), which says that the behavior at infinity is essentially the same everywhere in the complex plane even when the branch cuts extend to infinity, as long as we can expect dispersion relations at all.

In Sec. II we state the theorem and the simplest form of the dispersion relation when the function approaches finite limits along one of the cuts extending to infinity. We present the proof in Sec. III. In Sec. IV are given supplementary remarks on the theorem, applying to special cases when there is crossing symmetry and when only one cut extends to infinity. We remark also how to use the theorem to get dispersion relations in the case of asymptotic behaviors other than simple finite limits.

As examples of application of the theorem, we derive forward (Sec. V) and double (Sec. VI) pion-nucleon dispersion relations, assuming asymptotic behavior of scattering amplitudes which is consistent with the finite total cross section at infinite energy. The theorems due to Pomeranchuk[1] and Amati, Fierz, and Glaser[2] follow as immediate consequences of the present theorem. The double dispersion relation is essentially the same as, but

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[1] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 725 (1958) [Soviet Phys. J.E.T.P. **34(7)**, 499 (1958)].
[2] D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters **4**, 89 (1960).