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Unimportance of Surface Roughness Upon the Kapitza Resistance*

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A proof is given which shows that surface roughness which is less than the phonon mean free path can only increase the Kapitza resistance between liquid helium and a solid; and that this change in resistance is small.

SEVERAL people[1,2] have conjectured about the possible role surface roughness would have upon the Kapitza resistance between liquid helium and a solid. (The Kapitza resistance is the thermal resistance which gives rise to the discontinuous jump in temperature observed[3] at the solid-liquid interface when a solid is heated while immersed in liquid helium.) It has generally been believed that the heat flow would be enhanced by this roughness although this has not been proved. In an earlier paper[2] the present author showed that an increased heat flow and a variation of its temperature dependence would occur if the amplitude of the "roughness" was comparable to the phonon mean free path. We call this macroscopic roughness. The more tricky problem of microscopic roughness was not attempted, i.e., when the roughness is comparable to the phonon wavelength.

In this note we present a proof which shows that such microscopic roughness can only reduce the net heat flow across an interface. The proof is based upon the following argument. The rough surface is treated as a grating. The incoming phonons are diffracted in to various orders of the grating and only those which fall within the critical cone are not totally reflected. Phonons, which for a smooth surface would be totally reflected can now be diffracted in to the solid if the surface is rough, but, there is an equal chance of a phonon which was incident within the critical cone being diffracted out and thus being totally reflected. These effects do not cancel, however, because the phonons which are scattered into the cone have their polarization vectors less favorably oriented than those which are refracted out. Consequently, the net heat flow is reduced.

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IRONING TRANSFORMATION

Let us consider a rough surface depicted in Fig. 1. Let the deviation of the surface from the plane $z=0$ be given by some function $a(x,y)$. Let the incoming phonon $\mathbf{q}$ be incident with respect to the plane at an angle θ. We note first that the velocity of sound of practically any solid (medium 2) is very much greater than that of the fluid (liquid helium). Consequently, the effect of a bump on the surface at a point $P(x,y)$ is to make it appear to a point on the plane $z=0$ as if the phase of the incoming phonon wave was advanced locally by an amount approximately $qa(x,y)\cos\theta$. In view of this, we can obtain an equivalent physical situation by satisfying the boundary conditions on a "model" surface which is perfectly flat at $z=0$ but using, instead of the original phonon wave function, a new phonon function within the solid which contains the above phase modulation. That is, we have transformed the roughness away from the surface and incorporated it

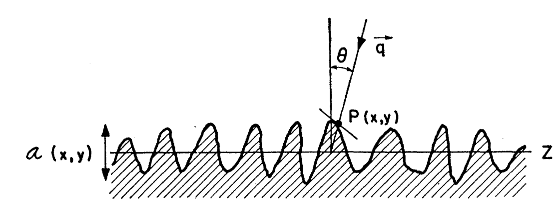

FIG. 1. Schematic representation of rough solid surface.

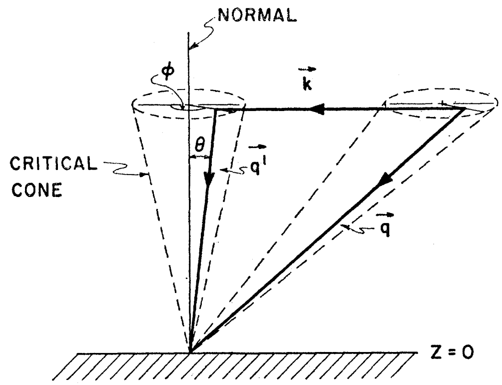

FIG. 2. The scatter of an incident phonon, $\mathbf{q}$ to $\mathbf{q}'$ by the Fourier component $\mathbf{k}$ of the rough surface.

in the phonon wave function. Of course, this analysis is not quite exact because we have ignored the fact that all parts of the real rough surface were not necessarily parallel to the plane $z=0$. This does affect the boundary conditions. However, the error introduced by this omission is small because at least on the average the real surface must be parallel to $z=0$.

We proceed then as follows. Let the displacement vector in the liquid be $\mathbf{s}$, where

$$\mathbf{s}=\text{grad}\phi_0, \tag{1.1}$$

and

$$\phi_0=\sum_{\mathbf{q}} b_{\mathbf{q}} \exp i(\mathbf{q}\cdot\mathbf{r}-\omega_q t)+\text{c.c.}, \tag{1.2}$$

so that at $z=0$

$$\phi_0=\sum_{\mathbf{q}} b_q \exp i(q_x x+q_y y-\omega_q t)+\text{c.c.} \tag{1.3}$$

We now replace the surface of the solid with the "model" flat surface and transform Eq. (1.3) at the surface $z=0$ to

$$\phi_m=\sum_{\mathbf{q}} b_{\mathbf{q}} \exp i(q_x x+q_y y+a(x,y)q\cos\theta-\omega_q t)+\text{c.c.} \tag{1.4}$$

$$=\sum_{\mathbf{q}} b_{\mathbf{q}} \exp i(q_x x+q_y y-\omega_q t)\exp iqa(x,y)\cos\theta+\text{c.c.} \tag{1.5}$$

We expand the exponent containing the roughness as a Fourier series:

$$\exp iqa(x,y)\cos\theta=\sum_{\mathbf{k}}\gamma_{\mathbf{k}}\exp i(k_x x+k_y y). \tag{1.6}$$

We note first an important sum rule by taking the complex conjugate of (1.6) and integrating over the surface. We find then that

$$\sum|\gamma_k|^2=1. \tag{1.7}$$

We rewrite (1.4) as

$$\phi_m=\sum_{\mathbf{q}}\sum_{\mathbf{k}} b_{\mathbf{q}}\gamma_{\mathbf{k}}\exp i(q_x'x+q_y'y-\omega_q t)+\text{c.c.}, \tag{1.8}$$

where

$$q_x'=q_x+k_x,$$
$$q_y'=q_y+k_y. \tag{1.9}$$

We can use Eq. (1.7) to compute the heat flow in the same way as has been done previously utilizing the true unmodified phonon expression.

HEAT FLOW

The heat flow per unit area across a flat, smooth interface can be shown to be given by

$$\dot{Q}_s=\frac{V\rho_1c_1}{(2\pi)^3}\int\int\int q^2\omega_q^2\{b_q^2(T_1)-b_q^2(T_2)\}\times f^2(\theta)\cos\theta d^3q. \tag{2.1}$$

V, ρ_1, and c_1, are the volume, density, and acoustic velocity of the fluid. $f(\theta)$ is the transmission coefficient of phonons incident at an angle θ.

$$b_q(T)=\text{phonon amplitude}=[\hbar\mathfrak{N}_q(T)/\omega_q q^2\rho_1 V]^{\frac{1}{2}}, \tag{2.2}$$

and in equilibrium

$$\mathfrak{N}_q(T)=[1/e^{(\hbar\omega_q/kT)}-1]. \tag{2.3}$$

Equation (2.1) is derived by evaluating the normal displacements at the interface, s_z and the normal stresses σ_{zz} which are given by

$$s_z=\sum_{\mathbf{q}} iq_z b_q \exp i[q_x x+q_y y-\omega_q t]+\text{c.c.}, \tag{2.4}$$

and

$$\sigma_{zz}=\lambda_1\sum_{\mathbf{q}} q^2 b_q \exp i[q_x x+q_y y-\omega_q t]+\text{c.c.} \tag{2.5}$$

From the similarity of Eqs. (1.8) and (1.3) it is clear that (2.5) may be generalized for the rough surface by replacing b_q by $b_q\gamma_k$ and replacing $\mathbf{q}$ by $\mathbf{q}+\mathbf{k}$ in the exponent.

The heat flow for the rough surface now becomes

$$\dot{Q}_R=\frac{Vc_1\rho_1}{(2\pi)^3}\int\int\int\sum_{\mathbf{k}}\gamma_{\mathbf{k}}^2q^2\omega_q^2[b_q^2(T_1)-b_q^2(T_2)]\times f^2(\theta')\cos\theta d^3q, \tag{2.6}$$

where θ is the angle of incidence of the phonon $\mathbf{q}$ and θ' is the angle of incidence of $\mathbf{q}'$, i.e., the resultant of $\mathbf{q}$ and $\mathbf{k}$.

Rewriting this, we obtain

$$\dot{Q}_R=\frac{c_1}{(2\pi)^3}\int\int\int\sum_{\mathbf{k}}\gamma_k^2\hbar\omega_q[\mathfrak{N}_{\mathbf{q}}(T_1)-\mathfrak{N}_{\mathbf{q}}(T_2)]\times f^2(\theta')\cos\theta d^3q. \tag{2.7}$$

We note first that $f(\theta')$ differs from zero only within the critical cone defined by $\sin\theta'=c_1/c_2$, where c_2 is the transverse acoustic velocity of the solid. This angle is extremely small, $\approx 6°$ for most solids.

Referring to Fig. 2, one can see that the integrand is only non-zero when $\mathbf{q}$ lies within a cone such that $\mathbf{q}'=\mathbf{q}+\mathbf{k}$ lies within the critical cone.

The integral over $\mathbf{q}$ can then be done by integrating over the different orientations of $\mathbf{q}'$ defined by θ' and

ϕ'. Also, because the cone is narrow, $|q'|\approx q_z'$,

$$\dot{Q}_R=\frac{c_1}{(2\pi)^3}\int\int\int\sum_{\mathbf{k}}\gamma_{\mathbf{k}}^2\hbar\omega_q[\mathfrak{N}_q(T_1)-\mathfrak{N}_q(T_2)]\times f^2(\theta')\cos\theta\sin\theta'd\theta'd\phi'dq_z, \quad (2.8)$$

$$\dot{Q}_R=\frac{c_1}{(2\pi)^3}\int\sum_{\mathbf{k}}\gamma_k^2\hbar\omega_q[\mathfrak{N}_q(T_1)-\mathfrak{N}_q(T_2)]\cos\theta q_z^2dq_z\times\int f^2(\theta')\sin\theta'd\theta'\int d\phi', \quad (2.9)$$

$$q_z=(q^2-k^2)^{\frac{1}{2}}=q[1-(k/q)^2]^{\frac{1}{2}}=q\cos\theta, \quad (2.10)$$

$$\dot{Q}_R=\frac{c_1}{(2\pi)^3}\int\sum_{\mathbf{k}}\gamma_k^2\hbar\omega_q[\mathfrak{N}_q(T_1)-\mathfrak{N}_q(T_2)]\times\left[1-\left(\frac{k}{q}\right)^2\right]^{\frac{3}{2}}q^2dq\int f^2(\theta')\sin\theta'd\theta'\int d\phi'. \quad (2.11)$$

If the surface were smooth, all γ_k's would be zero except for γ_0. The difference between the rough and the smooth surface lies only in the factor

$$\gamma_{\mathbf{k}}^2[1-(k/q)^2]^{\frac{3}{2}},$$

which is necessarily less than γ_k^2 for any finite value of k. Consequently,

$$\textstyle\sum_{\mathbf{k}}\gamma_{\mathbf{k}}^2[1-(k/q)^2]\leqslant\sum_{\mathbf{k}}\gamma_{\mathbf{k}}^2\leqslant 1. \quad (2.12)$$

The equality sign occurs for a truly smooth surface. Hence

$$\dot{Q}_R<\dot{Q}_s.$$

Thus we have shown that the heat flow across the solid-helium interface is necessarily smaller for a rough surface than for a smooth surface. This result is not to be confused with the increased heat flow which occurs when a surface is roughened due to the increase of the macroscopic area of the surface. The distinction between macroscopic roughening and microscopic roughening being given by the mean free path of the phonon. This effect has been discussed previously.[2]

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I wish to thank Professor Felix Bloch for two stimulating discussions: one which raised the question of the roughness of the surface, and one which contained most of the basis of the proof that it was not important for an understanding of the Kapitza resistance.

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Generalized Bardeen-Cooper-Schrieffer States and the Proposed Low-Temperature Phase of Liquid He^3

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Particle interactions in a Fermi gas may be such as to attract pairs near the Fermi surface more strongly in $l=1, 2, 3$ or higher states than in the simple spherically symmetrical s state. In that case the Bardeen-Cooper-Schrieffer condensed state must be generalized, and the resulting state is an anisotropic superfluid. We have studied the properties of this type of state in considerable detail, especially for $l=1$ and 2. We have derived expressions for the energy, the moment of inertia, the magnetic susceptibility and the specific heat. We also derive the density correlation function and the density-current density correlation; in some cases the latter implies that the liquid has net surface currents and a net orbital angular momentum. The ground state for $l=2$ is different from those previously considered, and has cubic symmetry and no net angular momentum. A general method for replacing the possibly rather complicated potential by a simple scattering matrix is given. A brief discussion of possible collective effects is included. We apply our results to liquid He^3; after correction for scattering by a method due to Suhl, it is found that the predicted transition should take place below 0.02°K. Other possible applications are suggested.

I. INTRODUCTION

SINCE the publication of the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity,[1] attempts have been made to extend their method to describe possible condensations of other interacting fermion systems, particularly liquid helium-3. It has been recently observed by several authors[2–4] that the problem of determining the ground state of a fermion

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