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Applications of Conformal Mapping to the Phenomenological Representation of Scattering Amplitudes

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A series representation of scattering amplitudes analytic in cut planes is derived by means of a conformal mapping. The new series has the advantage that in general it converges more rapidly than the conventional power series. An example is considered in which an angular distribution which requires $(\cos\theta)^4$ terms for a good fit requires only second-order terms in the new series. The resulting advantages in using this series in extrapolations to poles are discussed. As a second example, a modified effective-range formula is derived.

IN the application of dispersion relations one often needs a simple representation of a function which is analytic in a cut plane. For example, in extrapolating angular distributions to poles to measure coupling constants it has been customary to represent the angular distribution by a power series in $\cos\theta$. An improved representation, with a greatly increased region of validity and with, in general faster convergence, can be obtained by first mapping the cut plane conformally into a unit circle, and then expanding in a power series. We have applied this method to the extrapolation of angular distributions and to the construction of modified effective-range formulas, and have found a significant improvement over conventional methods.

The conformal mapping which carries the interior of the cut z plane with branch points at $z=a$ and $z=-b$ [see Fig. 1(a)] into the interior of the unit circle in the w plane [see Fig. 1(b)] is given by

$$w=\left\{1-\left[\frac{b}{a}\left(\frac{a-z}{b+z}\right)\right]^{\frac{1}{2}}\right\}\Big/\left\{1+\left[\frac{b}{a}\left(\frac{a-z}{b+z}\right)\right]^{\frac{1}{2}}\right\}. \quad (1)$$

The additional requirements that the origin transform into the origin and that the real axis between the branch points remain on the real axis make the mapping unique. The branch cuts lie along the unit circle in the w plane [see Fig. 1(b)]. The corresponding mapping for a function with a single cut is obtained by letting a or b approach infinity.

One can then expand a function $f(z)$ which is analytic in the cut z plane shown in Fig. 1(a) in a power series in w about the origin, i.e.,

$$f(z)=\sum_n a_n[w(z)]^n. \quad (2)$$

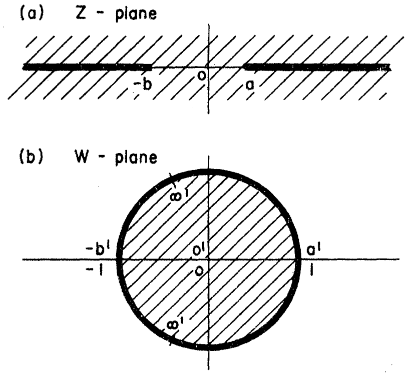

FIG. 1. The conformal mapping given by Eq. (1). The primed points are images of the corresponding unprimed points.

The analyticity of $w(z)$ guarantees that the analytic properties of $f(z)$ will be preserved in the w plane; hence that this series converges in the unit circle in the w plane. Therefore, it converges in the entire cut z plane, except possibly on the cuts themselves. Even in the singularity-free region along the real axis in the z plane this series will, in general, converge more rapidly than the ordinary power series in z. This follows from the fact that the rate of convergence of a Taylor series depends upon the distance to the nearest singularity, which is always relatively greater in the w plane. This argument of course breaks down if the discontinuity across the cut is concentrated far away from the origin, but this situation is unusual for the functions commonly considered in dispersion theory.

Applications of the w series are quite numerous; it should be advantageous to employ it in every case in which a conventional power series expansion has been used for a function which is analytic in a cut plane. The Mandelstam representation tells us that two-body scattering amplitudes have this property in both energy and momentum-transfer variables.[1] The physical basis for believing in the utility of the w series is the fact

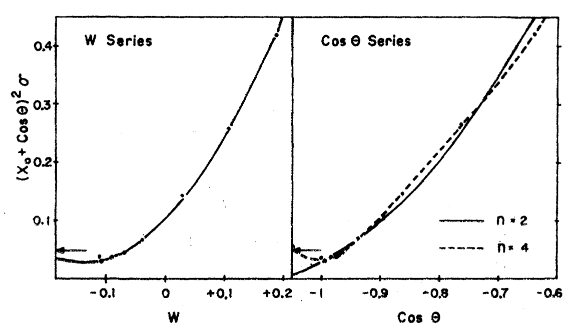

FIG. 2. Comparison of p-p angular distribution (with backward pole at x_0 removed) as function of w and of $\cos\theta$. In w, a parabola gave the good fit shown, whereas in $\cos\theta$ a fourth-order polynomial was needed for a good fit. The standard deviations on the points are roughly the size of the dots drawn. For a pion-nucleon coupling constant $f^2=0.08$, the curves should extrapolate to the point indicated by the arrow.

[1] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

TABLE I. Comparison of least-squares fits. The parameter Q is the usual weighted sum of the squares of the deviations of the fitting curve from the experimental points. The quantity f^2 is the pion-nucleon coupling constant, known to be ≈ 0.08, whereas n is the order of the polynomial.

	Power series in $\cos\theta$		w series	
n	Q	f^2	Q	f^2
1	158	imag.	349	imag.
2	38	0.022	13.7	0.066
3	14	0.060	9.4	0.086
4	8.9	0.084	8.4	0.11
5	7.9	0.11	8.4	0.11

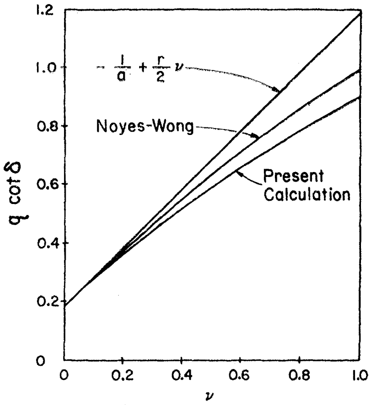

FIG. 3. The modified effective-range formula of the present calculation is compared with the standard formula and with the calculation of Noyes and Wong, for singlet p-p scattering.

that it, in contrast to the ordinary power series, makes use of the entire region of analyticity of the scattering amplitude.

The possible applications fall into two classes: the use of series expansions in dynamical theories, such as the Chew-Mandelstam equations for pion-pion scattering[2]; and the use of series expansions to provide a phenomenological fit to experimental data. The applications to dynamical calculations will not be considered in detail in this paper. They arise whenever it is necessary to make an analytic continuation to unphysical regions. This continuation has led to divergence difficulties when done by means of Legendre polynomial expansions.[3] Some of these difficulties would be removed by the use of the w series, since the individual terms remain finite when z approaches infinity. The rest of this paper will be concerned with two phenomenological applications of the w series; effective-range formulas and extrapolation to poles.

In extrapolating to poles in scattering amplitudes to determine coupling constants or cross sections, it is necessary to extrapolate experimental data to unphysical regions in the momentum-transfer variable. This extrapolation has usually been done by fitting the data with the first few terms of a power series in momentum transfer by the method of least squares.[4] We propose instead that the data be fit to the first few terms of a w series, which should result, according to the above arguments, in a better fit with no more (and often fewer) parameters. We have taken as an example the extrapolation to the one-meson exchange pole in $\cos\theta$ in neutron-proton scattering. In this case the scattering amplitude at fixed c.m. energy, $f(\cos\theta)$ has branch cuts as shown in Fig. 1(a), with $a=b=1+2m^2/k^2$, where k is the momentum in the c.m. system and m is the pion mass. In addition there are poles at $\cos\theta=\pm(1+m^2/2k^2)$. Because of the difficulty of making measurements near the forward direction it has been customary to extrapolate to the backward pole to determine the pion-nucleon coupling constant.[4] Suppose we wish to use experimental data in a certain range of $\cos\theta$ near the backward direction. Then we choose our mapping $w(z)$ such that all singularities, including the forward pole, are mapped onto the unit circle. In addition we shift the origin, $z=\cos\theta-z_0$, in such a way as to center the backward pole and the experimental data in the w plane. This is accomplished by choosing b/a in Eq. (1) according to the following prescription:

$$b/a=1/C_1C_2, \tag{3}$$

where

$$C_1^2=\frac{x_R-x_1}{x_L+x_1},\quad C_2^2=\frac{x_R-x_2}{x_L+x_2}. \tag{4}$$

The quantities x_R and x_L are the right- and left-hand branch points (in the example we are considering, x_R has been chosen to be the position of the forward pole), while x_1 and x_2 are the ends of the region to be centered about the origin in the w plane.

Having mapped our experimental data into the w plane (only the positions of the points are shifted, of course; the values of the cross section are unchanged), we can now fit the data with the first few terms of a power series in w by the method of least squares. The extrapolation can then be carried out in the usual manner.[4]

The comparative results with the w series and the ordinary power series are shown in Fig. 2 and Table I for an especially favorable case among those investigated. The data are at 400 Mev, and consist of eight points between 130° and 180°.[5] According to statistical criteria discussed by Cziffra and Moravcsik,[6] the lowest-order polynomial giving a good fit is *second* order (or possibly third; the choice is not clear-cut here) for the w series, whereas it is *fourth* order for the ordinary expansion in powers of $\cos\theta$. Thus the expansion behaves as expected on the basis of the analytic structure in $\cos\theta$ predicted by the Mandelstam representation.

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[3] G. F. Chew and S. Mandelstam, Lawrence Radiation Laboratory Report UCRL-9126, March, 1960 (unpublished).
[4] See, for example, P. Cziffra and M. J. Moravcsik, Phys. Rev. **116**, 226 (1959).
[5] The data are those of A. J. Hartzler, R. T. Siegel, and W. Opitz, Phys. Rev. **95**, 591 (1954).
[6] P. Cziffra and M. Moravcsik, Lawrence Radiation Laboratory Report UCRL-8523, October, 1958 (unpublished).

Several other cases were investigated. When twelve points at 400 Mev between 180° and 90° were fitted, the w series again tended to lower the order of the polynomial and give a better value of f^2 for the low-order polynomials, but not so much as in the eight-point case. When 23 points at 400 Mev between 12.7° and 180° were fitted, the w series gave no advantage over the $\cos\theta$ series. This behavior is to be expected theoretically, because when a wider region is used over which data must be fitted this region approaches closer to the branch cuts in the w plane. Thus the advantage gained from the mapping is greatest when data are used over only a small part of the physical region.

The success of the w series in lowering the order of the fitting polynomial strongly recommends it as a better representation to use in making extrapolations. It should be recognized, however, that in the mapping the distance of extrapolation is lengthened relative to the length of the region over which measurements are used. Thus the advantage gained by performing extrapolations in the w plane is reduced.

A closely related application of the w series would be to fit angular distributions with this series rather than the usual phase-shift analysis. Improvement in the fit, as well as a possible reduction in the number of parameters, might result. Such a program would be an extension of the modified phase-shift analysis of Cziffra, MacGregor, Moravcsik, and Stapp, in that it recognizes the existence of the branch points as well as the poles.[7]

We now turn to consideration of the w series for deriving modified effective-range formulas. Noyes and Wong have shown that the usual effective-range formula for singlet p-p scattering, for example, follows from the Mandelstam representation if the unphysical branch cut is simply replaced by a pole.[8] An alternative approximation is provided by the w series. The technique, which will be described below, can be used whenever a phenomenological approximation to a branch line is desired.

Let us consider in detail the case of nucleon-nucleon scattering. For any partial wave (except the triplet S n-p) the Mandelstam representation implies that the only singularities are two branch cuts on the real axis, as shown in Fig. 1(a). If we choose as our variable $\nu=q^2$, the square of the center-of-mass momentum, then $a=0$ and $b=\frac{1}{4}$. Unitarity tells us that in the elastic region the S-wave amplitude has the form (neglecting relativistic corrections)

$$f(\nu)=e^{i\delta}\sin\delta/q. \tag{5}$$

A convenient method for constructing such an amplitude is the familiar method of Chew and Mandelstam of expressing f as a quotient, $f=N/D$, such that N carries only the left cut and D only the right cut.[2] Then it follows from Eq. (5) that

$$D(\nu)=1-\frac{\nu}{\pi}\int_0^\infty d\nu'\frac{N(\nu')}{(\nu')^{\frac{1}{2}}(\nu'-\nu)}. \tag{6}$$

One can write down a similar equation for N; or, on a phenomenological level, one can obtain the usual effective-range formula by replacing N by a pole. The alternative method we propose is to expand N in a power series in w, where w is chosen appropriate to the singularities of N; i.e.,

$$w=\frac{1-(4\nu+1)^{\frac{1}{2}}}{1+(4\nu+1)^{\frac{1}{2}}}. \tag{7}$$

As emphasized above, such an expansion is valid in the entire ν plane except on the left cut. In particular, it is valid over the entire range of the integral in Eq. (6). Now the approximation we make is of course to keep only a few terms of the series. With two terms,

$$N=-a_0+a_1w, \tag{8}$$

we have a two-parameter formula, just as is the case with the usual effective-range formula. This approximation, however, has the advantage of being mathematically more straightforward than the replacement of a branch cut by a pole. Substituting Eq. (8) into Eq. (6), we obtain

$$q\cot\delta=\frac{1}{N}\left\{1-\frac{a_1}{\pi}\left[\left(\frac{4\nu+1}{4\nu}\right)^{\frac{1}{2}}\ln\frac{(4\nu+1)^{\frac{1}{2}}-2\nu^{\frac{1}{2}}}{(4\nu+1)^{\frac{1}{2}}-2\nu^{\frac{1}{2}}}+2\right]\right\}. \tag{9}$$

The parameters a_0 and a_1 can be fixed by requiring the correct value and slope of $q\cot\delta$ at $\nu=0$. Then a_0 is the usual scattering length, whereas

$$a_1=-\frac{ra_0}{2}\left(-\frac{1}{a_0}+\frac{8}{3\pi}\right)^{-1}. \tag{10}$$

The results for the singlet S, p-p scattering are compared in Fig. 3 with the usual effective-range curve and with the Noyes-Wong calculation.[9,10]

Further applications of the w series as a method for representing functions analytic in cut planes are very numerous. At the present stage of development of the program of construction of a dynamical theory of strong interactions on the basis of analyticity and unitarity, it is often necessary to confess ignorance by the phenomenological representation of branch cuts.

ACKNOWLEDGMENTS

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[7] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. 114, 880 (1959).

[8] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters 3, 191 (1959).

[9] In using the proton-proton phase shifts we assume that $q\cot\delta$ as calculated is to be compared with the usual Coulomb modification, $C^2q\cot\delta+2q\eta h(\eta)$. See, for example, G. Breit and R. L. Gluckstern, Ann. Rev. Nuclear Sci. 2, 365 (1953). Further Coulomb effects [H. P. Noyes and D. Y. Wong (to be published)] can be shown to have only a small effect on the calculation.

[10] Another conformal mapping has been derived independently by D. M. Greenberger and B. Margolis, Phys. Rev. Letters 6, 310 (1961), for the purpose of constructing a modified effective-range formula. Although the mapping is quite different, the results are similar to those obtained above.