independent. (This is seen most clearly in the momentum representation where the transversality condition becomes an algebraic constraint.) In terms of these the "reduced" Hamiltonian may be written in the form

$$H=\int d^3x\{\tfrac{1}{2}[\mathbf{E}^T(\mathbf{x})\overleftrightarrow{\partial}^0\mathbf{A}^T(\mathbf{x})+\mathbf{E}^T(\mathbf{x})\cdot\mathbf{E}^T(\mathbf{x})$$
$$-\mathbf{A}^T(\mathbf{x})\cdot\nabla^2\mathbf{A}^T(\mathbf{x})]+\mathbf{j}(\mathbf{x})\cdot\mathbf{A}^T(\mathbf{x})\}$$
$$+\int\int d^3xd^3x'\times\frac{1}{2}\frac{j^0(\mathbf{x})j^0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$$

Note the nonlocal interaction corresponding to the (instantaneous, nonretarded) Coulomb interaction between the sources. The apparent noncovariance of the theory is irrelevant and it can be shown that this theory is relativistically invariant. The explicit representation of the field operators can be obtained by proceeding to the momentum representation in which the operators satisfy the commutation relations:

$$[\tilde{E}_j(\mathbf{k}),\tilde{A}_{j'}(\mathbf{k}')]=-i\delta(\mathbf{k}-\mathbf{k}')\left(\delta_{jj'}-\frac{k_jk_{j'}}{\mathbf{k}^2}\right).$$

It is now straightforward to introduce the creation and destruction operators for "photons" with arbitrary momentum $\mathbf{k}$ and left- or right-circular polarizations.

The theory thus formulated does not contain either supplementary conditions or an indefinite metric; all the states entering in the formalism are physical states and all Hermitian operators are observables. However, this "reduced" theory is formally much more complicated; it is particularly interesting to note that there is now an instantaneous "action at a distance" which is consistent with relativistic invariance; and the true observables of the electromagnetic field, namely, the transverse field operators (and their functionals), are not "localizable" since transversality is a nonlocal condition.

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Quantum Mechanical Systems with Indefinite Metric. II*

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Several simple models, similar to that of Lee, involving indefinite metric are studied in this paper. In this connection, a dispersion-theoretic treatment is applied to a simple "equal-mass" model. It is shown that, at least for these models, the scattering amplitude is analytic in the upper-half energy plane provided time-reversal invariance holds; the rules of the dispersion-theoretic formulation in the case of an indefinite metric theory are given. The solution is reinterpreted as the exact solution of a slightly different model, which can also be obtained by Hamiltonian techniques; further techniques are generalized to include recoil in a relativistic no-pair model. Certain basic questions of interpretation are discussed in some detail in the concluding section.

1. INTRODUCTION

IN the preceding paper[1] it had been suggested that in a truly dynamical theory of quantized fields the principle of simplicity could be reinstated and a consistent theory formulated by the formal introduction of an indefinite metric. The systems discussed in that section were very simple and the important problem of interacting particles and the structure of the scattering amplitudes was not discussed in detail. Nor was it shown how the interpretive postulate restricting "physical" states to the subspace spanned by the eigenstates of the S matrix with positive definite norm could be reconciled with certain intuitive notions regarding asymptotic bare particle amplitudes, particularly in view of some recent discussions in the literature[2] regarding the lack of a consistent physical interpretation for such theories. This paper attempts to remedy these shortcomings and, in this sense, is to be considered as a sequel to the preceding paper. We choose for discussion certain models patterned after a simple example considered by Lee.[3] In the course of this analysis we formulate the rules for applying dispersion-theoretic techniques to a theory involving an indefinite metric. We also analyze, in the framework of this model, the construction of physical particle variables and physical configuration amplitudes.

In Sec. 2 we develop the dispersion-theoretic techniques to solve for the scattering amplitude in theories with an indefinite metric; and these are

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[1] E. C. G. Sudarshan, preceding paper [Phys. Rev. 123, 2183 (1961)].
[2] G. Kallen and W. Pauli, Kgl. Danske Videnskab Selskab, Mat-fys. Medd. 30, No. 7 (1955); G. Barton, Nuovo cimento 17, 864 (1960).
[3] T. D. Lee, Phys. Rev. 95, 1329 (1954).

applied to a simple "equal-mass model" in the "one-meson approximation." In Sec. 3 this solution is reinterpreted as the exact solution to a slightly different model; and hence the solution is generalized and rederived by a direct solution of the Hamiltonian equations. The subsequent section generalizes this to include nucleon recoil in the construction of a relativistic no-pair model. The questions of physical interpretation are discussed in Sec. 5; and Sec. 6 concludes the paper with several remarks.

2. EQUAL MASS MODEL

In this section we will first review the calculation of the V propagator in the Lee model, which illustrates the property common to all models of this type, where one is able to solve for the propagator exactly.

The Lee model[3] is characterized by the Hamiltonian $H=H_0+H_{\mathrm{I}}$ where

$$H_0=m_V\int d^3p\,\psi_V^\dagger(\mathbf{p})\psi_V(\mathbf{p})+m_N\int d^3p\,\psi_N^\dagger(\mathbf{p})\psi_N(\mathbf{p})+\int d^3k\,\omega(k)a^\dagger(\mathbf{k})a(\mathbf{k}),\quad(1)$$

$$H_{\mathrm{I}}=g_0\int\int d^3p\,d^3k\,\frac{f(\omega)}{(2\omega)^{\frac{1}{2}}}\times[\psi_V^\dagger(\mathbf{p}+\mathbf{k})\psi_N(\mathbf{p})a(\mathbf{k})+\text{h.c.}],\quad(2)$$

where $f(\omega)$ is a form factor and $\omega^2(k)=\omega^2=k^2+\mu^2$. The constants of motion are

$$B=N_V+N_N\quad\text{and}\quad Q=N_V+N_\theta.\quad(3)$$

The commutation relations are

$$\begin{aligned}&[a(\mathbf{k}),\,a^\dagger(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'),\\&\{\psi_V(\mathbf{p}),\,\psi_V^\dagger(\mathbf{p}')\}=\delta(\mathbf{p}-\mathbf{p}'),\\&\{\psi_N(\mathbf{p}),\,\psi_N^\dagger(\mathbf{p}')\}=\delta(\mathbf{p}-\mathbf{p}').\end{aligned}\quad(4)$$

We can calculate the exact V propagator by diagram techniques, as shown in Fig. 1. For convenience we set $m_N=0$.

$$\Sigma(k)=\frac{1}{\omega-m_V}+\frac{1}{\omega-m_V}\int_\mu^\infty\frac{d^3k'\,g_0^2f^2(\omega')}{(2\omega')(\omega-\omega'+i\epsilon)}\times\frac{1}{\omega-m_V}+\cdots\quad(5)$$

$$=\frac{1}{\omega-m_V}\left[1-\frac{1}{\omega-m_V}F_0(\omega+i\epsilon)+\cdots\right],$$

where

$$F_0(z)=(2\pi)g_0^2\int_\mu^\infty\frac{dk'\,k'f^2(\omega')}{\omega'-z}.\quad(6)$$

Since the V particle can only emit and absorb one meson at a time, it is clear that the right-hand side of Eq. (5) must be a geometric series, with the sum

$$\begin{aligned}\Sigma(k)&=\frac{1}{\omega-m_V}\left[\frac{1}{1+[1/(\omega-m_V)]F_0(\omega+i\epsilon)}\right]\\&=\frac{1}{\omega-m_V+F_0(\omega+i\epsilon)}.\end{aligned}\quad(7)$$

If the V particle can emit more than one kind of meson, but only one meson at a time, then a repetition of the above method would allow us to solve for the V-particle propagator. We will exploit this in constructing the models to be described in what follows.

We consider a model which has two static physical fermions V and V' of equal mass, with the V' a negative-norm state, and a boson θ, which has no antiparticle. We will consider the sector described by the quantum numbers $N_V+N_{V'}=1$, $N_\theta=1$. We will forbid intermediate states with two θ particles (the one-meson approximation). We will try to describe the scattering amplitudes in this sector by means of

FIG. 1. V propagator.

solutions of static dispersion equations. The Born approximation amplitudes, with *renormalized* coupling constants, are chosen as

$$\begin{aligned}f_{V\theta,V\theta}{}^B&\equiv f_{11}{}^B=1/\omega,\\f_{V'\theta,V'\theta}{}^B&\equiv f_{22}{}^B=-\beta^2/\omega,\\f_{V\theta,V'\theta}{}^B&\equiv f_{21}{}^B=-\beta/\omega,\\f_{V'\theta,V\theta}{}^B&\equiv f_{12}{}^B=\beta/\omega,\end{aligned}\quad(8)$$

where β is real and $\omega=(k^2+\mu^2)^{\frac{1}{2}}$ is the θ energy. The choice of signs reflects the use of indefinite metric.[1] We demand that the scattering amplitudes satisfy no-crossing (since there is no $\bar{\theta}$) static dispersion relations, and have the property that the scattering matrix in this representation for this sector be *pseudo*-unitary.

If A is a two-dimensional matrix, then the adjoint of A in the metric, $A^\dagger$, is defined as follows (the asterisk means complex conjugate);

$$\text{If}\quad A=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad\text{then}\quad A^\dagger=\begin{pmatrix}a^*&-c^*\\-b^*&d^*\end{pmatrix},\quad(9)$$

for a diagonal metric operator. For our model we can relate the S matrix to the matrix f by[4]

$$S=1+2ikf,\quad(10)$$

which enables us to define a T matrix through the

[4] C. Møller, Kgl. Danske Videnskab Selskab, Mat-fys. Medd. 23, No. 1 (1945).

equation

$$S=1+2iT. \tag{11}$$

The requirement that S be pseudo-unitary means $S^\dagger S=1$, which implies

$$T^\dagger T=\tfrac{1}{2}i(T^\dagger-T). \tag{12}$$

In general

$$T^\dagger=\eta^{-1}T^c\eta, \tag{13}$$

where η is the real metric operator and $\eta^2=1$, and T^c is the Hermitian conjugate of T, i.e., $(T_{\alpha\beta})^c\equiv(T_{\beta\alpha})^*$ with this choice of $\eta=\eta^{-1}$, $T^\dagger=(\eta T\eta)^c$. By using time-reversal invariance, we can show that

$$(T_{\alpha\beta})^\dagger=(T_{\alpha\beta})^*. \tag{14}$$

To prove this, we define the S matrix connecting state α to state β, with J and M as good quantum numbers, by

$$S_{\alpha\beta}=|\text{in}; JMLS\alpha\rangle\langle\text{out}; JMLS\beta|. \tag{15}$$

The adjoint is defined, consistent with (13), as

$$|\alpha\rangle^*=(\langle\alpha|\eta)^T. \tag{16}$$

Then if U is the time-reversal operator,

$$\begin{aligned} U|\text{in}; JMLS\alpha\rangle &= |\text{out}; J, -MLS\alpha\rangle^* \\ &= (\langle\text{out}; J, -MLS\alpha|\eta)^T, \end{aligned} \tag{17}$$

where an arbitrary phase factor has been set equal to unity; then, removing the explicit spin dependence,

$$\begin{aligned} &USU^{-1}=S, \\ &S_{\alpha\beta}=U|\text{in}; JM\alpha\rangle\langle\text{out}; JM\beta|U^{-1}. \end{aligned} \tag{18}$$

Using the rotational invariance, we have

$$\begin{aligned} &S_{\alpha\beta}=(\eta|\text{in}; JM\beta\rangle)(\langle\text{out}; JM\alpha|\eta), \\ &S_{\alpha\beta}=(\eta S\eta)_{\beta\alpha}, \end{aligned} \tag{19}$$

which means

$$T_{\beta\alpha}=(\eta T\eta)_{\alpha\beta}, \tag{21}$$

and so Eq. (14) is proved. Combining Eqs. (13) and (14), we find

$$T^\dagger T=\text{Im}T. \tag{22}$$

This is a general result which may be used for any theory with indefinite metric. One can then use (22), together with the definition of the adjoint operator, to work with dispersion theories[5] even when there is an indefinite metric.

To return to our problem, we define our Born matrix, by (8) as

$$f^B=\frac{1}{\omega}\begin{pmatrix} 1 & \beta \\ -\beta & -\beta^2 \end{pmatrix}, \tag{23}$$

which obviously satisfies Eq. (21). We are now able to define our "causality" condition in terms of a static, uncrossed dispersion relation, by virtue of (22). For example, for $f_{11}(\omega)$ we have

$$f_{11}(\omega)=f_{11}^B(\omega)+\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\text{Im}f_{11}(\omega')}{\omega'-\omega-i\epsilon}, \tag{24}$$

and similarly for the other amplitudes. We make the ansatz

$$f(\omega)=\begin{pmatrix} 1 & \beta \\ -\beta & -\beta^2 \end{pmatrix}f_{11}(\omega). \tag{25}$$

We will verify that this satisfies all our conditions. We define an inverse amplitude[5] by

$$f_{11}(\omega)=\frac{1}{\omega g_{11}(\omega)}. \tag{26}$$

If $f_{11}(\omega)$ has no zeros, then

$$g_{11}(\omega)=1+\frac{\omega}{\pi}\int_\mu^\infty\frac{d\omega'\,\text{Im}g_{11}(\omega')}{\omega'(\omega'-\omega-i\epsilon)}. \tag{27}$$

Then, by virtue of (22) and (25)

$$\text{Im}g_{11}(\omega)=-\frac{1}{\omega}\frac{\text{Im}f_{11}(\omega)}{|f_{11}(\omega)|^2}=-\frac{1}{\omega}(1-\beta^2)k. \tag{28}$$

Then

$$g_{11}(\omega)=1-\omega(1-\beta^2)I(\omega)-i(1-\beta^2)k/\omega, \tag{29}$$

where

$$I(\omega)=\frac{1}{\pi}P\oint_\mu^\infty\frac{d\omega'\,k'}{\omega'^2(\omega'-\omega)}, \tag{30}$$

so that

$$f_{11}(\omega)=\frac{1}{\omega-\omega^2(1-\beta^2)I(\omega)-i(1-\beta^2)k}, \tag{31}$$

and we can check that

$$\text{Im}f_{11}(\omega)=(1-\beta^2)k|f_{11}|^2. \tag{32}$$

We can check (22) by noting

$$T^\dagger T=\begin{pmatrix} 1 & \beta \\ -\beta & -\beta^2 \end{pmatrix}(1-\beta^2)|T_{11}|^2, \tag{33}$$

and

$$\text{Im}T=\begin{pmatrix} 1 & \beta \\ -\beta & -\beta^2 \end{pmatrix}\text{Im}T_{11}. \tag{34}$$

By virtue of (10), (11), and (32) we see that (22) is indeed satisfied. Similarly one can verify that the amplitudes f_{12}, f_{21}, and f_{22} with the ansatz satisfy the above conditions.

[5] The dispersion-theoretic techniques have been used for such models by T. D. Lee and R. Serber (unpublished); C. J. Goebel, Phys. Rev. **109**, 1846 (1958); M. L. Goldberger and S. B. Treiman, *ibid.* **113**, 1663 (1959).

We proceed to diagonalize the S matrix by means of a pseudo-unitary matrix; we find for the diagonalized matrix

$$S'=\begin{pmatrix}1 & 0\\ 0 & 1+2i(1-\beta^2)kf_{11}\end{pmatrix}. \tag{35}$$

Direct computation, using (31) shows that S' is unitary. There are two eigenstates of S', one with positive norm and one with negative norm. We choose the physical state as the one with positive norm for $\beta^2<1$. Then this state has scattering for all energies.[1]

We wish to emphasize that the *solution* to this model can be obtained by analytic continuation in β^2 from positive values for β^2 for the case with two *physical* V particles to negative values of β^2, which then yields the solution to our model.

When there are two physical V particles, the S matrix is unitary, and there are two observables, the relative phase between S_{11} and S_{22} and the "inelasticity." If one diagonalizes S, then the two variables are the phase of the state with scattering in S' and the "angle of rotation" of S to S', i.e., a mixing parameter. The eigenbeam is defined by the appropriate mixture of $V\theta$ and $V'\theta$ states, with transitions between $V\theta$ and $V'\theta$ measured by this mixing parameter. However when V' is a ghost, there is *only* one observable state, the state with positive definite norm, a pseudo-unitary transformation diagonalizes S. The mixing parameter is no longer an observable. The only observable in the $V\theta$, $V'\theta$ sector is the eigenbeam of $V\theta$ and $V'\theta$ states, which one can call the two-particle state $\tilde{V}\theta$.

The particularly simple results we have here are possible because, in the one-meson approximation, the denominator function $g(\omega)$ is independent of the state considered. The results we have can be considered to be the exact solution for the scattering of two mesons of equal mass θ and θ' and no crossing (with θ' a ghost) from an N. That is, for a model similar to that above in the θN, $\theta' N$ sector with a single $V(V\rightarrow\theta N$ or $\theta' N)$, our solution is exact if the sign of the Born terms are reversed. These lead us to consider a Hamiltonian model for this situation, which we will describe in the next section.

3. MODEL WITH MESON GHOST FIELDS

We will define our model by the Hamiltonian $H=H_0+H_{\rm I}$:

$$H_0=m_0\int d^3p\,\psi_V^\dagger(\mathbf{p})\psi_V(\mathbf{p})+\int d^3k\,\omega(k)a^\dagger(\mathbf{k})a(\mathbf{k})-\int d^3k\,\omega'(k)b^\dagger(\mathbf{k})b(\mathbf{k}), \tag{36}$$

$$\begin{aligned}H_{\rm I}=&\int\int d^3p\,d^3k\,\psi_V^\dagger(\mathbf{p}+\mathbf{k})\psi_N(\mathbf{p})\times\left[g_0\frac{f(\omega)}{(2\omega)^{\frac12}}a(\mathbf{k})+g_0'\frac{f(\omega')}{(2\omega')^{\frac12}}b(\mathbf{k})\right]\\&+\int\int d^3p\,d^3k\,\psi_N^\dagger(\mathbf{p})\psi_V(\mathbf{p}+\mathbf{k})\times\left[g_0\frac{f(\omega)}{(2\omega)^{\frac12}}a^\dagger(\mathbf{k})+g_0'\frac{f(\omega')}{(2\omega')^{\frac12}}b^\dagger(\mathbf{k})\right],\end{aligned} \tag{37}$$

where $f(\omega)$ is a form factor. We have set the mass of the N particle to be zero for convenience (which we may do since this is a static model).

$$\omega(k)=(k^2+\mu_1^2)^{\frac12},\qquad \omega'(k)=(k^2+\mu_2^2)^{\frac12},$$

where μ_1 is the mass of the θ and μ_2 is the mass of the θ'.

$$\begin{aligned}[a(\mathbf{k}),a^\dagger(\mathbf{k}')]&=[\psi_N(\mathbf{k}),\psi_N^\dagger(\mathbf{k}')]\\&=[\psi_V(\mathbf{k}),\psi_V^\dagger(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'),\\ [b(\mathbf{k}),b^\dagger(\mathbf{k}')]&=-\delta(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{38}$$

All other commutators vanish. There are two conserved quantum numbers,

$$B=\int d^3p[\psi_N^\dagger(\mathbf{p})\psi_N(\mathbf{p})+\psi_V^\dagger(\mathbf{p})\psi_V(\mathbf{p})], \tag{39}$$

$$Q=\int d^3k[a^\dagger(\mathbf{k})a(\mathbf{k})+\psi_V^\dagger(\mathbf{k})\psi_V(\mathbf{k})-b^\dagger(\mathbf{k})b(\mathbf{k})]. \tag{40}$$

The sector $\{B=0,Q=0\}$ is the vacuum, $\{B=1,Q=0\}$ the 1-baryon state, and $\{B=0,Q=1\}$ the 1-meson or 1-ghost state. The first nontrivial sector with interaction is $\{B=1,Q=1\}$ which we will solve by well-known methods.[6] We denote the wave function for the $B=1$, $Q=1$ sector by

$$|\Psi\rangle=\left|\begin{matrix}c\\ \phi(\mathbf{k})\\ \phi'(\mathbf{k})\end{matrix}\right\rangle. \tag{41}$$

Applying the Hamiltonian

$$(E-H_0)|\Psi\rangle=H_{\rm I}|\Psi\rangle, \tag{42}$$

which gives

$$\begin{aligned}(E-m_0)c&=\int d^3k\,g_0\frac{f(\omega)}{(2\omega)^{\frac12}}\phi(\mathbf{k})+\int d^3k\,g_0'\frac{f(\omega')}{(2\omega')^{\frac12}}\phi'(\mathbf{k}),\\(E-\omega)\phi(\mathbf{k})&=g_0cf(\omega)/(2\omega)^{\frac12},\\(E-\omega')\phi'(\mathbf{k})&=-g_0'cf(\omega')/(2\omega')^{\frac12}.\end{aligned} \tag{43}$$

[6] W. Heisenberg, Nuclear Phys. 4, 532 (1957).

Since the interaction takes place only in S waves we can put

$$\phi(\mathbf{k})=\frac{\varphi(\omega)}{k(4\pi)^{\frac{1}{2}}};\quad \phi'(\mathbf{k})=\frac{\varphi'(\omega')}{k(4\pi)^{\frac{1}{2}}}. \tag{44}$$

For a bound state $(E-\omega)$ and $(E-\omega')$ have no zeros; hence,

$$\phi(\mathbf{k})=\frac{cg_0f(\omega)}{(2\omega)^{\frac{1}{2}}(E-\omega)};\quad \phi'(\mathbf{k})=-\frac{cg_0'f(\omega')}{(2\omega')^{\frac{1}{2}}(E-\omega')}, \tag{45}$$

and

$$c\left\{(E-m_0)-\int\frac{g_0^2f^2(\omega)}{(2\omega)(E-\omega)}d^3k+\int\frac{g_0'^2f^2(\omega')}{(2\omega')(E-\omega')}d^3k'\right\}=0, \tag{46}$$

which is the eigenvalue equation for the bound states. For the scattering states, we have

$$\begin{aligned}\varphi(\omega)&=\alpha\delta(E-\omega)+\frac{cg_0f(\omega)k(4\pi)^{\frac{1}{2}}}{(2\omega)^{\frac{1}{2}}(E-\omega+i\epsilon)},\\ \varphi'(\omega')&=\alpha'\delta(E-\omega')-\frac{cg_0'f(\omega')k(4\pi)^{\frac{1}{2}}}{(2\omega')^{\frac{1}{2}}(E-\omega'+i\epsilon)}.\end{aligned} \tag{47}$$

We substitute in (43), obtaining

$$\begin{aligned}(E-m_0)c&=\alpha(4\pi)^{\frac{1}{2}}g_0\frac{Ef(E)}{(2E)^{\frac{1}{2}}}+\alpha'(4\pi)^{\frac{1}{2}}g_0'E\frac{f(E)}{(2E)^{\frac{1}{2}}}\\&\quad+g_0^2c\int\frac{d^3kf^2(\omega)}{(2\omega)(E-\omega+i\epsilon)}\\&\quad-g_0'^2c\int\frac{d^3k\,f^2(\omega')}{(2\omega')(E-\omega'+i\epsilon)},\end{aligned} \tag{48}$$

$$c=(4\pi)^{\frac{1}{2}}[\alpha g_0+\alpha'g'_0]\frac{Ef(E)}{(2E)^{\frac{1}{2}}}\Big/\left\{(E-m_0)+\int d^3k\times\left[\frac{g_0'^2f^2(\omega')}{(2\omega')(E-\omega'+i\epsilon)}-\frac{g_0^2f^2(\omega)}{(2\omega)(E-\omega+i\epsilon)}\right]\right\}. \tag{49}$$

Define

$$h(z)=(z-m_0)+\int d^3k\left[\frac{g_0'^2f^2(\omega')}{(2\omega')(z-\omega')}-\frac{g_0^2f^2(\omega)}{(2\omega)(z-\omega)}\right], \tag{50}$$

and

$$A=[\alpha g_0+\alpha'g_0'](4\pi)\frac{f(E)}{(2E)^{\frac{1}{2}}}E, \tag{51}$$

Then

$$c=\frac{A/4\pi}{h(E+i\epsilon)}. \tag{52}$$

The continuous wave channels have amplitudes

$$\begin{pmatrix}\varphi(\omega)\\ \varphi'(\omega')\end{pmatrix}=\begin{bmatrix}\alpha\delta(E-\omega)+\dfrac{g_0f(\omega)kA}{(2\omega)^{\frac{1}{2}}h(E+i\epsilon)(E-\omega+i\epsilon)}\\ \alpha'\delta(E-\omega')-\dfrac{g_0'f(\omega')kA}{(2\omega')^{\frac{1}{2}}h(E+i\epsilon)(E-\omega'+i\epsilon)}\end{bmatrix}. \tag{53}$$

By standard methods we find

$$S-1=\frac{4\pi^2if^2(E)}{h(E+i\epsilon)}\begin{pmatrix}g_0^2k_0 & g_0g_0'(k_0k_0')^{\frac{1}{2}}\\ -g_0g_0'(k_0k_0')^{\frac{1}{2}} & -g_0'^2k_0'\end{pmatrix}. \tag{54}$$

From (11) we have

$$T=\frac{2\pi^2f^2(E)}{h(E+i\epsilon)}\times\begin{pmatrix}g_0^2k_0 & g_0g_0'(k_0k_0')^{\frac{1}{2}}\\ -g_0g_0'(k_0k_0')^{\frac{1}{2}} & -g_0'^2k_0'\end{pmatrix}\quad \begin{aligned}k_0&=(E^2-\mu_1^2)^{\frac{1}{2}}\\ k_0'&=(E^2-\mu_2^2)^{\frac{1}{2}}.\end{aligned} \tag{55}$$

Since $\operatorname{Im}h(E+i\epsilon)=-2\pi^2(g_0^2k_0-g_0'^2k_0')f^2(E)$, it is easy to verify Eq. (22). If we diagonalize Eq. (55), we again find only one state with scattering:

$$T'=e^{i\delta(E)}\sin\delta(E)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{56}$$

where

$$e^{i\delta(E)}\sin\delta(E)=\frac{2\pi^2f^2(E)}{h(E+i\epsilon)}[g_0^2k_0-g_0'^2k_0']. \tag{57}$$

The single scattering state of T' has positive or negative square depending on whether $g_0^2k_0>g_0'k_0'$ or $g_0^2k_0<g_0'^2k_0'$. We note that $(g_0^2/g_0'^2)(k_0/k_0')\rightarrow g_0^2/g_0'^2$ monotonically for high energies, so we must have $g_0^2\geq g_0'^2$ if there is to be physical scattering for all energies.

We note that the original Hamiltonian contains bare coupling constants g_0 and g_0', and the bare V mass, m_0. We will sketch the renormalization of these quantities for the case of a cutoff, below the critical coupling. From Eq. (46), we have

$$m_V-m_0-g_0^2\int\frac{d^3k\,f^2(\omega)}{(2\omega)(m_V-\omega)}+g_0'^2\int\frac{d^3k\,f^2(\omega')}{(2\omega')(m_V-\omega')}=0, \tag{58}$$

as in implicit equation for the physical V mass m_V.

For convenience, we define

$$J_0(z)=g_0^2\int\frac{d^3k\,f^2(\omega)}{(2\omega)(\omega-z)}-g_0'^2\int\frac{d^3k\,f^2(\omega')}{(2\omega')(\omega'-z)};\qquad(59)$$

then

$$m_0-m_V=J_0(m_V).\qquad(60)$$

We also note that

$$h(z)=z-m_0+J_0(z),\qquad(61)$$

thus the "propagator"

$$h(E+i\epsilon)=E-m_V+[J_0(E+i\epsilon)-J_0(m_V)].\quad(62)$$

For the case with no cutoff, $J_0(m_V)$ diverges linearly; however, the bracketed term in (62) only diverges logarithmically. Now from (55) we have

$$\frac{T_{11}(E)}{k_0}=\frac{2\pi^2f^2(E)g_0^2}{E-m_V+[J_0(E+i\epsilon)-J_0(m_V)]},\qquad(63)$$

$$\frac{T_{11}(E)}{k_0}\xrightarrow[E\to m_V]{}\frac{2\pi^2f^2(m_V)g_0^2}{(E-m_V)[1+J_0'(m_V)]}.\qquad(64)$$

Thus it is natural to define the renormalized coupling constant by the relation;

$$g^2=g_0^2f^2(m_V)/[1+J_0'(m_V)].\qquad(65)$$

Similarly, we define

$$g'^2=g_0'^2f^2(m_V)/[1+J_0'(m_V)],\qquad(66)$$

thus, we note that $g^2/g'^2=g_0^2/g_0'^2$, so that our condition which specifies the physical states is not altered if formulated in terms of the renormalized coupling constants.

We can generalize this model to include the heavy-particle recoils. The advantage of this is that the self-mass only diverges logarithmically, with no cutoff, when the recoil is included.

4. MODEL WITH RECOIL

The model is defined by

$$\begin{aligned}H=\int d^3k\{&a^\dagger(\mathbf{k})a(\mathbf{k})E_1\\&+b^\dagger(\mathbf{k})b(\mathbf{k})E_2+c^\dagger(\mathbf{k})c(\mathbf{k})\omega_1-d^\dagger(\mathbf{k})d(\mathbf{k})\omega_2\}\\&+\frac{1}{(4\pi)^{\frac12}}\int\int d^3p\,d^3k\left(\frac{Mm}{E_1(\mathbf{p}+\mathbf{k})E_2(\mathbf{p})}\right)^{\frac12}\\&\times\left\{a(\mathbf{p}+\mathbf{k})b^\dagger(\mathbf{p})\left[\frac{g_0c^\dagger(\mathbf{k})}{[2\omega_1(\mathbf{k})]^{\frac12}}+\frac{g_0'd^\dagger(\mathbf{k})}{[2\omega_2(\mathbf{k})]^{\frac12}}\right]+\text{h.c.}\right\},\end{aligned}\qquad(67)$$

with the commutation rules

$$\begin{aligned}[a(\mathbf{k}),\,a^\dagger(\mathbf{k}')]=[b(\mathbf{k}),\,b^\dagger(\mathbf{k}')]&=[c(\mathbf{k}),\,c^\dagger(\mathbf{k}')]\\&=[d^\dagger(\mathbf{k}),\,d(\mathbf{k}')]=\delta(\mathbf{k}-\mathbf{k}'),\end{aligned}\qquad(68)$$

and all other commutators zero.

$$E_1^2-M^2=E_2^2-m^2=\omega_1^2-\mu_1^2=\omega_2^2-\mu_2^2=k^2.$$

Again there are two constants of motion:

$$\begin{aligned}B&=\int d^3k[a^\dagger(\mathbf{k})a(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})],\\Q&=\int d^3k[a^\dagger(\mathbf{k})a(\mathbf{k})+c^\dagger(\mathbf{k})c(\mathbf{k})-d^\dagger(\mathbf{k})d(\mathbf{k})],\end{aligned}\qquad(69)$$

with non-negative integral values. As before, a pair of their eigenvalues defines a sector. We again use an indefinite metric to define the states. As for our previous models, the first nontrivial sector is for $B=1$, $Q=1$. If

$$\psi=\begin{pmatrix}c\\\phi(\mathbf{k})\\\phi'(\mathbf{k})\end{pmatrix}$$

is the wave function, then applying the total Hamiltonian $H\psi=E\psi$ gives us a coupled set of equations which we may solve, as before. We will work in the center-of-mass system, with $\mathbf{p}+\mathbf{k}=0$. Then

$$\begin{aligned}(E-m)c=&\frac{g_0}{(4\pi)^{\frac12}}\int d^3k\left[\frac{m}{E_2(\mathbf{k})}\right]^{\frac12}\frac{1}{(2\omega_1)^{\frac12}}\phi(\mathbf{k})\\&+\frac{g_0'}{(4\pi)^{\frac12}}\int d^3k\left[\frac{m}{E_2(\mathbf{k})}\right]^{\frac12}\\&\times\frac{1}{(2\omega_2)^{\frac12}}\phi'(\mathbf{k}),\end{aligned}\qquad(70)$$

$$(E-\omega_1-E_2)\phi(\mathbf{k})=\frac{g_0}{(4\pi)^{\frac12}}c\left[\frac{m}{E_2(\mathbf{k})}\right]^{\frac12}\frac{1}{(2\omega_1)^{\frac12}},$$

$$(E-\omega_2-E_2)\phi'(\mathbf{k})=\frac{-g_0'}{(4\pi)^{\frac12}}c\left[\frac{m}{E_2(\mathbf{k})}\right]^{\frac12}\frac{1}{(2\omega_2)^{\frac12}}.$$

The method of solution is identical to that of the previous section, so we will not repeat the details. The solution may be written as

$$T=\begin{pmatrix}f_1^2&f_1f_2\\-f_1f_2&-f_2^2\end{pmatrix}\frac{e^{i\delta(E)}\sin\delta(E)}{f_1^2-f_2^2},1\qquad(71)$$

where

$$\begin{aligned}e^{2i\delta(E)}&=h(E-i\epsilon)/h(E+i\epsilon),\\f_1&=g_0[E^4-(m^2-\mu_1^2)^2]^{-\frac12},\\f_2&=g_0'[E^4-(m^2-\mu_2^2)^2]^{-\frac12},\\h(z)&=M-z+m/8\pi[g_0^2\alpha(\mu_1,z)-g_0'^2\alpha(\mu_2,z)],\end{aligned}\qquad(72)$$

$$\alpha(\mu,z)=\int_{\mu+m}^{\infty}dx\times\frac{[(x+\mu+m)(x-\mu+m)(x+\mu-m)(x-\mu+m)]^{\frac12}}{x^2(x-z)}.$$

The discussion of the previous section may be repeated here. Although the S matrix is only pseudo-unitary, there is only one state with scattering with positive or negative norm depending on $f_1^2 > f_2^2$ or $f_1^2 < f_2^2$. We project out the negative-norm states and have a two-particle system with only one channel and an energy-dependent phase shift.

Again the model is defined in terms of unrenormalized coupling constants and mass M. The renormalization may be carried out exactly. As we remarked, the renormalizations (without cutoff) are only logarithmically divergent *except* for the special case $g_0'^2 = g_0^2$. In this case the integral in (72) converges, and all renormalizations are finite. This occurs because of the cancellation of the high energy components in $[\alpha(\mu_1,z) - \alpha(\mu_2,z)]$. For the case $g_0^2 = g_0'^2$, $f_1^2 > f_2^2$ implies:

$$(m^2-\mu_1^2)^2 > (m^2-\mu_2^2)^2$$

The "success" of these models was made possible by the fact that the "reduced" matrix elements (corresponding to diagrams with external lines amputated) is the same irrespective of the external lines. In each of these models there are questions which arise in relation to asymptotic conditions, two-particle states, and the measuring process. This will be discussed in the following section.

5. ASYMPTOTIC CONDITION AND PARTICLE VARIABLES

In the last section it was mentioned that while on the one hand the amplitudes are analytic functions of the coupling constants and the amplitudes for a theory with indefinite metric can be obtained by analytic continuation from a theory with a definite metric, the number of observables changes discontinuously, and simultaneously the operator structure changes discontinuously. This decrease in the number of "observables" is an essential element of our theory; while the interpretive postulate[1] restricting physical states to the subspace spanned by positive-norm eigenstates of the scattering matrix is a self-consistent postulate, the physical interpretation of such a theory presents certain new features; therefore it is worthwhile to discuss this question in some detail.

For the consistency of the interpretation of the positive-norm scattering state in the $B=1$, $Q=1$ sector as a physical two-particle state of a system with physical one-particle states in the $B=1$, $Q=0$ and $B=0$, $Q=1$ sectors with suitable masses, it is necessary and sufficient to show that these states can be made to correspond to the scattering states of a two-particle system (with positive-definite metric) with the *same* energy-dependent phase-shift and scalar products. This we proceed to do by employing the "wave matrix" of Møller.[4] To avoid unessential complications we deal with the S-wave amplitudes alone. (Equivalently, they may be thought of as describing a one-dimensional two-particle system.) The "wavefunctions" of the physical V-particle state $|V\rangle$, and the two-particle scattering states $|\mathrm{I};E\rangle$, $|\mathrm{II};E\rangle$ corresponding to only "incident waves" of $N\theta_1$ and $N\theta_2$, respectively, are given by

$$|V\rangle \to \begin{cases} \phi_0{}^0 = \dfrac{1}{[\alpha'(M)]^{\frac{1}{2}}}, \\[2ex] \phi_1{}^0(\omega) = \dfrac{1}{[\alpha'(M)]^{\frac{1}{2}}} \dfrac{F_1(\omega)}{E-\omega}, \\[2ex] \phi_2{}^0(\omega) = \dfrac{1}{[\alpha'(M)]^{\frac{1}{2}}} \dfrac{F_2(\omega)}{E-\omega}, \end{cases} \tag{73}$$

$$|\mathrm{I};E\rangle \to \begin{cases} \phi_0{}^{\mathrm{I}}(E) = -\dfrac{F_1(E)}{\alpha^+(E)}, \\[2ex] \phi_1{}^{\mathrm{I}}(E;\omega) = \delta(E-\omega) + \dfrac{F_1(E)}{\alpha^+(E)} \dfrac{F_1(\omega)}{E-\omega+i\epsilon}, \\[2ex] \phi_2{}^{\mathrm{I}}(E;\omega) = \dfrac{F_1(E)}{\alpha^+(E)} \dfrac{F_2(\omega)}{E-\omega+i\epsilon}, \end{cases} \tag{74}$$

$$|\mathrm{II};E\rangle \to \begin{cases} \phi_0{}^{\mathrm{II}}(E) = \dfrac{-F_2(E)}{\alpha^+(E)}, \\[2ex] \phi_1{}^{\mathrm{II}}(E;\omega) = \dfrac{F_2(E)}{\alpha^+(E)} \dfrac{F_1(\omega)}{E-\omega+i\epsilon}, \\[2ex] \phi_2{}^{\mathrm{II}}(E;\omega) = \delta(E-\omega) + \dfrac{F_2(E)}{\alpha^+(E)} \dfrac{F_2(\omega)}{E-\omega+i\epsilon}, \end{cases} \tag{75}$$

where the ϕ are the three components of the indicated wave functions and where

$$\alpha^{\pm}(z) = z - m_0 - \int \frac{F_1{}^2(\omega) - F_2{}^2(\omega)}{z-\omega\pm i\epsilon}, \tag{76}$$

with

$$F_1(\omega) = \theta(\omega-\mu_1) g_1 \frac{f(\omega)}{(2\omega)^{\frac{1}{2}}}, \qquad F_2(\omega) = \theta(\omega-\mu_2) g_2 \frac{f'(\omega)}{(2\omega)^{\frac{1}{2}}}. \tag{77}$$

We have normalized the wave functions to be on the energy scale. We now assert that these wave functions are the elements of the Møller wave matrix. For this purpose it is necessary and sufficient to show that these wave functions are orthonormal and complete.[4] The demonstration of this result is straightforward and is omitted. Let us now construct the "physical"

and "unphysical" scattering states:

$$|A;E\rangle=\frac{F_1(E)|\mathrm{I};E\rangle+F_2(E)|\mathrm{II};E\rangle}{[F_1^2(E)-F_2^2(E)]^{\frac{1}{2}}},\qquad |B;E\rangle=\frac{F_2(E)|\mathrm{I};E\rangle+F_1(E)|\mathrm{II};E\rangle}{[F_1^2(E)-F_2^2(E)]^{\frac{1}{2}}},\tag{78}$$

which diagonalize the phase shift matrix of these states corresponding to the phase shifts

$$\delta^A(E)=\frac{1}{2i}\ln\frac{\alpha^-(E)}{\alpha^+(E)};\quad E>\mu_1,\qquad \delta^B(E)=0.\tag{79}$$

For constructing the physical particle variables, we now construct a "comparison theory" consisting of only positive-norm states and having one bound state $|v\rangle$ and one set of scattering states $|E\rangle$ with $E>\mu_1$ which corresponds to the scattering of a meson of mass μ_1 and which has the same phase shifts; the "wave functions" are now two-component entities which may be interpreted as the bare one-particle and two-particle amplitudes. These wave functions have the components:

$$|v\rangle\rightarrow\begin{cases}\chi_0{}^0=\dfrac{1}{[\alpha'(M)]^{\frac{1}{2}}},\\[2mm] \chi_1{}^0(\omega)=\dfrac{-1}{[\alpha'(M)]^{\frac{1}{2}}}\dfrac{F(\omega)}{E-\omega},\end{cases}\tag{80}$$

$$|E\rangle\rightarrow\begin{cases}\chi_0{}^{\mathrm{I}}(E)=-\dfrac{F(E)}{\alpha^+(E)},\\[2mm] \chi_1{}^{\mathrm{I}}(E;\omega)=\delta(E-\omega)+\dfrac{F(E)}{\alpha^+(E)}\dfrac{F(\omega)}{E-\omega+i\epsilon},\end{cases}\tag{81}$$

where $\alpha^+(z)$ is the same expression as before and $F(\omega)=[F_1{}^2(\omega)-F_2{}^2(\omega)]^{\frac{1}{2}}$. It is easily verified that these wave functions do predict exactly the same phase shift as given by $\delta^A(E)$ in Eq. (79). One observes that the physical states $|V\rangle$, $|A;E\rangle$ of the original theory involving physical and unphysical states are in one-to-one correspondence with the states $|v\rangle$, $|E\rangle$ of the comparison theory and this correspondence preserves scalar products (and phase shifts). Hence dynamical variables defined by linear operations on these wave functions of the two systems can be put in one-to-one correspondence with each other.

We may now employ this correspondence to define the relative momentum and position variables[7] for the original system; and it is then easily seen that *the configuration wave function for the "physical" scattering state is given by the Fourier transform of* $\chi_1{}^{\mathrm{I}}(E;\omega)$ (rather than of $\phi_1{}^{\mathrm{I}}(E;\omega)$ or of $[F_1(E)\phi_1{}^{\mathrm{I}}(E;\omega)+F_2(E)\phi_1{}^{\mathrm{II}}(E;\omega)]/[F_1{}^2(E)-F_2{}^2(E)]^{\frac{1}{2}}$). Hence it is apparent that the system involving two kinds of mesons and an indefinite metric is equivalent, in this sector, to an alternative theory involving only one kind of meson with an appropriate form factor and no indefinite metric.[1]

The essential restriction imposed on the construction of a particle interpretation is that all "physical" dynamical variables operating on any physical state generate a combination of *physical* states only; in the terminology of Foldy and Wouthuysen,[8] all physical variables should be "even operators." All intuitive objections to our interpretive postulate regarding physical states are based on *gedanken* experiments which violate the characterization of physically measurable dynamical variables.

The use of the indefinite metric thus leads to the construction of models which are equivalent to models with a definite metric but an effective form factor. They are thus trivial in nonrelativistic theories except perhaps to introduce "simple" form factors; but they are far from trivial for relativistic theories where nonlocal theories are normally beset with conceptual and consistency difficulties.[9] The advantages of generating effective form factors is seen quite transparently in the model involving the heavy-particle recoil in the previous section.

For completeness we should perhaps mention that in the above demonstration we had assumed that $F_1{}^2(E)-F_2{}^2(E)>0$ for all E. If this is not so, we must note that at some critical energy E_c the quantity $F_1{}^2(E)-F_2{}^2(E)$ changes sign and vanishes at $E=E_c$; the role of the physical and unphysical states are interchanged for $E>E_c$. One has the scattering phase shift for the physical state

$$\delta(E)=\frac{1}{2i}\ln\left(\frac{\alpha^-(E)}{\alpha^+(E)}\right)\theta(E_c-E),$$

and $\alpha^+(E_c)$ is real. These circumstances do not add anything essentially new to the interpretation.

The analysis of measurements presented above shows that one has to be very careful in dealing with the "asymptotic conditions" in any realistic theory with interaction, since if the interactions are local the theory may involve an indefinite metric; on the other hand, if the theory involves only a positive-definite metric then the effective interactions are nonlocal and appeal to covariance may be unwarranted. In any case the physical characterization of asymptotic fields

[7] Compare T. D. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949); R. Acharya and E. C. G. Sudarshan, J. Math. Phys. (to be published).

[8] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[9] See, for example, *Proceedings of the International Conference of Theoretical Physics, Kyoto and Tokyo, 1953* (Science Council of Japan, Tokyo, 1954).

in manifestly covariant quantum theories of interacting fields may not be as simple as the characterizations in current use.[10]

6. DISCUSSION

In this paper we have dealt in detail with several simple model theories which are solvable and at the same time exhibit scattering. There are three major points demonstrated by this study:

(1) The nature of the "physical" solutions of a quantum-mechanical system with an indefinite metric; and the equivalence of the Hamiltonian and dispersion-theoretic treatments of the problem.

(2) The analytic structure of the amplitudes in such models; the upper half-plane analyticity ("causality") of the amplitude continues to be true provided time-reversal invariance holds. Further the exact solutions are *analytic* functions of the coupling strength parameters and (while the operator structure and physical interpretation changes discontinuously when one of the coupling strength parameters changes sign) the solutions can be obtained by analytic continuation from a "normal" case.[11]

(3) The interpretive postulate of the present theory and its realization in terms of physical particle variables illustrates the new physical principles involved and shows that notions relating to asymptotic conditions have to be handled very carefully. In particular, if one wants to interpret every singularity of a scattering amplitude in terms of many-particle states[12] at least some of these states may be unphysical; a particularly interesting case in point is the analysis of the Møller (electron-electron) scattering amplitude in terms of a "complete set of intermediate" states, where if one wants to get the Coulomb potential unphysical states have to be included.[13]

Admittedly all the systems discussed in the present paper are highly idealized models chosen only by virtue of the fact that they could be solved exactly. The success of the new physical principles involved for a quantitative discussion of physical phenomena can only be tested by the study of more elaborate and realistic systems. But the relevance of the general questions discussed here is already apparent from the solutions to the model and may have to be considered in any *fundamental* theory.

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[10] See, for example, A. S. Wightman, *Proceedings of the Conference on Mathematical Problems of Quantum Field Theory, Lille, 1957* (unpublished).

[11] Contrast this with the discussion by F. J. Dyson, Phys. Rev. **85**, 631 (1952).

[12] See, for example, G. F. Chew, University of California Lawrence Radiation Laboratory Report UCRL-9289 (unpublished).

[13] We wish to thank Professor C. J. Goebel for a discussion of this topic.