(*b*) *Small-κ approximation:*

For sufficiently small values of κ the ratio $(\kappa/|\delta|)^4$ becomes negligible, which is equivalent to uncoupling. There are two dispersion relationships: $A(\omega,k)=0$ represents the transverse mode, and $F(\omega,k)=0$ represents the longitudinal mode. The corresponding instabilities will be illustrated by means of "$\omega-k$" diagrams.

(1) Longitudinal oscillations. The corresponding "$\omega-k$" diagram shown in Fig. 10(b) is qualitatively similar to the one shown in Fig. 10(a). There is, however, an essential difference between them. The curve of Fig. 10(a) represents the dispersion relationship for the entire range of ω and k. On the other hand, the curve of Fig. 10(b) represents the "small-κ approximation," and, therefore, the only range of ω and k that is quantitatively significant is in the regions for which there exists a solution for δ satisfying the inequalities $\kappa\ll|\delta|\ll\tilde{\omega}$. It has been shown that there exists such a solution at least in the interior of the two rectangular "windows." The remainder of the graph is inappropriate and has been included only to show more clearly the character of the graph inside the "windows." The shaded portions again show the region of instability and the unshaded portion shows the region of stable oscillations. The instability is "strong" in this region since the inequality $\kappa\ll|\mathrm{Im}(\delta)|$ is satisfied. The instability is convective.

(2) Transverse oscillations. Figure 11 shows the "$\omega-k$" diagram which is applicable only in the regions where there exists a solution for δ satisfying the inequalities $\kappa\ll|\delta|\ll\tilde{\omega}$. As in the previous case, such a solution has been shown to exist inside the rectangular "windows." The remainder of the graph is shown only in order to illustrate more clearly the behavior of the small portion of the graph inside the "windows." The instability shown in the shaded area is associated with Vavilov-Cherenkov radiation and is shown to be convective.

If one examines the graphs in Figs. 9, 10(b), and 11, it is clear how the Vavilov-Cherenkov radiation is "extended" into region of hybrid instability of the type "*t*" and the Bohr radiation is "extended" into the hybrid instability of type "*l*".

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Effective Depth of X-Ray Production*

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By measuring x-ray emission profiles as a function of voltage, the depth of penetration of the cathode electrons involved may be calculated. Similarly, the effective depth of x-ray production may be obtained. Uncertainties exist because of conditions at the surface, so the rate of change of these depths with voltage may be established with greater confidence than the values themselves. For copper, the continuum yields values increasing from 300 A/kv to 480 A/kv over the 10–30 kv range. An average value of 450 A/kv is obtained from analysis of the less trustworthy line data. The effective depth obtained from the conventional calculation on the absorption edge is shown to have little physical meaning.

INTRODUCTION

INFORMATION about the passage through matter of electrons having energies in the range of tens of kilo-electron volts is needed for the analysis of several physical phenomena. Among them are (1) the correction of the low-energy portion of the continuous β-ray spectrum, and (2) the correction for target self-absorption of x-ray emission spectra involving transitions of the valence electrons. Some information pertinent to the problem is available from measurements on the intensity of emission of x-ray lines and also from the discontinuities in the continuum. This can yield a measure of the depth of penetration of the cathode electron and also a measure of the effective depth of x-ray production. The information is necessarily limited in scope since it involves cathode electrons which have lost little energy through inelastic collisions prior to the event which produced the x rays.

The physical processes involved in x-ray production are numerous and complex. The path of the cathode ray in the target is tortuous, although the electrons involved in this particular process are probably traversing a reasonably linear path. These electrons may eject inner electrons from the target atoms giving rise to directly produced line radiation; or they may be abruptly deflected so that continuous radiation is produced. The bremsstrahlung of sufficient energy has the further capacity to produce line radiation by being photoelectrically absorbed. Because of the plethora of

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physical processes involved, the situation is not readily amenable to analysis. It is doubtful that a truly definitive study could be made at present; nevertheless, information of value can be obtained, and reasonable agreement may be found with quasi-theoretical formulas.

THEORETICAL

Thomson,[1] through classical arguments, Whiddington,[2] by experimental measurements, and Bohr,[3] by crude quantum theoretical calculations, arrived at a common mathematical expression for electron energy as a function of penetration. They obtained $V_0^2-V^2=cx$, where V_0 is the original energy of the impinging electron, V is the energy at the depth x, and c is effectively a constant for a given material.

There are obvious weaknesses in the assumptions that underlie the theoretical derivations, and Wilson's[4] measurements, made at about the same time as Whiddington's, favored an equation of the form, $V_0-V=c'x$. In any event, the data do not unambiguously support a particular functional form.

The concept of an absorption coefficient for beta particles has been used frequently. For beta rays having a continuous range of energy, good exponential transmission curves are obtained.[5] On the other hand, Glendenin's[6] work on monoenergetic beta particles favors a linear relation, of negative slope, between the number of transmitted electrons and absorber thickness.

In our particular problem involving low voltage cathode particles and the production of x rays, the important feature is not the decrease in energy of the particles; rather, it is the attrition in the number of particles capable of producing x radiation. Over the range of voltage employed, the cross section for bremsstrahlung production will not vary with the energy of the electron, nor will the cross section for K excitation be a strong enough function of the electron energy to markedly affect the result. Consequently, we need concern ourselves principally with the variation of the number of impinging electrons as a function of depth of penetration. Thomson obtained an expression for this also, writing the number of corpuscles crossing unit area as $N\exp(-kx/V_0^2)$, V_0 being the original energy. There is good reason to believe that an exponential law would be quite adequate under the conditions considered in this paper. An exponential law obtains when the particle is extracted from the beam in a single discrete process; this is virtually always the case when the incoming electron loses enough energy to produce the x ray.

[1] J. J. Thomson, *Conduction of Electricity Through Gases* (Cambridge University Press, New York, 1906), 2nd ed., p. 375.
[2] R. Whiddington, Proc. Roy. Soc. (London) **A86**, 360 (1911–1912).
[3] N. Bohr, Phil. Mag. **25**, 360 (1913).
[4] W. Wilson, Proc. Roy. Soc. (London) **A84**, 141 (1910–1911).
[5] W. Paul and H. Steinwedel, *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955), pp. 20–23.
[6] L. E. Glendenin, Nucleonics **2**, 12 (1948).

Using this concept, we define a linear absorption coefficient, b, so that

$$N=\int_{x=0}^{x=\infty} dN=\int_0^\infty Nbe^{-bx}dx, \tag{1}$$

where N is the total number of quanta produced directly per unit solid angle per unit time and may refer to the number of quanta in an emission line or to the number of quanta in a range of wavelengths in the continuum. As will be established, the indirectly produced line radiation is relatively small and will not be considered here. If the takeoff target angle is ϵ, the number of quanta observed per unit time and solid angle is

$$N_0=\int_0^\infty Nbe^{-bx}e^{-\mu x\csc\epsilon}dx=\frac{b}{b+\mu\csc\epsilon}N, \tag{2}$$

where μ is the linear absorption coefficient appropriate to the wavelength of the x rays observed. The electronic absorption coefficient, b, is a function of voltage, and may be evaluated in the following way. One may measure the ratio of intensities on the two sides of an absorption edge; for a given voltage this will be

$$r_E=(b+\mu_2\csc\epsilon)/(b+\mu_1\csc\epsilon). \tag{3}$$

Measurement of r_E as a function of voltage will determine $b(V)$.

It is of value to establish a representative depth at which the x rays may be considered to be produced. We can obtain a mean depth of electron penetration by averaging; thus

$$X_m=\left(\frac{1}{N}\right)\int_0^\infty Nbe^{-bx}xdx=1/b. \tag{4}$$

This is also the mean depth of x-ray production for both the continuous radiation and the directly produced line radiation. However, the x rays are attenuated on passage through the anode material so that we define a weighted depth of production from

$$X_w=\int_0^\infty be^{-bx}xe^{-\mu x\csc\epsilon}dx, \tag{5}$$

so that we have

$$X_w=b/(b+\mu\csc\epsilon)^2. \tag{6}$$

Furthermore, we obtain the relation

$$X_w/X_m=b^2/(b+\mu\csc\epsilon)^2. \tag{7}$$

From the dependence on μ, we see that the weighted depth for production of x rays is dependent on the wavelength under consideration.

Of perhaps more significance is the effective depth of

x-ray production, X_e, which we shall define so that

$$\int_0^\infty Nbe^{-bx}e^{-\mu x \csc\epsilon}dx = Ne^{-\mu X_e \csc\epsilon}. \tag{8}$$

That is, if the x rays were all produced at X_e, they would have suffered the same net attenuation as when they are produced according to the exponential law. We see then that

$$b/(b+\mu \csc\epsilon) = e^{-\mu X_e \csc\epsilon}. \tag{9}$$

Solving under the reasonable assumption that $\mu \csc\epsilon$ will be relatively small in most cases, one gets

$$X_e = (1/b) - (3\mu \csc\epsilon/2b^2). \tag{10}$$

Obviously, X_e is not greatly different from X_m unless target self-absorption is excessive.

It has been common practice[7] to calculate an effective depth of x-ray production, X_c, from the observed jump; thus,

$$e^{(\mu_2-\mu_1)X_c \csc\epsilon} = r_E. \tag{11}$$

The value of X_c computed from this relation is of virtually no significance. In fact, it can be readily shown that the value of X_c is less than X_e for the wavelengths on either side of the edge. Consider X_{e1} to be the effective depth of penetration on the low absorption side of the edge and X_{e2} to be the effective depth of penetration on the high absorption side. Then we have

$$e^{(\mu_2-\mu_1)X_c \csc\epsilon} = (b+\mu_2 \csc\epsilon)/(b+\mu_1 \csc\epsilon). \tag{12}$$

But from (8) we can obtain

$$e^{(\mu_2 X_{e2}-\mu_1 X_{e1}) \csc\epsilon} = (b+\mu_2 \csc\epsilon)/(b+\mu_1 \csc\epsilon). \tag{13}$$

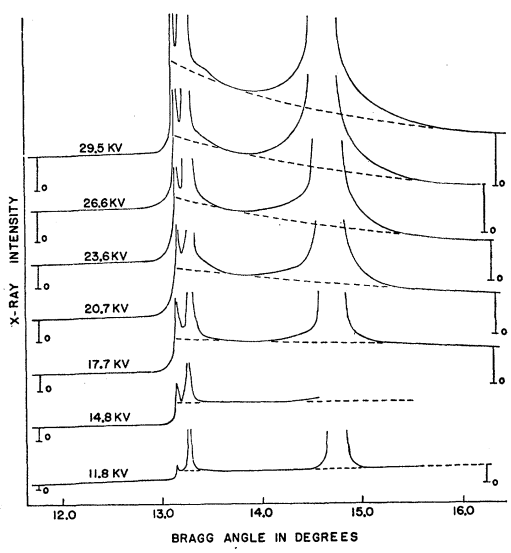

FIG. 1. Emission profile of Cu target x-ray tube operated at various voltages.

[7] D. L. Webster and A. E. Hennings, Phys. Rev. **21**, 301 (1923). H. P. Hanson and J. Herrera, *ibid.* **105**, 1483 (1957).

Equating, we find that

$$(\mu_2-\mu_1)X_c = \mu_2 X_{e2} - \mu_1 X_{e1}. \tag{14}$$

This can be written in the form

$$X_c = X_{e2} - \frac{\mu_1}{\mu_2-\mu_1}(X_{e1}-X_{e2}). \tag{15}$$

Thus, the "true" effective depth of x-ray production may be considerably greater than the jump at the absorption edge indicates.

We shall also be interested in a comparison of the intensities of the K_α and the K_β lines. In this case, the ratio is given by

$$r = \left[\frac{b+\mu_\beta \csc\epsilon}{b+\mu_\alpha \csc\epsilon}\right]\frac{N_\alpha}{N_\beta}. \tag{16}$$

This expression could not yield a reliable value of b since the value of N_α/N_β is not available. However, N_α/N_β will presumably not be a function of the accelerating potential so the ratio may be taken of the r's at various values of the voltage, i.e.,

$$\frac{r(V_1)}{r(V_2)} = \frac{b(V_1)+\mu_\beta \csc\epsilon}{b(V_1)+\mu_\alpha \csc\epsilon}\,\frac{b(V_2)+\mu_\alpha \csc\epsilon}{b(V_2)+\mu_\beta \csc\epsilon}. \tag{17}$$

To a first order of approximation, this yields

$$\frac{r(V_1)}{r(V_2)} = 1 + (\mu_\alpha-\mu_\beta)\csc\epsilon\left[\frac{b(V_1)-b(V_2)}{b(V_1)b(V_2)}\right]. \tag{18}$$

Furthermore, for increments of voltage that are not too large, we see that

$$\Delta X_e = \frac{-1+r(V_1)/r(V_2)}{(\mu_\alpha-\mu_\beta)\csc\epsilon}. \tag{19}$$

Here the difference in μ_α and μ_β is not enough to invalidate the use of a common effective depth for production of the lines. Starting with this concept, one can readily show that

$$\Delta X_e = \frac{\ln r(V_1) - \ln r(V_2)}{(\mu_\alpha-\mu_\beta)\csc\epsilon}, \tag{20}$$

which is the same as the previous expression in the approximation that $r(V_1)/r(V_2)$ is very nearly 1.

EXPERIMENTAL

Measurements were taken on the intensity of the line radiation as a function of voltage and on the K edge jump as a function of voltage. The target materials were copper and nickel. The continuum was measured with a continually aligned Economy model[8] two-crystal

[8] H. P. Hanson and R. Economy Rev. Sci. Instr. **29**, 420 (1958).

spectrometer; the bulk of the data on line intensities was obtained with a Philips goniometer although some measurements were done with the two-crystal unit. Intensity measurements were made on continually pumped tubes and on sealed-off commercial tubes. The geometry of the former could be adjusted to our desires, but these tubes lacked the inherent stability of the commercial tubes and they provided data over a limited range. In the continuously pumped tube, the targets were truncated so as to give 15° and 20° takeoff angles for horizontal beams of x radiation.[9] These targets were also plated (copper on nickel) for purposes of examining the indirectly produced line radiation.

The line intensity was measured by ordinary counting techniques using Philips geiger counters; the continuum was continuously recorded by a Berkeley rate meter in conjunction with a Varian recorder. The absorption edge ratios were obtained as functions of voltage on the 4° targets. In the 15°–20° targets, the jumps are too small to observe changes with a usable degree of accuracy. In addition, the commercial tubes provided more stable operation, and this is essential since a meaningful measurement of the jump requires measurement over a considerable wavelength region of the continuum.

Two likely sources of error lay in target inhomogeneities and tungsten contaminations of the anode. Care was taken to make the targets as smooth as possible with the residual striations of the final polishing left in the direction of the emergent x-ray beam. The anode of the commercial tube could, of course, have pits of which we were not aware; however, the tubes had always been operated, in our laboratory, far below the rated voltage.

Tungsten plating on the surface will result in an increased value of the intensity in the continuum but a smaller value of the line intensity. While a thin layer would change the absolute value of the depth of electron penetration, it would not markedly change the variation of this quantity with voltage. Nevertheless, it is certainly advisable to obtain information about the rate of tungsten buildup. It was found that the intensity of the $WL_{\alpha 1}$ line increased linearly in time over relatively short periods. For runs of hundreds of hours at constant voltage and current, the curve fit the following expression: $I=I_0(1-e^{-pt})$, a form which may be derived on the basis of very simple assumptions. The precise value of p depends at least in part on the vacuum conditions in the x-ray tube. To keep the intensities due to the tungsten contaminant less than that observed on the sealed-off tube, we were not able to operate for more than a few hours without disassembling and cleaning the unit. No apparent difficulty was experienced due to backstreaming of diffusion pump oil in our well-baffled system although an invisible film of oil might have been present on the target surface. The carbon contaminant would tend to decrease both the line and continuum intensities.

[9] Obviously the cot rather than the csc should be employed in our previously derived equation for this geometry, but for this angle the values are as close as the theory warrants.

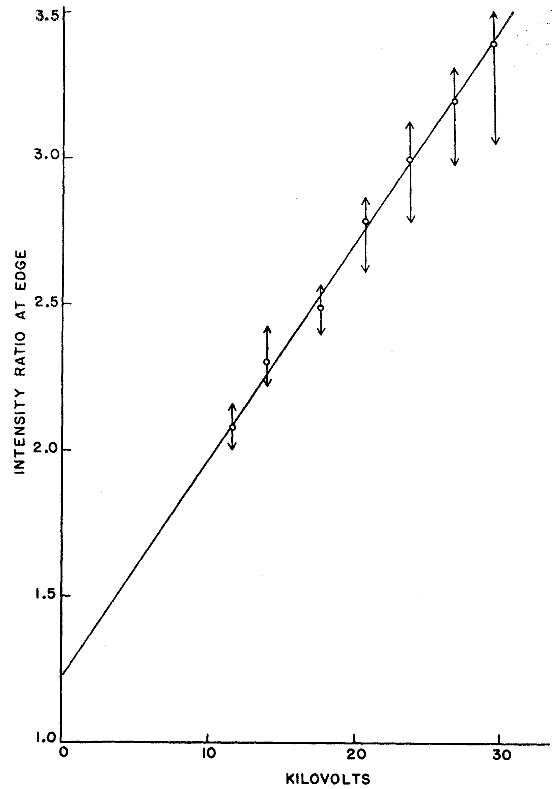

FIG. 2. Absorption jumps, r_E values as a function of voltage.

The current and voltage were regulated by conventional techniques. The amount of ripple in the rectified and smoothed voltage was certainly less than 5 parts in 1000.

DISCUSSION

The emission profiles showing the absorption discontinuities are plotted to a common intensity scale in Fig. 1. Particularly for the low voltages, the figures do not indicate the true degree of precision to which the profiles may be measured. However, large variations are possible in estimating the magnitude of the continuous radiation on the long-wavelength side of the edge. The dotted curve shown represents a rather crude guess as to the magnitude; the values of the estimated ratios, r_E, are plotted in Fig. 2. The points are presumably unbiased estimates of the r_E values, and the vertical bars represent *a posteriori* estimates of the range of values that might be reasonable from the data. The values of b, X_m, and X_c which are tabulated in Table I were computed from the straight line drawn in Fig. 2, although it is clear that other values of r_E are well within the realm of probability. One can state unequivocally, however, that a pronounced increase in r_E occurs. With the personal factor involved in judging the level of the continuous radiation, it is obviously

TABLE I. Absorption jump, mean depth of electron penetration, and depths of x-ray production defined according to the text. Obviously the numbers given in last six columns are accurate to no more significant figures than those in the second. The numbers are carried merely to show the trend. The units of b are reciprocal angstroms, the depths are in angstroms, and the last column is in angstroms/kilovolt.

Voltage (kv)	r_E	b	X_m	X_{e1}	X_{e2}	X_c	$\Delta X_m/\Delta V$
10	1.97	30 785	3248	3022	2095	1970	
12	2.12	26 029	3842	3529	2344	2180	297
14	2.27	22 393	4466	4051	2581	2370	312
16	2.42	19 527	5121	4584	2809	2560	328
18	2.57	17 209	5811	5132	3028	2730	345
20	2.72	15 295	6538	5685	3241	2900	364
22	2.87	13 688	7306	6273	3448	3060	384
24	3.02	12 319	8118	6868	3650	3210	406
26	3.17	11 140	8977	7479	3848	3350	430
28	3.32	10 113	9888	8109	4042	3490	456
30	3.47	9211	10 856	8758	4232	3620	484

possible to draw a curve for the r_E which would yield a truly straight line relationship for the values of X_m listed in Table I. Even with the r_E values selected, a log log plot of X_m versus applied voltage reveals a relationship that is nearly linear as is shown in Fig. 3.

The term in the last column of Table I, $\Delta X_m/\Delta V$, has the most physical significance of the items listed, but considering the uncertainty in r_E is probably only reliable to about 20%.

A value for $\Delta X_e/\Delta V$ may also be obtained by examining the relative intensities of the K_α and K_β lines as a function of voltage. The variation is small and statistically significant data could not be obtained by merely shifting between two voltage settings at each of the two lines. This was presumably due to our inability to reproduce the voltage settings with sufficient precision. The technique successfully employed[10] was to make a series of point-by-point measurements over the range from 10 to 20 kv. One may then find a difference in slope of the intensities of the K_α and the K_β line when plotted as a function of voltage on logarithmic scale coordinates. Such plots are known to be approximate straight lines. The change of an effective depth, X_e, may be written from Eq. (20), as

$$\Delta X_e = \frac{\ln[I_\alpha(V_1)/I_\alpha(V_2)] - \ln[I_\beta(V_1)/I_\beta(V_2)]}{(\mu_\alpha - \mu_\beta)\csc\epsilon}. \quad (21)$$

The convergence of the two lines as they tend toward higher voltage is too slight to illustrate, but the estimated value of $\Delta X_e/\Delta V$ is 450 A/kv averaged over the 10–20 kv range. This figure was obtained for both copper and nickel targets on the continuously pumped tubes.

The values of X_c, which were shown to be of virtually no significance, are seen from Table I to be smaller than X_m by factors varying from about $\frac{1}{3}$ to $\frac{2}{3}$. This undoubtedly also accounts for the relatively small depth of cathode ray penetration reported by Webster and Hennings who used the absorption edge to find X_c.

Table I also presents the effective depth of x-ray production on the two sides of the x-ray edge when calculated according to Eq. (9), and a plot of the various depths are found in Fig. 3. It is to be noted that the X_m and X_{e1} values plotted against voltage have slopes which do not differ greatly from one. This bespeaks a linear dependence of depth of penetration at this low voltage in keeping with the results of early workers.[11]

In general, empirical relations that describe the penetration of β particles through matter are only valid at high energies. The Flammersfeld[12] equation, which purports to be accurate over the range of 0 to 3 Mev, yields a figure of about $\frac{1}{2}$ of our value at 10 kv but is 15% too high at 30 kv. On the other hand, the $\Delta X_m/\Delta V$ computed from Flammersfeld's equation conforms almost exactly to ours at 10 kv but is high by a factor of 1.7 at 30 kv.

The value of X_c obeys an approximate $V^{\frac{1}{2}}$ relation. This is not consistent with other measures of the penetration of electrons or of the effective depth of x-ray production.

The straight line of Fig. 2 relating r_E and V is observed to pass above the origin. This may be inter-

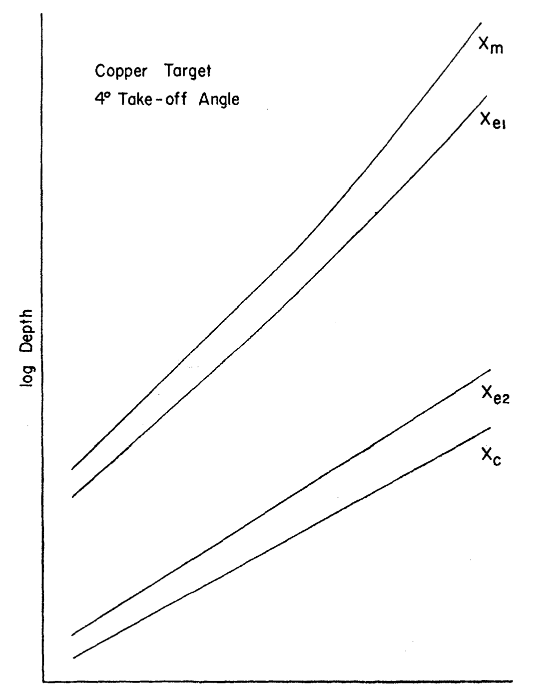

FIG. 3. Depth of x-ray production as a function of voltage. Depths defined according to text.

[10] S. I. Salem, Ph.D. thesis, The University of Texas, Austin, Texas, 1959 (unpublished).

[11] W. R. Ham, Phys. Rev. 30, 96 (1910). W. P. Davey, J. Franklin Inst., March (1911).

[12] A. Flammersfeld, Naturwissenchaften 33, 280 (1946).

preted in either of two ways. One might suggest that there is an effective, zero-voltage, mean free path which the electrons traverse before a photon-producing encounter occurs. From the extrapolated r_E value of 1.225, the mean free path would be 580 A. This is not greatly different from the calculated value[13] of 420 A given for the mean free path of conduction electrons in copper. Indeed the agreement could readily be made exact by a minor modification of Fig. 2.

However, this value is probably smaller than irregularities that are to be expected in the surface. Since the tube on which this particular measurement was made is sealed off, no information is available about the apparent smoothness of the target. Target pitting represents a distinct possibility as a source of the residual r_E value.

In principle, one might hope to glean information about the indirectly produced line radiation from measurements on plated targets. The plating should presumably be of sufficient thickness that only indirect radiation is produced in the backing material. In practice, one finds that the data are not reliable. For example, the slopes of the $\ln I$ vs $\ln(V-V_K)$ curves for unplated nickel anodes were about 1.9 in our 15° targets. When they were plated with copper, similar curves for the nickel radiation had slopes that ranged from 1.6 to 3.0. There are explanations which one may offer for either an increased or a decreased slope. For example, the effective depth for continuous x-ray production moves closer to the underlying material as the voltage increases, thus tending to produce an increase of slope. On the other hand, the deeper penetration of the bremsstrahlung with voltage will tend to decrease the slope. It was found that a target that is deliberately damaged gives high values of the slopes; consequently, it was assumed that the satisfactory curves were those with low values. It is found that $\Delta X_e/\Delta V$ values obtained from such curves were about four times as great as in the unplated targets. Since the $\Delta X_e/\Delta V$ found for the lines in unplated targets was not greatly different from $\Delta X_m/\Delta V$, the indicated portion of indirectly produced line radiation is relatively small.

However, geometric factors may well tend to produce an erroneous figure for the percentage of indirect radiation when measured in this fashion. Presumably, the cathode electrons involved in this process have not been appreciably deflected from their original direction. The Sommerfeld expression for the radiation field would indicate that even at these low voltages the intensity maxima lean appreciably forward. Thus, the measured value may not represent the true effect. Furthermore, the uncertainty in measuring the differences in slopes precludes any serious quantitative analysis by this technique. Only a direct comparison such as is made in the following paper would produce reliable results.

[13] Charles Kittel, *Introduction to Solid-State Physics* (John Wiley & Sons, Inc., New York, 1957), 2nd ed., p. 240.