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Vavilov-Cherenkov Effect and "Bohr Radiation" Produced by a Beam of Charged Particles in a Dispersive Medium

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An electron beam interacting with a dispersive (atomic or molecular) medium produces two intense sources of instability that are represented by a growing longitudinal wave and a growing transverse wave. The longitudinal wave has frequencies that are equal to the atomic binding frequencies of the surrounding medium and is designated as the "Bohr wave." The transverse wave has frequencies determined by the Vavilov-Cherenkov criterion and is similar to the Vavilov-Cherenkov wave produced by a single particle interacting with the medium. These sources of instability are "continued" into lower frequency ranges in which they produce growing waves of a "hybrid" type that are characterized by an electric vector having both longitudinal and transverse components. The longitudinal and transverse waves represent the "fundamental modes" that exist in the medium in the absence of the beam. The perturbation produced by the beam is responsible for the instability of the fundamental modes and for the occurrence of the coupling between these modes. The coupling produces electromagnetic waves in which the electric field has a longitudinal component. The conditions for coupling and the character of the instabilities are investigated.

INTRODUCTION

THE study of interaction of moving charged particles with a dispersive (atomic or molecular) medium is one of the oldest and one of the most active areas of research in modern physics. During the last 47 years a considerable literature has been accumulated on this subject. It includes the pioneer investigations of Bohr,[1] Bethe,[2] Bloch,[3] Vavilov,[4] Cherenkov,[5] Frank and Tamm,[6] and others. It is of interest to note that these investigations dealt exclusively with the interaction of an individual charged particle with an atomic or molecular medium. Thus no consideration has been given to the collective interactions of a beam of charged particles with such a medium. The collective interactions may be of interest in the study of gaseous discharges, in investigations of the ionosphere, and in astrophysics.

There are new and physically significant effects that result from such collective interaction, and, apparently, very few investigations have been devoted to the study of these effects. Some remarks on this subject have been made by Akhiezer and Fainberg,[7] and the nature of the interaction under some specific conditions has been studied by Getmantsev,[8] Getmantsev and Rapoport,[9] and Neufeld.[10] (The latter reference shall be designated as Paper I.) Both Getmantsev and Getmantsev and Rapoport dealt with a nondispersive medium. Getmantsev assumed that there is no external magnetic field, while Getmantsev and Rapoport took such field into account.

This investigation deals with the interaction of a beam with a dispersive medium and brings into evidence certain physically significant effects that are associated with the frequency dispersion.

A beam of charged particles passing through a dispersive (atomic or molecular) medium produces

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[1] N. Bohr, Phil. Mag. **25** (6) 10 (1913); **30** (6) 581 (1915).

[2] H. A. Bethe, Ann. Physik **5**, 325 (1930). Z. Physik **76**, 293 (1932).

[3] F. Bloch, Ann. Physik **16**, 285 (1933).

[4] S. I. Vavilov, Doklady Akad. Nauk. SSSR **2**, 457 (1934).

[5] P. A. Cherenkov, Doklady Akad. Nauk. SSSR **2**, 451 (1934).

[6] I. M. Frank and I. E. Tamm, Doklady Akad. Nauk. SSSR **14**, 107 (1937).

[7] A. I. Akhiezer and Ia. B. Fainberg, Zhur. Eksptl. i Teoret. Fiz. **21**, 1262 (1951).

[8] G. G. Getmantsev, Zhur. Eksptl. i Theoret. Fiz. **37**, 843 (1959) [translation: Soviet Phys.—JETP **10**, 600 (1960)].

[9] G. G. Getmantsev and V. O. Rapoport, Zhur. Eksptl. i Teoret. Fiz. **38**, 1205 (1960) [translation: Soviet Phys.—JETP **11**, 871 (1960)].

[10] J. Neufeld, Phys. Rev. **116**, 785 (1959).

radiation of a type that apparently was not previously observed and is different from the Vavilov-Cherenkov radiation. The occurrence of this new radiation and some of its properties were discussed in Paper I. It was pointed out in Paper I that the radiated frequencies are the atomic binding frequencies (Bohr frequencies) and not those that satisfy the Vavilov-Cherenkov criterion. This new radiation designated as "Bohr radiation" is propagated in the form of longitudinal waves. In order to transmit the Bohr radiation in free space, a mechanism for the conversion of longitudinal into transverse oscillations is necessary. This mechanism may be similar to the one reported by Ginzburg and Zhelezniakov.[11] Thus, in the case of a beam passing through a gaseous medium the conversion could be effected by the interaction of the longitudinal waves with the density fluctuation of the medium.

In Paper I an assumption was made that $\theta=0$, where θ is the angle formed by the direction of the wave and the direction of the beam. Under these conditions there are no electromagnetic "transverse" instabilities and the only instability that exists appears in the form of a growing longitudinal wave. However, when $\theta\neq 0$, electromagnetic instabilities appear, and these have been investigated by Getmantsev.[8] Since the investigation of Getmantsev concerned nondispersive media, it did not bring into evidence an instability associated with Bohr radiation.

The formulation of our problem shows that there exists a certain duality in the response of a frequency-dispersive dielectric medium to a perturbation produced by a beam. There are two "fundamental modes" characterized by a relatively intense instability. One of these is associated with longitudinal "Bohr waves" and the other with transverse waves. The transverse waves result from the interaction of a beam with the surrounding medium, i.e., they represent essentially a "beam effect." Nevertheless, there is a similarity between these transverse oscillations and the Vavilov-Cherenkov radiation due to the "particle effect." Thus, the transverse mode is associated with a Vavilov-Cherenkov cone which is structurally similar to the corresponding cone produced by a single particle.

Both the longitudinal and the transverse modes appear independently as uncoupled oscillations. The coupling of these waves produces "supplementary modes" characterized by "hybrid waves" (i.e., electromagnetic waves in which the electrical vector has a longitudinal component).[12] Each of these supplementary modes is associated with, and forms a "continuation" of, the corresponding fundamental mode. One of these supplementary modes represents the continuation of the longitudinal instability, and it occurs at frequencies for which the dielectric constant of the medium $\epsilon(\omega)<0$. This mode is represented by waves which do not grow as rapidly as the longitudinal waves for which $\epsilon(\omega)\sim 0$. The other supplementary mode represents the continuation of the transverse instability, and it occurs outside of the Vavilov-Cherenkov cone. It exhibits a less intense instability than the transverse wave aligned along the Vavilov-Cherenkov cone.

This paper is subdivided into the following sections: I. Formulation of the Problem; II. Uncoupled Modes; III. Coupled Modes; IV. Response of the System for a Fixed Value of θ; V. Directional Selectivity of Unstable Oscillations; VI. Small-Angle Approximation; and VII. Graphical Representation of the Dispersion Relations.

I. FORMULATION OF THE PROBLEM

A. Dispersion Equation

Consider a dispersive system comprising an electron beam moving with the velocity $\mathbf{v}$ in a dielectric medium. The medium can be defined by its dielectric constant $\epsilon(\omega)$ or by its electric susceptibility $\chi_e(\omega)$. These are expressed as follows:

$$\epsilon(\omega)=1+4\pi\chi_e(\omega)=1-\frac{\omega_1^2}{\omega^2-\omega_a^2}, \tag{1}$$

where $\omega_1^2=4\pi ne^2/m$. We are considering an idealized and simplified form of a dissipationless medium in which harmonic oscillators of binding frequency ω_a are uniformly distributed with a density n.

The beam and the dielectric interpenetrate each other and form a composite medium, the macroscopic properties of which shall be described. Under the effect of an electromagnetic perturbation, this composite medium becomes polarized. The electric polarization $\mathbf{P}$ and magnetic polarization $\mathbf{M}$ can be represented as

$$\mathbf{P}=\mathbf{P}_d+\mathbf{P}_b, \tag{2}$$

$$\mathbf{M}=\mathbf{M}_b. \tag{3}$$

The term $\mathbf{P}_d$ represents the electric polarization produced in the dielectric and the terms $\mathbf{P}_b$ and $\mathbf{M}_b$ represent the electric and magnetic polarization due to the presence of the beam. The term $\mathbf{P}_d$ is directly determined from Eq. (1), i.e.,

$$\mathbf{P}_d=\frac{1}{4\pi}[1-\epsilon(\omega)]\mathbf{E}. \tag{4}$$

In accordance with the accepted convention, we shall refer to the microscopic field quantities $\mathbf{E}$ and $\mathbf{B}$ as the electric field strength and "magnetic induction" and the corresponding macroscopic quantities $\mathbf{D}$ and $\mathbf{H}$ shall be identified as the electric displacement and the "magnetic field strength." The terms $\mathbf{P}_b$ and $\mathbf{M}_b$ are obtained from the equations for the moving media

[11] V. L. Ginzburg and V. V. Zhelezniakov, Astron. Zhur. **35**, 694 (1958).

[12] A similar situation occurs in a plasma beam system. Hybrid waves produced by coupling have been described by Jacob Neufeld and P. H. Doyle, Phys. Rev. **121**, 654 (1961).

as formulated by Minkowski.[13] These are as follows:

$$\mathbf{D}_b+\frac{1}{c}(\mathbf{v}\times\mathbf{H}_b)=\epsilon_b(\omega)\left[\mathbf{E}+\frac{1}{c}(\mathbf{v}\times\mathbf{B})\right], \tag{5}$$

$$\mathbf{H}_b-\frac{1}{c}(\mathbf{v}\times\mathbf{D}_b)=\frac{1}{\mu_b}\left[\mathbf{B}-\frac{1}{c}(\mathbf{v}\times\mathbf{E})\right], \tag{6}$$

where $\mathbf{D}_b$ and $\mathbf{H}_b$ represent the electric displacement and the magnetic field strength produced by $\mathbf{E}$, $\mathbf{B}$ in an electron beam moving with velocity $\mathbf{v}$. The expressions $\epsilon_b(\omega)$ and μ_b represent the dielectric constant and the magnetic permeability of a medium consisting of the electron beam. The magnetic permeability is constant ($\mu_b=1$) and in determining the capacitivity of our composite medium we take into account the Doppler shift due to the observer moving with relative velocity $\boldsymbol{\beta}=\mathbf{v}/c$ with respect to the medium. We put

$$\mathbf{P}_b=\frac{1}{4\pi}(\mathbf{D}_b-\mathbf{E});\quad \mathbf{M}_b=\frac{1}{4\pi}(\mathbf{H}_b-\mathbf{B});$$

$$\epsilon_b(\omega)=\omega_0^2(1-\beta^2)^{\frac{3}{2}}/(\omega-c\mathbf{k}\cdot\boldsymbol{\beta})^2; \tag{7}$$

where $\mathbf{P}_b$ and $\mathbf{M}_b$ represent the electric and magnetic polarizations in the beam. The term ω_0 is expressed as

$$\omega_0^2=4\pi n_b e^2/m, \tag{8}$$

where n_b is the density of the beam. It should be noted in this connection that n_b is measured by the stationary observer.

Taking into account the relationships (7), we obtain from (5) and (6) the following expressions[14]:

$$\mathbf{P}_b(1-\beta^2)+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{P}_b)=-\frac{\omega_0^2(1-\beta^2)^{\frac{3}{2}}}{(\omega-c\mathbf{k}\cdot\boldsymbol{\beta})^2}[\mathbf{E}+(\boldsymbol{\beta}\times\mathbf{B})], \tag{9a}$$

$$\mathbf{M}_b(1-\beta^2)+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{M}_b)=-\frac{\omega_0^2(1-\beta^2)^{\frac{3}{2}}}{(\omega-c\mathbf{k}\cdot\boldsymbol{\beta})^2}[\beta^2\mathbf{B}-(\boldsymbol{\beta}\times\mathbf{E})-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]. \tag{9b}$$

The relationships (8), (9a), and (9b) determine $\mathbf{P}_b$ and $\mathbf{M}_b$ as functions of $\mathbf{E}$ and $\mathbf{B}$. These values for $\mathbf{P}_b$ and $\mathbf{M}_b$ and the value $\mathbf{P}_d$ given by (4) are substituted in (2) and (3) so as to determine the electric and magnetic polarization of the composite medium as functions of $\mathbf{E}$ and $\mathbf{B}$. The expressions (2) and (3) are then applied to Maxwell's equations and by using the standard procedure we obtain a dispersion equation giving the relationship between the frequency ω and wave vector $\mathbf{k}$ of an electromagnetic field having the form $\mathbf{E}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, $\mathbf{B}\exp i(\mathbf{kr}-\omega t)$.

We use rectangular coordinates in which the $\mathbf{k}$ vector is aligned along the z axis, $\beta_x=\beta\sin\theta$, $\beta_y=0$, and $\beta_z=\beta\cos\theta$. The oscillatory modes of the system can be expressed as

$$\begin{vmatrix} A(\omega,k) & 0 & G(\omega,k) \\ 0 & C(\omega,k) & 0 \\ G(\omega,k) & 0 & F(\omega,k) \end{vmatrix} \quad \begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix}=0, \tag{10}$$

where

$$A(\omega,k)=\frac{c^2k^2}{\omega^2}-\epsilon(\omega)+\frac{\omega_0^2(1-\beta^2)^{\frac{1}{2}}}{\omega^2}+\frac{\omega_0^2(1-\beta^2)^{\frac{1}{2}}\beta^2\sin^2\theta(c^2k^2-\omega^2)}{(\omega-ck\beta\cos\theta)^2\omega^2}, \tag{11}$$

$$G(\omega,k)=\frac{\omega_0^2(1-\beta^2)^{\frac{1}{2}}\beta\sin\theta(ck-\beta\omega\cos\theta)}{(\omega-ck\beta\cos\theta)^2\omega}, \tag{12}$$

$$C(\omega,k)=\frac{c^2k^2}{\omega^2}-\epsilon(\omega)+\frac{\omega_0^2(1-\beta^2)^{\frac{1}{2}}}{\omega^2}, \tag{13}$$

and

$$F(\omega,k)=-\epsilon(\omega)+\frac{\omega_0^2(1-\beta^2\cos^2\theta)(1-\beta^2)^{\frac{1}{2}}}{(\omega-ck\beta\cos\theta)^2}. \tag{14}$$

Equation (10) describes two waves. One of these is a transverse wave $C(\omega,k)E_y=0$ in which the electric vector is perpendicular to $\boldsymbol{\beta}$. This wave is stationary. The other wave can be represented in the form

$$\begin{aligned} A(\omega,k)E_x+G(\omega,k)E_z&=0,\\ G(\omega,k)E_x+F(\omega,k)E_z&=0, \end{aligned} \tag{15}$$

The wave (15) is "hybrid," i.e., its electric vector has a transverse component E_x and a longitudinal component E_z. The dispersion equation has the form

$$A(\omega,k)F(\omega,k)-[G(\omega,k)]^2=0. \tag{16}$$

In his investigations of various instabilities produced by beams, Sturrock[15] differentiated between "convective instability" which moves with the flow and amplifies at the same time, "absolute instability" if the oscillation amplitude at a given point increases with time, and "evanescent waves" which decay with time. The occurrence of these instabilities and of evanescent waves can be determined from the dispersion equation. This equation usually contains a finite or denumberably infinite number of solutions and these can be expressed either as

$$\omega=\Omega_\alpha(k), \tag{17a}$$

or as

$$k=K_\alpha(\omega), \tag{17b}$$

where $\alpha=1, 2, \cdots$ designates various modes of the system.

[13] H. Minkowski and M. Born, Math. Ann. 68, 526 (1910).
[14] Jacob Neufeld, Phys. Rev. 123, 1 (1961).
[15] P. A. Sturrock, Phys. Rev. 112, 1488 (1958).

The procedure for determining the character of various modes is as follows: We consider the expression (17a) in which we assign to k real values and ascertain whether or not ω is complex. Similarly, in Eq. (17b) we assign to ω real values and ascertain whether or not k is complex. We shall identify the term $\mathrm{Im}(\omega)>0$ that may be obtained from (17a) as the "excitation coefficient." Similarly, the term $\mathrm{Im}(k)<0$ that may be obtained from (17b) shall be identified as the "amplification coefficient."

According to Sturrock, the occurrence of the excitation coefficient may indicate either a convective or an absolute instability. On the other hand, the occurrence of the amplification coefficient may indicate either a convective instability or evanescent waves. It should, therefore, be noted that the presence of an excitation coefficient is always associated with an unstable condition. On the other hand the occurrence of an amplification coefficient may not always be associated with an unstable condition for the particular mode to which it refers. Furthermore, the convective instability occurs if for a given mode there exists a region represented by both "excitation coefficients" and "amplification coefficients."

A particularly important term characterizing the expressions (11) through (14) is

$$\kappa=\omega_0(1-\beta^2)^{\frac{1}{4}}. \tag{18}$$

This term expresses an invariant property of the beam, i.e., it gives the Langmuir frequency in the framework of an observer traveling with the beam. Our investition is concerned with small values of κ and we shall seek solutions of the dispersion Eq. (16) in the neighborhood of $\kappa=0$ ("small κ approximation"). We are concerned with the solutions that are close to $\omega=ck\beta\cos\theta$ for values of κ that are sufficiently small. Equation (16) may exhibit "excited waves" and "amplified waves." In order to determine the occurrence of excited waves, we assume that for small κ we have

$$\omega=\tilde{\omega}+\delta, \tag{19}$$

where

$$\tilde{\omega}=ck\beta\cos\theta, \tag{20}$$

and

$$|\delta|\ll\tilde{\omega}. \tag{21}$$

The term $\tilde{\omega}$ is real and shall be designated as the "characteristic frequency" of the wave. The expression δ may be complex and $\mathrm{Im}(\delta)$, if positive, designates the excitation coefficient. If $\mathrm{Re}(\delta)\neq 0$, the wave has an effective frequency

$$\omega_{\mathrm{eff}}=\tilde{\omega}+\mathrm{Re}(\delta), \tag{22}$$

and is characterized by a relative frequency shift

$$\xi=(\omega_{\mathrm{eff}}-\tilde{\omega})/\tilde{\omega}. \tag{23}$$

The excitation coefficient and the relative frequency shift are functions of κ and tend to zero when $\kappa\rightarrow 0$. The characteristic frequency is the limiting case of the effective frequency for $\kappa\rightarrow 0$.

The instabilities associated with amplified waves are treated in a similar manner. We assume that for κ sufficiently small we have

$$k=\tilde{k}-\gamma, \tag{24}$$

where

$$\tilde{k}=\omega/\beta c\cos\theta, \tag{25}$$

and

$$|\gamma|\ll\tilde{k}. \tag{26}$$

The term $\tilde{k}$ is real and $\mathrm{Im}(\gamma)$, if it is negative, designates the amplification coefficient.

The term κ represented by expression (18) is significant since it permits us to extend the results of our investigation to highly relativistic beams of high intensity. Various types of instabilities discussed in the literature deal with the "small-beam approximation," i.e., they are specifically applied to those cases for which $\omega_0\ll\tilde{\omega}$ (for excited waves). It is noted that the magnitude of κ depends upon the density of the beam and its velocity. Therefore, the "small-κ approximation" used by us applies to a more general situation that includes not only weak beams at nonrelativistic velocities, but also intense beams at relativistic velocities.

In the case of excited waves, the small κ approximation requires the inequality $\kappa\ll|\delta|$ and the corresponding inequality for the amplified waves is $\kappa\ll|\gamma c\beta\cos\theta|$.

B. Classification of Oscillatory Modes

(1) Region of Coupling and Uncoupling

We are considering the effects of a beam of small intensity or of a highly relativistic beam on a frequency-dispersive and isotropic dielectric medium. There are two "fundamental" oscillatory modes in such a medium: a longitudinal mode in which the frequencies are determined by the roots of $\epsilon(\omega)=0$ and a transverse mode having a dispersion relation $c^2/\epsilon(\omega)=\omega^2/k^2$. In the absence of the beam ($\kappa=0$) these modes are stationary. The perturbation produced by the beam causes an instability in the fundamental modes and it introduces additional "supplementary modes" that are unstable. The supplementary modes are characterized by hybrid waves.

The occurrence of fundamental and supplementary modes is dependent on the magnitude of the "coupling term" $G(\omega,k)$ in the dispersion formula (16). When $G(\omega,k)=0$ the system is "uncoupled" and the two fundamental modes coexist independently. On the other hand, when $G(\omega,k)\neq 0$ there is an interdependence, or "coupling" between the longitudinal and transverse modes. The regions of coupling and uncoupling will be separately investigated for the case of excited and amplified waves.

The term $G(\omega,k)$ for the excited waves is obtained from the substitution of (19) in (12). Assuming $\kappa\to 0$, we have

$$G(\omega,k)\to L^2\beta\sin\theta(ck/\omega-\beta\cos\theta), \tag{27}$$

where

$$L=\lim_{\kappa\to 0}(\kappa/|\delta|). \tag{28}$$

If δ approaches zero more slowly than κ, we have $L=0$ and if δ approaches zero at least as fast as κ, we have $L\neq 0$. If $L=0$, then for sufficiently small values of κ the inequality

$$\kappa\ll|\delta| \tag{29}$$

is satisfied. For $\kappa\neq 0$ but small, the coupling term $G(\omega,k)=O(\kappa/|\delta|)^2$ and it appears in the dispersion Eq. (16) in the second power, i.e., in the form of $[G(\omega,k)]^2=O(\kappa/|\delta|)^4$. Therefore, when $L=0$ we may assume that for sufficiently small values of κ the terms containing $(\kappa/|\delta|)^4$ may be neglected. This is equivalent to the assumption $G(\omega,k)=0$. In such a case the system is uncoupled. The dispersion Eq. (16) is then separated into the equations $F(\omega,k)E_z=0$ and $A(\omega,k)E_x=0$. The expression $F(\omega,k)E_z=0$ represents the longitudinal mode and the expression $A(\omega,k)E_x=0$ represents the transverse mode. If $L\neq 0$ the system is coupled and the interdependence between the two fundamental modes produces the auxiliary mode represented by hybrid waves.

We shall, therefore, refer to $L=0$ as "the criterion for uncoupling" and to $L\neq 0$ as the "criterion for coupling."

Consider now the amplified waves. Substituting (24) in (12) we obtain for $\kappa\to 0$ an expression similar to (27) in which

$$L=\lim_{\kappa\to 0}(\kappa/|\gamma|\beta c\cos\theta). \tag{30}$$

We have here a region of uncoupling for $L=0$ and a region of coupling for $L\neq 0$.

(2) "Strong" and "Weak" Instability

We introduce an expression

$$N=\lim_{\kappa\to 0}(\kappa/|\mathrm{Im}(\delta)|), \tag{31}$$

which serves as an index of the strength of the instability. Thus when $N=0$, the excitation coefficient $|\mathrm{Im}(\delta)|$ approaches zero more slowly than κ, and the instability is "strong". On the other hand, when $N\neq 0$ the excitation coefficient $|\mathrm{Im}(\delta)|$ approaches zero at least as fast as κ and the instability is "weak." It will be shown that in the region of uncoupling where $L=0$, we may have $N=0$ or $N\neq 0$, i.e., the instability in some instances may be strong and in other instances it may be weak.

For amplified waves, the strength of the instability is determined in a similar manner by means of the expression

$$N=\lim_{\kappa\to 0}(\kappa/|\mathrm{Im}(\delta)|).$$

II. UNCOUPLED MODES

We neglect in the dispersion Eq. (16) the terms containing $(\kappa/|\delta|)^4$ and those containing $(\kappa/\omega)^2$ but retain the terms containing $(\kappa/|\delta|)^2$. This approximation is equivalent to an assumption $L=0$, for which the transverse and longitudinal modes are uncoupled. We obtain for the longitudinal wave

$$\epsilon(\omega)-\frac{\kappa^2(1-\beta^2\cos^2\theta)}{(\omega-ck\beta\cos\theta)^2}=0, \tag{32}$$

and for the transverse wave

$$\epsilon(\omega)-\frac{c^2k^2}{\omega^2}-\frac{\kappa^2\beta^2\sin^2\theta(c^2k^2-\omega^2)}{\omega^2(\omega-ck\beta\cos\theta)^2}=0. \tag{33}$$

A. Longitudinal Oscillations

(1) Excited Waves

(a) Occurrence of uncoupling and instability:

Substituting (1) and (19) in (32), we obtain the following equation for δ:

$$\delta^4+2\tilde{\omega}\delta^3+[\tilde{\omega}^2-\omega_a^2-\omega_1^2-\kappa^2(1-\beta^2\cos^2\theta)]\delta^2 +2\kappa^2\tilde{\omega}(\beta^2\cos^2\theta-1)\delta+\kappa^2(\omega_a^2-\tilde{\omega}^2)(1-\beta^2\cos^2\theta)=0. \tag{34}$$

We take into account $|\delta|\ll\tilde{\omega}$ and $\kappa\ll|\delta|$ and neglect terms that are small when compared with $|2\tilde{\omega}\delta^3|$. We obtain then an equation of the type

$$\delta^3+p_l\delta^2+r_l=0, \tag{35}$$

where

$$p_l=S_l/2\tilde{\omega}, \tag{36}$$

and

$$r_l=-(\tilde{\omega}^2-\omega_a^2)\kappa^2(1-\beta^2\cos^2\theta)/2\tilde{\omega}. \tag{37}$$

The term

$$S_l=\tilde{\omega}^2-\omega_1^2-\omega_a^2 \tag{38}$$

is designated as the "characteristic parameter." The solution of Eq. (35) can be written as

$$\delta=-\frac{A_l^{\frac{1}{3}}+B_l^{\frac{1}{3}}}{2}\pm i\sqrt{3}\frac{A_l^{\frac{1}{3}}-B_l^{\frac{1}{3}}}{2}-\frac{S_l}{6\tilde{\omega}}, \tag{39}$$

where

$$\begin{matrix}A_l\\B_l\end{matrix}=\frac{1}{4\tilde{\omega}}\Bigg\{-\frac{S_l^3}{54\tilde{\omega}^2}+\kappa^2(\tilde{\omega}^2-\omega_a^2)(1-\beta^2\cos^2\theta) \pm\Bigg[\left(\frac{S_l^3}{27\tilde{\omega}^2}-\kappa^2(\tilde{\omega}^2-\omega_a^2)(1-\beta^2\cos^2\theta)\right) \times(-\kappa^2(\tilde{\omega}^2-\omega_a^2)(1-\beta^2\cos^2\theta))\Bigg]^{\frac{1}{2}}\Bigg\}. \tag{40}$$

Not every expression for δ has a physically acceptable meaning, and only those solutions that satisfy the criterion for uncoupling $(L=0)$ shall be considered. The expression for δ, and consequently the term L, are functions of the characteristic parameter S_l. Using the criterion $L=0$, we shall determine the range of values for S_l for which there is uncoupling. Once the range of uncoupling is ascertained, we shall investigate the stability of oscillations within this range. In the event there is an instability, an appropriate criterion shall be applied in order to determine whether the instability is strong or weak. The instability occurs when Eq. (35) has complex solutions, i.e., when the discriminant

$$\Delta_l=(S_l/432\tilde{\omega}^3+r_l/4)r_l>0. \qquad (41)$$

The inequality (41) determines a range of values of S_l for which there is an instability. It is seen by direct substitution that the critical value $S_l=0$ is contained within this range. It will be shown that this range is limited to the immediate neighborhood of $S_l=0$.

(*b*) *Response of the system for* $S_l=0$:

For $S_l=0$ the characteristic frequency $\tilde{\omega}$ has a resonance value $\tilde{\omega}_l$, i.e.,

$$\tilde{\omega}=\omega_l=(\omega_1^2+\omega_a^2)^{\frac{1}{2}}. \qquad (42)$$

Our equation for δ has then a complex solution,

$$\delta=\kappa^{\frac{2}{3}}\left[\frac{\omega_1^2(1-\beta^2\cos^2\theta)}{2(\omega_1^2+\omega_a^2)^{\frac{1}{2}}}\right]^{\frac{1}{3}}\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right), \qquad (43)$$

which represents an instability.

We see that $\delta=O(\kappa^{\frac{2}{3}})$ and hence it satisfies the condition for uncoupling $L=0$. Also, since $\mathrm{Im}(\delta)=O(\kappa^{\frac{2}{3}})$, we have $N=0$ and the instability is strong. We have also a frequency shift, the magnitude of which is

$$\xi=\left[\frac{\kappa^2\omega_1^2(1-\beta^2\cos^2\theta)}{16(\omega_1^2+\omega_a^2)^2}\right]^{\frac{1}{3}}. \qquad (44)$$

(*c*) *Range of uncoupling:*

We shall determine the "width" of the range in which the uncoupling occurs, i.e., the range of S_l above and below the critical value $S_l=0$. It is convenient to introduce a constant α_l defined by the relation

$$S_l^3=\alpha_l\kappa^2[F_l(\tilde{\omega})]^3, \qquad (45)$$

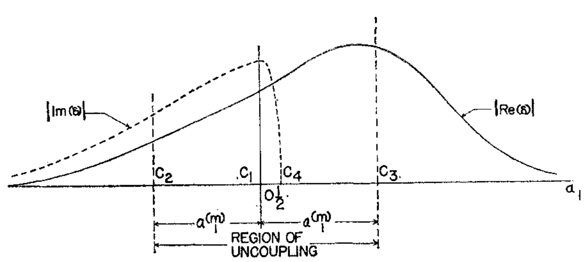


FIG. 1. Dependence between $|\mathrm{Im}(\delta)|$, $|\mathrm{Re}(\delta)|$, and α_l.

where

$$F_l(\tilde{\omega})=[54\tilde{\omega}^2(\tilde{\omega}^2-\omega_a^2)(1-\beta^2\cos^2\theta)]^{\frac{1}{3}}. \qquad (46)$$

The solution for δ can now be written as

$$\delta=\phi(\alpha_l)\kappa^{\frac{2}{3}}(\tilde{\omega}^2-\omega_a^2)^{\frac{1}{3}}(1-\beta^2\cos^2\theta)^{\frac{1}{3}})/4^{\frac{1}{3}}\tilde{\omega}^{\frac{1}{3}}, \qquad (47)$$

where

$$\begin{aligned}\phi(\alpha_l)=-0.5\{&[1-\alpha_l+(1-2\alpha_l)^{\frac{1}{2}}]^{\frac{1}{3}}\\&+[1-\alpha_l-(1-2\alpha_l)^{\frac{1}{2}}]^{\frac{1}{3}}\}\pm\{[1-\alpha_l+(1-2\alpha_l)^{\frac{1}{2}}]^{\frac{1}{3}}\\&-[1-\alpha_l-(1-2\alpha_l)^{\frac{1}{2}}]^{\frac{1}{3}}\}(-0.75)^{\frac{1}{2}}-\alpha_l^{\frac{1}{3}}. \qquad (48)\end{aligned}$$

Consider the range of uncoupling as a function of α_l. For $\alpha_l=0$ we have $S_l=0$ and the system is uncoupled as shown above. For positive and negative values of α_l that are sufficiently large in absolute magnitude, the system is coupled which can be shown as follows:

Assume that $|\alpha_l|\gg(2|\alpha_l|)^{\frac{1}{2}}$. Then we may neglect $(2\alpha_l)^{\frac{1}{2}}$ or the number 1 when either is added to α_l in the expression (48). We obtain $\delta=0$ and, consequently, $L\neq0$. Therefore, for the values of α_l satisfying the inequality $|\alpha_l|^{\frac{1}{2}}\gg\sqrt{2}$ the system is coupled.

There are obviously no sharp boundaries between the regions of coupling and uncoupling. It is apparent that the coupling term $G(\omega,k)$ is not zero when $\kappa\neq0$. Our uncoupling procedure is based on neglecting terms of the order of $(\kappa/|\delta|)^4$ and, therefore, its accuracy is based on approximations that are dependent on the magnitude of the term κ. We choose a suitable value $\alpha_l^{(m)}$ that represents the "boundaries" of the range of uncoupling. This value should not satisfy the relationship $|\alpha_l^{(m)}|^{\frac{1}{2}}\gg\sqrt{2}$. The region of uncoupling will then be associated with the following range of the characteristic parameter S_l:

$$-(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega})<S_l<(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega}). \qquad (49)$$

(*d*) *Range of the instability:*

Using the expression (45), the discriminant (41) may be written in a form

$$\Delta_l=(1-2\alpha_l)\kappa^4(\tilde{\omega}^2-\omega_a^2)^2(1-\beta^2\cos^2\theta)^2/16\tilde{\omega}. \qquad (50)$$

We have an instability if $\Delta_l>0$ and this inequality determines the range for α_l and S_l. We shall consider separately the case of $\alpha_l\leqslant0$ and $\alpha_l>0$.

For $\alpha_l\leqslant0$ the inequality $\Delta_l>0$ is satisfied and in such case the instability is expressed by an excitation coefficient $\mathrm{Im}(\delta)=O(\kappa^{\frac{2}{3}})$. Hence the criterion $N=0$ is fulfilled and we have a strong instability.

For $\alpha_l>0$ we have $\Delta_l>0$ only if $\alpha_l<0.5$. The criterion $N=0$ is not fulfilled for all values of α_l within this range. For $\alpha_l=0$ we have $N=0$ and consequently a strong instability. However, for increasing values of α_l the instability decreases in intensity. It is seen that $\mathrm{Im}(\delta)\rightarrow0$ as $\alpha_l\rightarrow0.5$. Consequently, we have $N\neq0$ which indicates that the instability is weak. There is no instability for $\alpha_l\geqslant0.5$ since $\Delta_l<0$ and hence the solutions of Eq. (35) are real.

Therefore, the range of instability for uncoupled longitudinal oscillations is defined by the inequalities

$$-(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega})<S_l<0.5^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega}). \qquad (51)$$

It is noted that for $S_l>0$ the region of uncoupling extends beyond the instability range, i.e., it comprises the range of values defined by the inequalities

$$0.5^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega})\leqslant S_l\leqslant(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega}), \qquad (52)$$

for which there is no instability.

Figure 1 illustrates graphically the relationship (47) expressed in a form $\delta=K_l\phi(\alpha_l)$, where $K_l=\kappa^{\frac{2}{3}}(\tilde{\omega}^2-\omega_a^2)^{\frac{1}{3}}\times(1-\beta^2\cos^2\theta)^{\frac{1}{3}}/4^{\frac{1}{3}}\tilde{\omega}^{\frac{2}{3}}$ is assumed to be a constant. The region of uncoupling is contained within the range C_2C_3 and the curves $|\mathrm{Im}(\delta)|$ and $|\mathrm{Re}(\delta)|$ are plotted as functions of α_l. The curve $|\mathrm{Im}(\delta)|$ shows the resonance behavior which corresponds to the maximum instability for $\alpha_l=0$. For $\alpha_l<0$ the values of $|\mathrm{Im}(\delta)|$ are relatively large, indicating a strong instability. For $\alpha_l>0$ the value of $|\mathrm{Im}(\delta)|$ rapidly decreases and there is no instability in the range C_4C_3. The curve $|\mathrm{Re}(\delta)|$ indicates a frequency shift of the effective frequency ω_{eff} with respect to the characteristic frequency $\tilde{\omega}$. This frequency shift reaches a maximum for some positive value of α_l and not the critical value $\alpha_l=0$.

(e) Relationship between $|\mathrm{Im}(\delta)|$ *and* $\tilde{\omega}$ *in the region of uncoupling:*

We shall refer now to Fig. 2 which represents graphically some of the main results of this investigation. The curves M_l and M_t show the relationship between the excitation coefficient $|\mathrm{Im}(\delta)|$ and the characteristic frequency $\tilde{\omega}$ for all the oscillatory modes resulting from the interaction of a beam with the dielectric medium. The resonance frequency ω_l given by (42) is represented by the point A_1 and the range of uncoupling for longitudinal waves is expressed by the inequality

$$\omega_l-(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega})<\tilde{\omega}<\omega_l+(\alpha_l^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_l(\tilde{\omega}). \qquad (53)$$

This range is contained between the points A_2 and A_3. The strong instability corresponding to $\alpha_l<0$ is represented by the segment of the curve M_l between the points B_2 and B_1. For $\alpha_l>0$ the excitation coefficient decreases to zero within the range A_1A_4 and the oscillation is stationary in the range A_4A_3. It should be noted that the width of the range of uncoupling is relatively small, i.e., it is of the order of $\kappa^{\frac{2}{3}}$. In the diagrammatic representation this region has been enlarged in order to show more clearly the behavior of the instability.

(2) Amplified Waves

Substituting (24) and (1) into (32) we obtain

$$\gamma=\frac{\kappa}{\beta c\cos\theta}\left[\frac{(1-\beta^2\cos^2\theta)(c^2\tilde{k}^2\beta^2\cos^2\theta-\omega_a^2)}{c^2\tilde{k}^2\beta^2\cos^2\theta-\omega_1^2-\omega_a^2}\right]^{\frac{1}{2}}. \qquad (54)$$

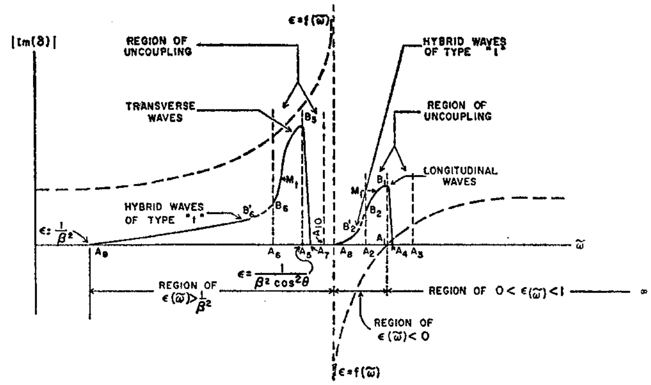


FIG. 2. Dependence between $|\mathrm{Im}(\delta)|$ and $\tilde{\omega}$ for various modes in the system (the value of θ is fixed).

This solution will be complex provided the expression in the brackets is negative, i.e., provided

$$(\omega_a/c\beta\cos\theta)^2<\tilde{k}^2<(\omega_1^2+\omega_a^2)/\beta^2c^2\cos^2\theta. \qquad (55)$$

However, the solution is applicable only if it satisfies the criterion $\kappa\ll|\gamma\beta c\cos\theta|$ for uncoupling and the assumption (26). The criterion for uncoupling will be satisfied only if

$$\tilde{k}^2\approx(\omega_1^2+\omega_a^2)/\beta^2c^2\cos^2\theta. \qquad (56)$$

It is of interest to note that $\gamma c\beta\cos\theta=O(\kappa^{\frac{2}{3}})$ provided $\tilde{k}^2\beta^2c^2\cos^2\theta-\omega_1^2-\omega_a^2=O(\kappa^{\frac{2}{3}})$. For sufficiently small values of κ this would give an instability satisfying both of the inequalities $\kappa\ll|\gamma c\beta\cos\theta|$ and $|\gamma|\ll\tilde{k}$.

If $\tilde{k}^2=(\omega_1^2+\omega_a^2)/\beta^2c^2\cos^2\theta$, then the expression (54) would give $\gamma=\infty$ and hence would fail to satisfy the condition $|\gamma|\ll\tilde{k}$. Therefore, we must exclude values of $\tilde{k}^2$ in the immediate neighborhood of $(\omega_1^2+\omega_a^2)/\beta^2c^2\cos^2\theta$.

The instability, therefore, occurs in the rather narrow subrange of (55) for which both the condition $\kappa\ll|\gamma\beta c\cos\theta|$ for uncoupling and the inequality $|\gamma|\ll\tilde{k}$ are satisfied.

B. Transverse Oscillations

(1) Excited Waves

(a) Occurrence of uncoupling and of instability:

The transverse waves are described by the dispersion equation (33) in which $\epsilon(\omega)$ is given by the expression (1). We approximate $\epsilon(\omega)$ by a Taylor series in the neighborhood of $\tilde{\omega}$ and obtain

$$\epsilon(\omega)=\epsilon(\tilde{\omega})+\frac{2\tilde{\omega}\omega_1^2}{(\tilde{\omega}^2-\omega_a^2)^2}\delta-\frac{\omega_1^2(\omega_a^2+3\tilde{\omega}^2)}{(\tilde{\omega}^2-\omega_a^2)}\delta^2+\cdots. \qquad (57)$$

We use the first term in the expression (57) as an approximation for $\epsilon(\omega)$. In order that this approximation be reasonable, we impose the restriction

$$\left|\frac{2\tilde{\omega}\omega_1^2\delta}{(\tilde{\omega}^2-\omega_1^2-\omega_a^2)(\tilde{\omega}^2-\omega_a^2)}\right|\ll1. \qquad (58)$$

Therefore, we exclude values of $\tilde{\omega}^2$ which are near ω_a^2 and those which are near $\omega_1^2+\omega_a^2$. If $\tilde{\omega}^2 \sim \omega_a^2$, the exciting frequency is at resonance with the binding frequency and we have $\epsilon(\tilde{\omega}) \rightarrow \infty$. If $\tilde{\omega}^2 \sim \omega_1^2+\omega_a^2$ we have $\epsilon(\tilde{\omega}) \sim 0$ which expresses the longitudinal instability discussed above.

Substituting $\omega = \tilde{\omega}+\delta$ in (33) and using for $\epsilon(\omega)$ the first term of (57), we obtain

$$\epsilon(\tilde{\omega})\delta^4+2\tilde{\omega}\epsilon(\tilde{\omega})\delta^3+[\tilde{\omega}^2\epsilon(\tilde{\omega})-\kappa^2\beta^2 \sin^2\theta - c^2k^2]\delta^2 + [2\tilde{\omega}\kappa^2\beta^2 \sin^2\theta]\delta + \kappa^2\beta^2 \sin^2\theta(\tilde{\omega}^2-c^2k^2)=0. \tag{59}$$

We take into account $|\delta| \ll \tilde{\omega}$ and $\kappa \ll |\delta|$, and we neglect terms which are small when compared with $|2\tilde{\omega}\epsilon(\tilde{\omega})\delta^3|$. We obtain

$$\delta^3+p_t\delta^2+r_t=0, \tag{60}$$

where

$$p_t=\tilde{\omega}S_t/2(S_t+1), \tag{61}$$

$$r_t=-\kappa^2\tilde{\omega}\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta)/2(S_t+1), \tag{62}$$

and

$$S_t=\beta^2\epsilon(\tilde{\omega}) \cos^2\theta-1. \tag{63}$$

The solution of Eq. (60) can be written as

$$\delta=-\frac{A_t^{\frac{1}{3}}+B_t^{\frac{1}{3}}}{2} \pm i\sqrt{3}\frac{A_t^{\frac{1}{3}}-B_t^{\frac{1}{3}}}{2}-\frac{\tilde{\omega}S_t}{6(S_t+1)}, \tag{64}$$

where

$$\begin{aligned} \frac{A_t}{B_t}=\frac{\tilde{\omega}}{4(S_t+1)}\Bigg\{&-\frac{\tilde{\omega}^2S_t^3}{54(S_t+1)^2}+\kappa^2\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta) \\ &\pm\Bigg[\left(\frac{\tilde{\omega}^2S_t^3}{27(S_t+1)^2}-\kappa^2\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta)\right) \\ &\times(-\kappa^2\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta))\Bigg]^{\frac{1}{2}}\Bigg\}. \end{aligned} \tag{65}$$

The term S_t is designated as the "characteristic parameter for transverse oscillations." This term is analogous to S_l expressed in (38). The parameter S_t is, however, a pure number whereas S_l has dimension of the square of frequency. We proceed here in the same manner as in the case of longitudinal waves. We determine the range of uncoupling for which $L=0$ and the instability range for which the discriminant

$$\Delta_t=\left(\frac{2\tilde{\omega}^3S_t^3}{(S_t+1)^2}+26r_t\right)r_t>0. \tag{66}$$

Both ranges are in the immediate neighborhood of the the critical value $S_t=0$. An estimate will be given of the extent of these ranges.

(b) Response of the system for $S_t=0$:

For $S_t=0$ the solution for δ is

$$\delta=\left[\frac{\kappa^2\tilde{\omega}\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta)}{2}\right]^{\frac{1}{3}}\left(\frac{-1\pm i\sqrt{3}}{2}\right). \tag{67}$$

The above expression can be written in a form showing the unique dependence between δ and $\tilde{\omega}$ that exists for $S_t=0$:

$$\delta=\kappa^{\frac{2}{3}}\left\{\frac{\tilde{\omega}[\epsilon(\tilde{\omega})\beta^2-1][\epsilon(\tilde{\omega})-1]}{2[\epsilon(\tilde{\omega})]^2}\right\}^{\frac{1}{3}}\left(\frac{-1\pm i\sqrt{3}}{2}\right). \tag{68}$$

A similar solution was obtained for a nondispersive medium by Getmantsev.[8] Since $\delta=O(\kappa^{\frac{2}{3}})$ the above expressions satisfy the criterion for uncoupling ($L=0$) and the criterion for strong instability ($N=0$).

There is a relative frequency shift which can be expressed as

$$\xi=-\frac{\kappa^{\frac{2}{3}}}{2\tilde{\omega}}\left\{\frac{\tilde{\omega}[\epsilon(\tilde{\omega})\beta^2-1][\epsilon(\tilde{\omega})-1]}{2[\epsilon(\tilde{\omega})]^2}\right\}^{\frac{1}{3}}. \tag{69}$$

(c) Range of uncoupling:

We shall define a number α_t by the relationship

$$S_t^3/(S_t+1)^2=\alpha_t\kappa^2[F_t(\tilde{\omega})]^3, \tag{70}$$

where

$$F_t(\tilde{\omega})=[54\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta)/\tilde{\omega}^2]^{\frac{1}{3}}. \tag{71}$$

The solution for δ can be written in the form

$$\delta=\phi(\alpha_t)\left[\frac{\tilde{\omega}\kappa^2\beta^2 \sin^2\theta(1-\beta^2 \cos^2\theta)}{4(S_t+1)}\right]^{\frac{1}{3}}, \tag{72}$$

where ϕ is given by the expression (48).

We assume now that $|S_t| \ll 1$. Then S_t can be neglected in the denominator in the left-hand side of the expression (70) and we obtain $S_t=\alpha_t^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega})$. By applying the same arguments as we did for longitudinal oscillations we can limit the range of uncoupling by means of a suitably chosen number $\alpha_t^{(m)}$ that does not satisfy the inequality $|\alpha_t^{(m)}| \gg \sqrt{2}$. The values of the characteristic parameter that are contained within this range satisfy the inequality

$$-(\alpha_t^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega})<S_t<(\alpha_t^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega}). \tag{73}$$

Our discussion is concerned with small values of κ. Therefore, our assumption that $|S_t| \ll 1$ is in agreement with the inequalities (73).

(d) Range of the instability:

The discriminant of Eq. (60) can be put in the form

$$\Delta_t=(1-2\alpha_t)\tilde{\omega}^2\kappa^4\beta^4 \sin^4\theta(1-\beta^2 \cos^2\theta)^2/16(S_t+1)^2. \tag{74}$$

The discriminant is positive for $\alpha_t<0.5$. Therefore, we have instability when

$$-(\alpha_t^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega})<S_t<(0.5)^{\frac{1}{3}}\kappa^{\frac{2}{3}}F,(\tilde{\omega}), \tag{75}$$

and stationary waves when

$$(0.5)^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega})<S_t<(\alpha_t^{(m)})^{\frac{1}{3}}\kappa^{\frac{2}{3}}F_t(\tilde{\omega}). \tag{76}$$

Assuming that

$$K_t = [\tilde{\omega}\kappa^2\beta^2 \sin^2\theta(1-\beta^2\cos^2\theta)/4(S_t+1)]^{\frac{1}{3}}$$

is a constant, we can express the relationship $\delta = K_t\phi(\alpha_t)$ given by (72) in the form of the graph shown in Fig. 1, provided the abscissa are labeled α_t instead of α_l.

(*e*) *Dependence between* $Im(\delta)$ *and* $\tilde{\omega}$ *in the region of uncoupling:*

The curve M_t representing in Fig. 2 the transverse oscillations is similar to the previously discussed curve M_l for the longitudinal oscillations. The value $S_t=0$ corresponds to the resonance frequency

$$\omega_t = [\omega_a^2 - \omega_1^2\beta^2\cos^2\theta/(1-\beta^2\cos^2\theta)]^{\frac{1}{2}}, \tag{77}$$

which is represented by the point A_5. The range of uncoupling is contained between the points A_6 and A_{10}. The instability extends over the range A_6A_7, and it does not appear in the range A_7A_{10}.

(*f*) *Vavilov-Cherenkov radiation:*

(1) Similarity between the "particle effect" and the "beam effect." It is noted that the characteristic parameter S_t may have any value within the range (75). This parameter expresses a functional relationship between $\tilde{\omega}$ and θ and, therefore, to each value of S_t within the range (75) corresponds a different relationship between $\tilde{\omega}$ and θ. Assume that S_t has a determined and fixed value and consider (1) the excitation coefficient $|\mathrm{Im}(\delta)|$ as a function of $\tilde{\omega}$ for a given value of θ, and (2) the excitation coefficient $|\mathrm{Im}(\delta)|$ as a function of θ for a given value of $\tilde{\omega}$. In the first case we obtain a relationship represented by the curve M_t which shows a pronounced frequency selectivity at $\tilde{\omega}=\omega_t$. In the second case we have a directional selectivity. The instability is contained within an appropriate range for θ and reaches its maximum value when the wave vector is aligned along a definite "resonance" direction within this range. This directional resonance is characterized by an angle θ_1, satisfying the relationship

$$\cos\theta_1 = [(1+S_t)/\beta^2\epsilon(\tilde{\omega})]^{\frac{1}{2}}, \tag{78}$$

and defines a cone that is a locus of all resonance directions corresponding to a given value of S_t. This cone is structurally similar to the Vavilov-Cherenkov cone that is associated with a single particle moving with velocity $\boldsymbol{\beta}=\mathbf{v}/c$ through a dielectric medium. The Vavilov-Cherenkov cone for a single particle is expressed by the relationship

$$\cos\theta = [1/\beta^2\epsilon(\omega)]^{\frac{1}{2}}, \tag{79}$$

which is obtained from (78) by putting $S_t=0$ and $\tilde{\omega}=\omega$.

There is, therefore, an analogy between the instability produced by a beam and the "conventional" Vavilov-Cherenkov effect produced by a single particle. This analogy is due to the character of the waves and to the directional behavior of radiation associated with both effects. In both cases the wave vectors are aligned along conical surfaces of a similar structure. In both effects the radiated waves are transverse. The occurrence of a transverse wave in the "particle effect" is without any particular significance, since in a conventional isotropic dielectric medium the propagation of energy can be associated with transverse waves only. On the other hand, in a medium comprising an electron beam, the energy flux is not necessarily associated with purely transverse waves.

(2) Differences between the "particle effect" and the "beam effect." There are significant distinctions in the formulation of the two problems. In the "particle problem" there are no stability considerations, and the radiation is associated with the occurrence of a nonzero Poynting vector directed outwardly from the particle track at infinite lateral distances from the track. The energy flux is represented by a real number and there are no complex quantities that would indicate growth or decay. On the other hand, the "beam problem" is based primarily on stability considerations and the Vavilov-Cherenkov effect is expressed by the emergence of a growing electromagnetic wave that exhibits an instability.

There is also a difference in the directional characteristics of both effects. In the particle effect there is a single cone defined by (79) and there is a unique relationship between the radiated frequency ω and the direction θ. Such a uniqueness does not exist in the beam effect. In the latter case the parameter S_t may have any value within the range (75), and, therefore, there is a family of Vavilov-Cherenkov cones defined by this range. It should be noted, however, that the width of the range (75) is proportional to $\kappa^{\frac{2}{3}}$. Therefore for $\kappa \to 0$, this range becomes coincident with a single point and we obtain a Vavilov-Cherenkov cone corresponding to $S_t=0$. We shall designate it as the "characteristic cone." The characteristic cone for the beam is the same as the conventional Vavilov-Cherenkov cone for the particle provided the velocities of the particle and of the beam are the same, and the characteristic frequency for the beam is equal to the frequency ω radiated by the particle.

Another significant difference between the particle effect and the beam effect concerns the magnitude of the radiated frequency for waves aligned along the same direction. There is a relative frequency shift associated with the beam effect and given by the expression (69).

(2) *Amplified Waves*

We consider again the dispersion Eq. (33). We substitute in this equation the expression $k=\tilde{k}-\gamma$ subject to the inequalities $\kappa \ll |\gamma\beta c\cos\theta|$ and $|\gamma| \ll \tilde{k}$. Neglecting κ^2/ω^2 and terms that are small when compared with $|2c^2\tilde{k}\alpha^3|$, we obtain again an equation of

the type $\gamma^3+p_t'\gamma^2+r_t'=0$, where

$$p_t'=\tfrac{1}{2}\tilde{k}[\beta^2\epsilon(\omega)\cos^2\theta-1], \quad (80)$$

$$r_t'=-(1/2c^2)\tilde{k}\kappa^2\tan^2\theta(1-\beta^2\cos^2\theta). \quad (81)$$

This equation is of the same type as previously discussed. The equality $\beta^2\epsilon(\omega)\cos^2\theta-1=0$ represents the Vavilov-Cherenkov cone.

III. COUPLED MODES

Since $L\neq 0$, the transverse and the longitudinal modes are coupled and the dispersion equation has the form (16). The oscillations are of hybrid type.

A. Excited Waves

Neglecting κ^2/ω^2, the dispersion equation (16) can be put in a form

$$\epsilon(\omega)\left[\epsilon(\omega)-\frac{c^2k^2}{\omega^2}\right]-\frac{\kappa^2}{\delta^2}\left[\epsilon(\omega)\left(1-\beta^2+\frac{c^2k^2\beta^2\sin^2\theta}{\omega^2}\right)-\frac{c^2k^2}{\omega^2}(1-\beta^2\cos^2\theta)\right]=0. \quad (82)$$

We substitute in (82) $\omega=\tilde{\omega}+\delta$ subject to the inequality $|\delta|\ll\tilde{\omega}$. We approximate $\epsilon(\omega)$ by Taylor series about $\tilde{\omega}$ and retain the first term of this series. We obtain then

$$\delta=\pm\kappa D^{\frac{1}{2}}, \quad (83)$$

where

$$D=\frac{[1-\epsilon(\tilde{\omega})\beta^2][1-\beta^2\cos^2\theta]}{\epsilon(\tilde{\omega})[1-\epsilon(\tilde{\omega})\beta^2\cos^2\theta]}. \quad (84)$$

An expression similar to the above was obtained by Getmantsev[8] for a dispersionless medium. The approximation based on the Taylor expansion is reasonable if the inequality (58) is satisfied. We must, therefore, exclude those values of $\tilde{\omega}$ which are near ω_a and which are near $\omega_l=(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$. The equality $\tilde{\omega}=\omega_a$ corresponds to $\epsilon(\tilde{\omega})=\infty$ and the equality $\tilde{\omega}=\omega_l$ corresponds to $\epsilon(\tilde{\omega})=0$. The expression (83) is also subject to the criterion for coupling ($L\neq 0$). Therefore, the solution should satisfy the relationship $\delta\leqslant O(\kappa)$. In order to satisfy these requirements, we should exclude from our consideration the region for which $\epsilon(\tilde{\omega})\sim 0$ and the region for which $1-\epsilon(\tilde{\omega})\beta^2\cos^2\theta\sim 0$. These two excluded regions have already been discussed. For $\epsilon(\tilde{\omega})\sim 0$ we have longitudinal oscillations contained within the frequency range A_2A_3 in Fig. 2. For $1-\epsilon(\tilde{\omega})\beta^2\cos^2\theta\sim 0$ we have transverse oscillations contained within the frequency range A_6A_{10}. We shall consider two frequency regions corresponding to $\epsilon(\tilde{\omega})<0$ and $\epsilon(\tilde{\omega})>0$, respectively.

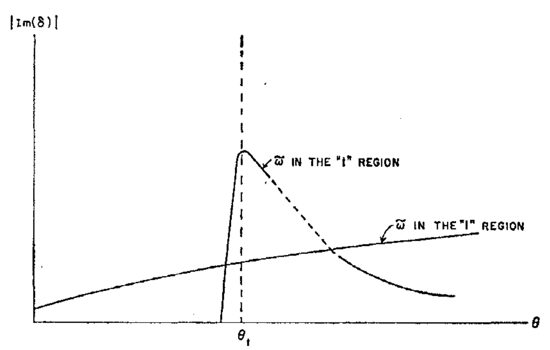


FIG. 3. Relationship between $|\mathrm{Im}(\delta)|$ and θ for various values of $\tilde{\omega}$.

For $\epsilon(\tilde{\omega})<0$ we have $D<0$ and consequently there is an instability. This instability is represented by a portion of the range A_8A_1 for which the solution (83) satisfies the criterion for coupling and the condition $|\delta|\ll\tilde{\omega}$. Points in the immediate neighborhood of A_8 are excluded from this range in order that the inequality (58) be satisfied.

If $\epsilon(\tilde{\omega})>0$ we have $D<0$ and consequently an instability if

$$1/\beta^2\cos^2\theta>\epsilon(\tilde{\omega})>1/\beta^2. \quad (85)$$

This instability is represented by a portion of the range A_9A_5 for which the solution (83) satisfies the criterion for coupling and the condition $|\delta|\ll\tilde{\omega}$.

B. Amplified Wave

We consider real values of ω and look for complex values of k in Eq. (82). We use the criterion $L\neq 0$ which gives $\gamma\beta c\cos\theta\leq O(\kappa)$ for sufficiently small values of κ. We neglect κ^2/ω^2 and write $k=\tilde{k}-\gamma\approx\tilde{k}$ by making the assumption $|\gamma|\ll\tilde{k}$ where $\tilde{k}=\omega/\beta c\cos\theta$.

The solution of Eq. (82) can be put in a form

$$\gamma=\pm\frac{\kappa}{\beta c\cos\theta}\left\{\frac{[1-\epsilon(\omega)\beta^2][1-\beta^2\cos^2\theta]}{\epsilon(\omega)[1-\epsilon(\omega)\beta^2\cos^2\theta]}\right\}^{\frac{1}{2}}, \quad (86)$$

which exhibits an instability similar to the one that occurs in the case of the excited wave.

IV. RESPONSE OF THE SYSTEM FOR A FIXED VALUE OF θ

The response of the system shown in Fig. 2 is characterized by two intense centers of instability, the longitudinal instability that emerges in the form of Bohr waves and the transverse instability that emerges in the form of Vavilov-Cherenkov waves. The longitudinal instability is contained within the range of uncoupling A_2A_3 and is centered at the resonance frequency $\omega=\omega_l$. The transverse instability is contained within the range of uncoupling A_6A_{10} and is centered at the resonance frequency $\omega=\omega_t$. The

response of the system is also characterized by less intense instabilities that occur in the region of coupling. This latter region is represented by frequencies that are outside of the range A_2A_3 and A_6A_{10}. These less intense instabilities emerge in the form of hybrid waves and are represented by lines A_9B_6' and A_8B_2'.

A characteristic feature of our description is associated with the presence of two "continuation curves" represented by dashed lines in Fig. 2. One of these, represented by the line $B_2'B_2$, is the continuation of the longitudinal instability within the uncoupled region into the hybrid waves of the type "l". The other represented by the line $B_6'B_6$ is the continuation of the transverse instability into the hybrid waves of the type "t." In our subsequent discussion we shall refer to the "l" region and the "t" region. The "l" region comprises the longitudinal wave together with the corresponding hybrid wave. Similarly, the "t" region comprises the transverse wave together with the corresponding hybrid wave.

For a high-velocity electron beam passing through a gaseous medium of low density, we have $\omega_l \ll \omega_a$. In such case the frequency range represented by the "l" region should be considerably narrower than the frequency range represented by the "t" region.

V. DIRECTIONAL SELECTIVITY OF UNSTABLE SCILLATIONS

An unstable oscillatory mode that increases in amplitude as time progresses is characterized by two parameters: the frequency ω and the wave vector $\mathbf{k}$, and each of these parameters is represented by two numbers. The frequency ω is complex, i.e., $\omega=\omega_{\text{eff}}+i\,\text{Im}(\delta)$, and the wave vector $\mathbf{k}$ is defined by its magnitude k and direction θ. It would seem appropriate to describe such an instability in the form of an expression of type $|\text{Im}(\delta)|=f(\omega_{\text{eff}},k,\theta)$ in which ω_{eff}, k, θ are independent variables. There is, however, another independent variable that determines the behavior of the instability. This additional independent variable is expressed as $\kappa=\omega_0(1-\beta^2)^{\frac{1}{2}}$ and the behavior of our system has been described for small values of κ. It is noted that for $\kappa\to 0$ we have $\lim \omega_{\text{eff}}=\tilde{\omega}$. For the values of κ that are sufficiently small, ω_{eff} differs very little from $\tilde{\omega}$ and, therefore, we have assumed that $\tilde{\omega}\sim\omega_{\text{eff}}$. The replacement of the actual effective frequency by the characteristic frequency, as an independent variable, introduces some inaccuracy which does not change substantially the functional behavior of the excitation coefficient.

We shall consider two different expressions for the directional behavior of the instability. One of these has the form $|\text{Im}(\delta)|=F(\tilde{\omega},\theta)$ and describes the directional properties of the excitation coefficient as a function of the characteristic frequency. The other has the form $|\text{Im}(\delta)|=\phi(k,\theta)$ and describes the directional properties as a function of the wave number.

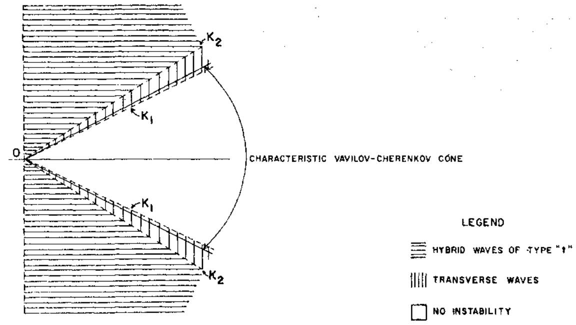


FIG. 4. Directional characteristics of the waves for a fixed value of $\tilde{\omega}$ in the "t" region.

A. Behavior of $|\text{Im}(\delta)|=F(\tilde{\omega},\theta)$

(1) *Dependence between* $|Im(\delta)|$ *and* θ *for a Fixed Value of* $\tilde{\omega}$

The character of the function expressing the directional behavior of $|\text{Im}(\delta)|$ depends upon the value of $\tilde{\omega}$. When $\tilde{\omega}$ is in the "l" region, this behavior is significantly different from the corresponding behavior when $\tilde{\omega}$ is in the "t" region.

In the "t" region, the excitation coefficient shows a pronounced selectivity for $\theta=\theta_t$. This is shown in Fig. 3. Thus the waves in the "t" region are characterized not only by frequency resonance but also by a "directional resonance." The directional resonance is associated with conical surfaces shown in the "geometrical" representation of Fig. 4. Within the cone K_1, the oscillations are stationary. There is a transverse instability between the cones K_1 and K_2 and hybrid instability outside the cone K_2.

Figure 3 also shows the behavior of the instability in the "l" region. The graph representing this instability increases slowly with θ without exhibiting any strong directional selectivity.

(2) *General Form of the Relationship* $|Im(\delta)|=F(\tilde{\omega},\theta)$

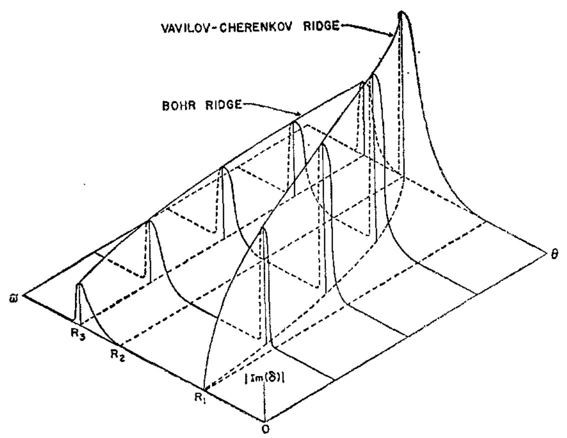
Using three rectangular coordinates we can represent the relationship between the excitation coefficient $|\text{Im}(\delta)|$ and the two variables $\tilde{\omega}$ and θ in the form of a three-dimensional surface shown in perspective in Fig. 5(a). A characteristic feature of this surface is represented by the "Vavilov-Cherenkov ridge" and the "Bohr ridge." These two ridges are loci of points for which there is a maximum in the excitation coefficients for the Vavilov-Cherenkov and for the Bohr instability. The projection of the two ridges on the $\text{Im}(\delta),\tilde{\omega}$ plane is shown in Fig. 5(b).

As shown in Fig. 5(a), the Bohr ridge and the Vavilov-Cherenkov ridge increase gradually in height without exhibiting any peak. The absence of a peak can be determined directly from the corresponding expressions for the Bohr instability and the Vavilov-Cherenkov instability. The gradual increase in the

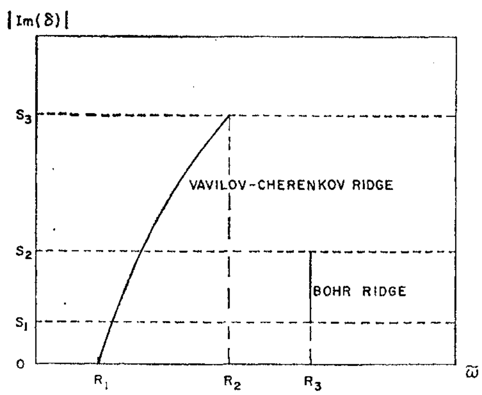
height of the Bohr ridge is evident from inspection of the formula (43). In the Vavilov-Cherenkov instability the absence of a peak can be determined from the expression (68). Using this expression we obtain

$$\frac{d(\delta^3)}{d\tilde{\omega}}=\frac{\kappa^2}{2[\epsilon(\tilde{\omega})]^3}\left\{\epsilon(\tilde{\omega})[\epsilon(\tilde{\omega})\beta^2-1][\epsilon(\omega)-1]+\frac{2\tilde{\omega}^2\omega_1^2}{(\tilde{\omega}^2-\omega_a^2)^2}([\epsilon(\tilde{\omega})]^2\beta^2-1)\right\}. \quad (87)$$

Since $\epsilon(\tilde{\omega})\sim 1/\beta^2\cos^2\theta$ the value $d(\delta^3)/d\tilde{\omega}$ as given by (87) is always positive. Hence the Vavilov-Cherenkov ridge increases gradually in height and there is no peak.



(a)



(b)

FIG. 5. (a) Perspective view of $|\mathrm{Im}(\delta)|=F(\tilde{\omega},\theta)$.

$$OR_1=[\omega_a^2-\omega_1^2\beta^2(1-\beta^2)]^{\frac{1}{2}};$$
$$OR_2=\omega_a;$$
$$OR_3=(\omega_1^2+\omega_a^2)^{\frac{1}{2}}.$$

(b) Projection of the Bohr ridge and the Vavilov-Cherenkov ridge on the $|\mathrm{Im}(\delta)|$, $\tilde{\omega}$ plane.

$$OS_1=(\sqrt{3}/2)[\kappa^2\omega_1^2(1-\beta^2)/2(\omega_1^2+\omega_a^2)^{\frac{1}{2}}]^{\frac{1}{3}};$$
$$OS_2=(\sqrt{3}/2)[\kappa^2\omega_1^2/2(\omega_1^2+\omega_a^2)^{\frac{1}{2}}]^{\frac{1}{3}};$$
$$OS_3=(\sqrt{3}/2)[\kappa^2\omega_a\beta^2/2]^{\frac{1}{3}}.$$

(3) *Instability for $\theta\sim\pi/2$*

For $\theta\sim\pi/2$ our original assumption $|\delta|\ll ck\beta\cos\theta$ is not valid. We shall investigate this case under the assumption that the term κ may not be necessarily small.

(a) Excited waves:

We substitute $\theta=\pi/2$ and Eq. (1) into the dispersion Eq. (16) and obtain

$$(\omega^2-\omega_1^2-\omega_a^2)^2\omega^4-(\omega^2-\omega_1^2-\omega_a^2)(\omega^2-\omega_a^2)\times[\omega^2\kappa^2(1-\beta^2)^{\frac{1}{2}}+\omega^2c^2k^2+\omega^2\kappa^2(1-\beta^2)+\kappa^2c^2k^2\beta^2]+(\omega^2-\omega_a^2)^2[\kappa^4(1-\beta^2)+\kappa^2c^2k^2]=0. \quad (88)$$

We notice that only even powers of ω appear in (88) and, therefore, this expression is of the form

$$f(\omega^2)=0. \quad (89)$$

Consider the inequalities

$$\omega_1^2+\omega_a^2>\kappa^2\beta^2, \quad (90)$$

and

$$\epsilon(0)=1+\frac{\omega_1^2}{\omega_a^2}>\frac{1}{\beta^2}+\frac{\kappa^2(1-\beta^2)}{c^2k^2\beta^2}. \quad (91)$$

These inequalities represent a sufficient condition for Eq. (89) to have a solution for ω^2 satisfying the inequality $-(\omega_1^2+\omega_a^2)<\omega^2<0$. The existence of this solution can be shown by computing

$$f(0)=\omega_a^4\kappa^2\left[-c^2k^2\beta^2\left(1+\frac{\omega_1^2}{\omega_a^2}\right)+c^2k^2+\kappa^2(1-\beta^2)\right], \quad (92)$$

and

$$f(-\omega_1^2-\omega_a^2)=4(\omega_1^2+\omega_a^2)^4+2(\omega_1^2+\omega_a^2)(\omega_1^2+2\omega_a^2)\times[c^2k^2(\omega_1^2+\omega_a^2-\kappa^2\beta^2)+(\omega_1^2+\omega_a^2)\kappa^2(2-\beta^2)]+(\omega_1^2+2\omega_a^2)^2[\kappa^4+\kappa^2c^2k^2]. \quad (93)$$

If (91) holds we have $f(0)<0$. If (90) holds we have $f(-\omega_1^2-\omega_a^2)>0$. In this case Eq. (88) has a purely imaginary solution. If θ differs from $\pi/2$ by some small amount, there appears a real component in the solution. To show this, one separates the odd and even powers of ω in Eq. (88) and finds that a purely imaginary value for ω could not satisfy the equation.

(b) Amplified waves:

The dispersion equation (16) can be written

$$k=\pm\frac{1}{c}\left\{\frac{[\omega^2\epsilon(\omega)-\kappa^2][\omega^2\epsilon(\omega)-\kappa^2(1-\beta^2)]}{\omega^2\epsilon(\omega)+\kappa^2[\beta^2\epsilon(\omega)-1]}\right\}^{\frac{1}{2}}. \quad (94)$$

Thus k is always either real or purely imaginary. For values of ω satisfying $\omega_a^2<\omega^2<\omega_1^2+\omega_a^2$ we have $\epsilon(\omega)<0$ and hence k is imaginary.

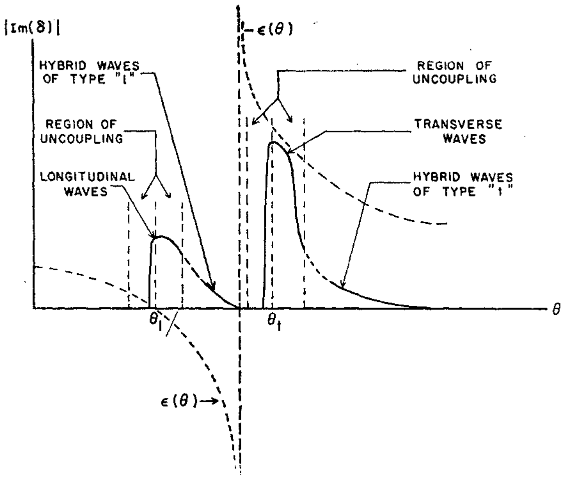


FIG. 6. Relationship between $|\mathrm{Im}(\delta)|$ and θ for a fixed $k>(\omega_1^2+\omega_a^2)^{\frac{1}{2}}/\beta c$.

B. Behavior of $|\mathrm{Im}(\delta)|=\phi(\kappa,\theta)$

(1) *Dependence between $|Im(\delta)|$ and θ for a Fixed Value of k*

Assume k fixed and large enough so that $k^2>(\omega_1^2+\omega_a^2)/\beta^2c^2$. The directional behavior of the excitation coefficient is represented in Fig. 6 by two curves representing the regions "l" and "t". These curves exhibit directional resonance at points $\theta=\theta_l$ and $\theta=\theta_t$. The corresponding "geometrical representation" is given in Fig. 7. The transverse instabilities in the neighborhood of $\theta=\theta_t$ are represented by a region of Vavilov-Cherenkov cones and the corresponding hybrid instabilities are in an adjacent region outside of the Vavilov-Cherenkov cones. Similarly, the longitudinal instabilities in the neighborhood of $\theta=\theta_l$ are represented by a region of "Bohr cones," and the corresponding hybrid instabilities are in an adjacent region outside of the Bohr cones.

(2) *General Form of the Relationship $|Im(\delta)|=\phi(k,\theta)$*

The relationship between the excitation coefficient $|\mathrm{Im}(\delta)|$ and the two variables k and θ is shown in perspective in Fig. 8(a). The surface representing this relationship shows the two characteristic ridges representing the Bohr and the Vavilov-Cherenkov instabilities. The projection of the two ridges on the plane of $|\mathrm{Im}(\delta)|$ and k is shown in Fig. 8(b).

C. Relative Magnitudes of the Bohr Instability and the Vavilov-Cherenkov Instability

There is a considerable difference in the directional behavior of the two instabilities. One can estimate the relative magnitudes of these instabilities by determining the expression

$$|\mathrm{Im}(\delta)|_{\mathrm{B}}/|\mathrm{Im}(\delta)|_{\mathrm{V\text{-}Ch}}=f(\theta), \tag{95}$$

in which $|\mathrm{Im}(\delta)|_{\mathrm{B}}$ representing the excitation coefficient due to Bohr radiation is given by (43). The term $|\mathrm{Im}(\delta)|_{\mathrm{V\text{-}Ch}}$ corresponding to the Vavilov-Cherenkov effect can be expressed by means of (67) in the following form:

$$|\mathrm{Im}(\delta)|_{\mathrm{V\text{-}Ch}}=(\sqrt{3}/2^{4/3}) \times\{\kappa^{2/3}[\omega_a^2+(\omega_1^2-\omega_a^2)\beta^2\cos^2\theta]^{1/6} \times(1-\beta^2\cos^2\theta)^{1/6}\beta^{2/3}(\sin\theta)^{2/3}\}. \tag{96}$$

We shall consider the expression (96) for a physically important case in which $\omega_1\ll\omega_a$, and we shall exclude from our consideration the values of β and θ satisfying the relationship $\beta\cos\theta\sim1$. This relationship is satisfied if θ is very small and the electron beam has velocities in a highly relativistic range. The expression (95) can then be represented as

$$|\mathrm{Im}(\delta)|_{\mathrm{B}}/|\mathrm{Im}(\delta)|_{\mathrm{V\text{-}Ch}}\sim K^{\frac{1}{3}}\sin^{\frac{2}{3}}\theta, \tag{97}$$

where

$$K=\omega_1^2/\omega_a^2\beta^2. \tag{98}$$

The expression (97) gives an index of the relative magnitudes of the two instabilities and shows that this index is strongly directional. Thus the Bohr radiation is dominant for small values of θ, but the relative intensity of the Bohr radiation as compared to the Vavilov-Cherenkov radiation decreases very rapidly for increasing values of θ.

VI. SMALL-ANGLE APPROXIMATION

In the dispersion Eq. (16) the assumption that $G(\omega,k)=0$ gives an uncoupling of the longitudinal oscillations described by $F(\omega,k)E_z=0$ and the trans-

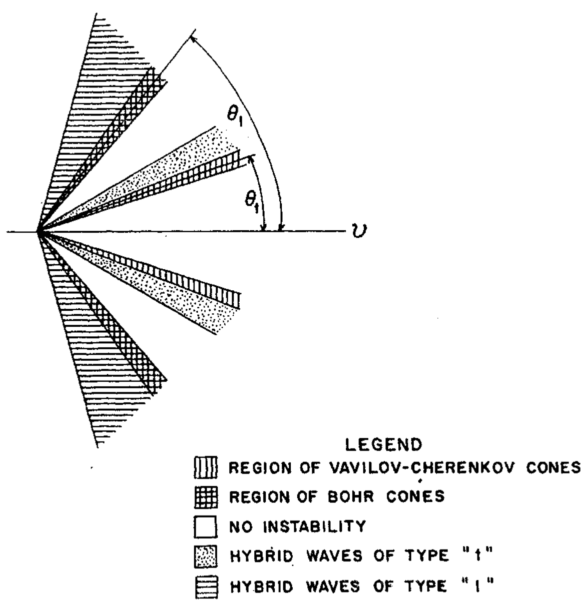


FIG. 7. Directional characteristics of the waves for a fixed $k>(\omega^2+\omega_a^2)^{\frac{1}{2}}/\beta c$.

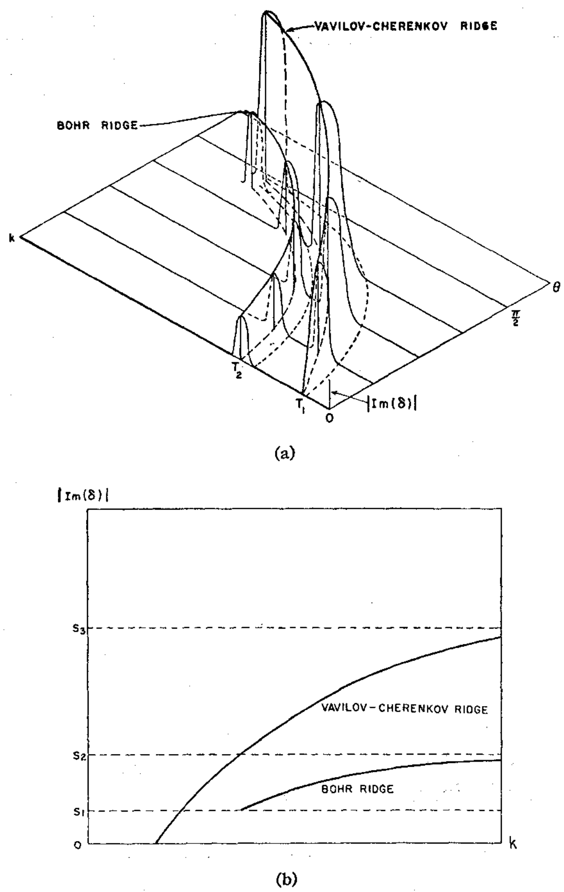


FIG. 8. (a) Perspective view of $|\mathrm{Im}(\delta)|=\phi(k,\theta)$.

$OT_1=[\omega_a^2-\omega_1^2\beta^2/(1-\beta^2)]^{\frac{1}{2}}(1/\beta c)$;
$OT_2=(\omega_a^2+\omega_1^2)^{\frac{1}{2}}/\beta c$.

(b) Projection of the Bohr ridge and the Vavilov-Cherenkov ridge on the $|\mathrm{Im}(\delta)|$, k plane.

verse oscillations described by $A(\omega,k)E_x=0$. Heretofore we have discussed the behavior of the system for small values of κ by assuming $\kappa\ll|\omega-\beta ck\cos\theta|$ and neglecting terms of the order of $[\kappa/(\omega-\beta c\cos\theta)]^4$. This was equivalent to assuming $G(\omega,k)=0$ and hence gave uncoupling.

If $\theta=0$ we will again have $G(\omega,k)=0$ and uncoupling. We shall now examine the behavior of the system for values of θ in the neighborhood of zero (small angle approximation). We assume $\sin\theta$ sufficiently small that we may neglect terms involving $\sin^2\theta$. This is equivalent to assuming $G(\omega,k)=0$

A. Longitudinal Oscillations

(*1*) *Excited Waves*

We put $\omega=ck\beta+\delta$ in the dispersion Eq. (32). We assume that $|\delta|\ll ck\beta$ and obtain

$$\delta=\pm\kappa(1-\beta^2)^{\frac{1}{2}}[(c^2k^2\beta^2-\omega_a^2)/(c^2k^2\beta^2-\omega_1^2-\omega_a^2)]^{\frac{1}{2}}. \quad (99)$$

It is apparent that δ is imaginary for values of $ck\beta$ satisfying the inequality $\omega_a^2<c^2k^2\beta^2<(\omega_1^2+\omega_a^2)$ and $\delta=O(\kappa)$ except for values of $ck\beta\approx(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$. Therefore, if we assume $\kappa\ll ck\beta$, we have a solution satisfying our assumption and hence an instability.

(*2*) *Amplified Waves*

The dispersion Eq. (32) yields the following expression for k:

$$k=\frac{\omega}{c\beta}\pm\frac{\kappa}{c\beta}(1-\beta^2\cos^2\theta)^{\frac{1}{2}}\left[\frac{\omega^2-\omega_a^2}{\omega^2-\omega_1^2-\omega_a^2}\right]^{\frac{1}{2}}. \quad (100)$$

It is apparent that no real value of ω satisfying the inequality $\omega_a<\omega<(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$ can satisfy Eq. (32) since the term in brackets will be negative. Thus there is an instability for frequencies between ω_a and $(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$.

B. Transverse Oscillations

Neglecting the terms containing $\sin^2\theta$, the dispersion Eq. (33) can be expressed in a quadratic form as follows:

$$\omega^4[2\omega_a^2+c^2k^2+\kappa^2]\omega^2-\omega_a^2c^2k^2-\omega_a^2\kappa^2=0. \quad (101)$$

It can be shown that if $c^2k^2>\kappa^2$ the equation (101) gives four real solutions for ω. On the other hand, if $c^2k^2<\kappa^2$ there will be two real and two pure imaginary solutions. In either case, the transverse wave is stable.

If θ is sufficiently small so that we can consider $G(\omega,k)=0$, we must also neglect the last term of Eq. (33). We have then the transverse wave stable. However, the longitudinal wave is unstable for values of ω satisfying the inequality $\omega_a^2<\omega^2<(\omega_1^2+\omega_a^2)$.

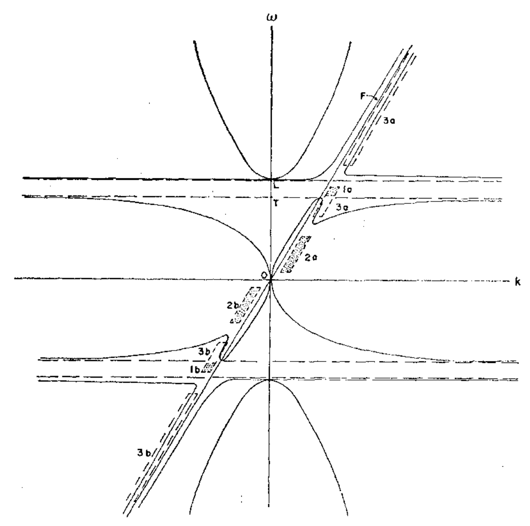


FIG. 9. "$\omega-k$" diagram for hybrid oscillations (small κ approximation). $OT=\omega_a$; $OL=(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$. F is the line $\omega=ck\beta\cos\theta$.

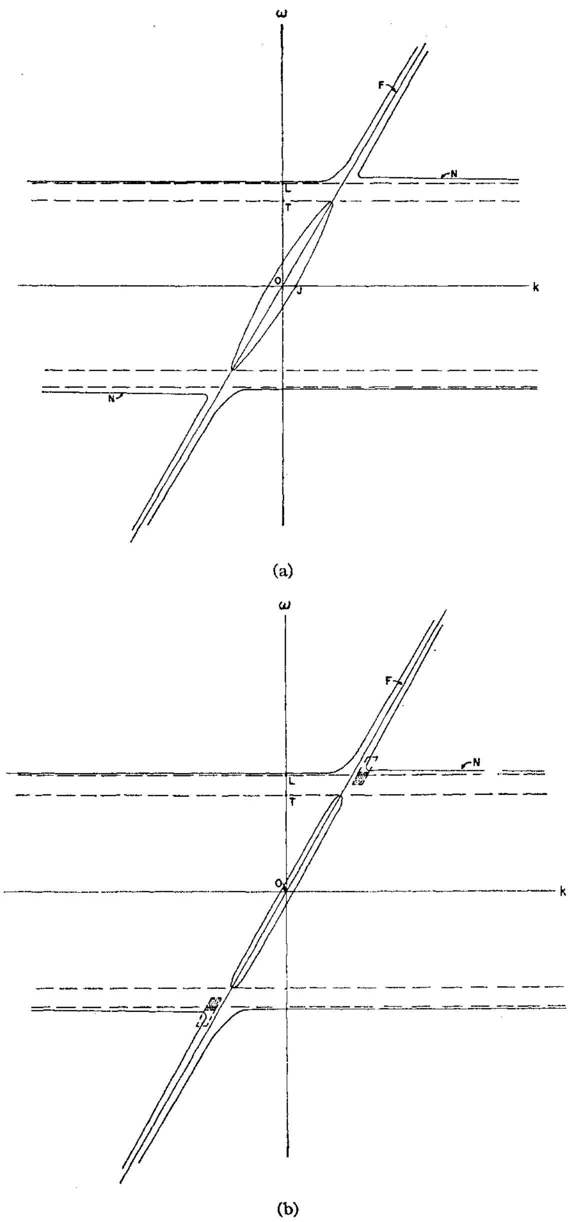


FIG. 10. (a) "$\omega-k$" diagram for "uncoupled" longitudinal oscillations (small-angle approximation). $OJ=\kappa\omega_a/\beta c(\omega_1^2+\omega_a^2)^{\frac{1}{2}}$. (b) "$\omega-k$" diagram for "uncoupled" longitudinal oscillations (small κ approximation).

VII. GRAPHICAL REPRESENTATION OF THE DISPERSION RELATION

We shall now illustrate some of the results of our discussion in the form of "$\omega-k$" diagrams as used by Sturrock.

(1) *Coupled Oscillations*

The dispersion equation has the form (16). The "$\omega-k$" diagram representing this equation for a small value of κ is shown in Fig. 9. In such case (small κ approximation) we have a solution of the form $\omega=ck\beta\cos\theta+\delta$ where $|\delta|\ll ck\beta\cos\theta$. The system is coupled if there exists a solution for δ of the dispersion equation which is subject to the condition $|\delta|=O(\kappa)$. If this solution is complex the resulting instability has been described as "weak" instability of the hybrid type. It was shown that the system does in fact have a solution satisfying the above conditions. Since in the region of instability k is real and ω complex, these solutions cannot be represented by points in the real "$\omega-k$" plane. The shaded rectangular "windows" have been drawn close to the line $\omega=ck\beta\cos\theta$ and are used symbolically here. These "windows" indicate that for a fixed real value of k in the region of instability, the point in the complex plane which corresponds to the solution for ω will be close to the real number $ck\beta\cos\theta$. There are six rectangular windows shown in Fig. 9. The shaded sections within the windows 1a, 1b represent hybrid instability of type "l" while the shaded sections within windows 2a and 2b represent hybrid instability of type "t." The unshaded portion of the graph inside windows 3a and 3b represents the region of stable oscillations. It is noted from the diagram that both "t" and "l" instabilities are convective.

(2) *Uncoupled Oscillations*

(*a*) *Small-angle approximation:*

We assume that $\theta\sim 0$, but κ is not necessarily small. The "$\omega-k$" diagram for longitudinal oscillation is shown in Fig. 10. The instability associated with the branch "N" of the curve represents the "Bohr radiation." This instability is shown to be convective. There are no instabilities associated with the transverse oscillations and a diagram representing the transverse mode has not been shown.

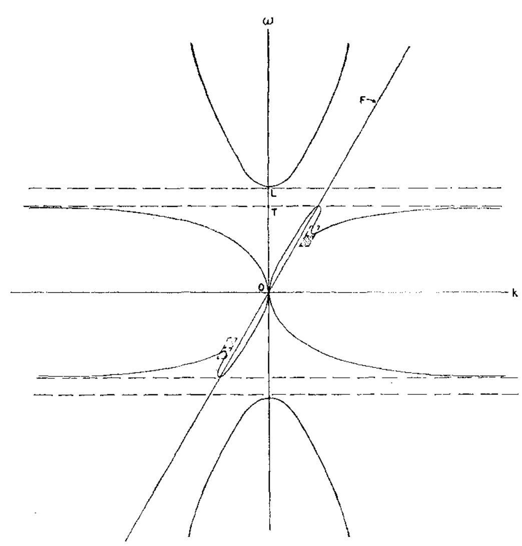


FIG. 11. "$\omega-k$" diagram for "uncoupled" transverse oscillations (small κ approximation).

(*b*) *Small-κ approximation:*

For sufficiently small values of κ the ratio $(\kappa/|\delta|)^4$ becomes negligible, which is equivalent to uncoupling. There are two dispersion relationships: $A(\omega,k)=0$ represents the transverse mode, and $F(\omega,k)=0$ represents the longitudinal mode. The corresponding instabilities will be illustrated by means of "$\omega-k$" diagrams.

(1) Longitudinal oscillations. The corresponding "$\omega-k$" diagram shown in Fig. 10(b) is qualitatively similar to the one shown in Fig. 10(a). There is, however, an essential difference between them. The curve of Fig. 10(a) represents the dispersion relationship for the entire range of ω and k. On the other hand, the curve of Fig. 10(b) represents the "small-κ approximation," and, therefore, the only range of ω and k that is quantitatively significant is in the regions for which there exists a solution for δ satisfying the inequalities $\kappa \ll |\delta| \ll \tilde{\omega}$. It has been shown that there exists such a solution at least in the interior of the two rectangular "windows." The remainder of the graph is inappropriate and has been included only to show more clearly the character of the graph inside the "windows." The shaded portions again show the region of instability and the unshaded portion shows the region of stable oscillations. The instability is "strong" in this region since the inequality $\kappa \ll |\mathrm{Im}(\delta)|$ is satisfied. The instability is convective.

(2) Transverse oscillations. Figure 11 shows the "$\omega-k$" diagram which is applicable only in the regions where there exists a solution for δ satisfying the inequalities $\kappa \ll |\delta| \ll \tilde{\omega}$. As in the previous case, such a solution has been shown to exist inside the rectangular "windows." The remainder of the graph is shown only in order to illustrate more clearly the behavior of the small portion of the graph inside the "windows." The instability shown in the shaded area is associated with Vavilov-Cherenkov radiation and is shown to be convective.

If one examines the graphs in Figs. 9, 10(b), and 11, it is clear how the Vavilov-Cherenkov radiation is "extended" into region of hybrid instability of the type "t" and the Bohr radiation is "extended" into the hybrid instability of type "l".

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Effective Depth of X-Ray Production*

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By measuring x-ray emission profiles as a function of voltage, the depth of penetration of the cathode electrons involved may be calculated. Similarly, the effective depth of x-ray production may be obtained. Uncertainties exist because of conditions at the surface, so the rate of change of these depths with voltage may be established with greater confidence than the values themselves. For copper, the continuum yields values increasing from 300 A/kv to 480 A/kv over the 10–30 kv range. An average value of 450 A/kv is obtained from analysis of the less trustworthy line data. The effective depth obtained from the conventional calculation on the absorption edge is shown to have little physical meaning.

INTRODUCTION

INFORMATION about the passage through matter of electrons having energies in the range of tens of kilo-electron volts is needed for the analysis of several physical phenomena. Among them are (1) the correction of the low-energy portion of the continuous β-ray spectrum, and (2) the correction for target self-absorption of x-ray emission spectra involving transitions of the valence electrons. Some information pertinent to the problem is available from measurements on the intensity of emission of x-ray lines and also from the discontinuities in the continuum. This can yield a measure of the depth of penetration of the cathode electron and also a measure of the effective depth of x-ray production. The information is necessarily limited in scope since it involves cathode electrons which have lost little energy through inelastic collisions prior to the event which produced the x rays.

The physical processes involved in x-ray production are numerous and complex. The path of the cathode ray in the target is tortuous, although the electrons involved in this particular process are probably traversing a reasonably linear path. These electrons may eject inner electrons from the target atoms giving rise to directly produced line radiation; or they may be abruptly deflected so that continuous radiation is produced. The bremsstrahlung of sufficient energy has the further capacity to produce line radiation by being photoelectrically absorbed. Because of the plethora of

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