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Renormalization of Time-Ordered Green's Functions*

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The renormalization of time-ordered Green's functions is carried out without reference to Feynman diagrams. The arguments are entirely based on the generalized unitarity condition and the parametric dispersion relations. The renormalization of the meson-nucleon interaction is studied, and then a close examination is given of the renormalization of quantum electrodynamics in a special gauge. Finally the connection between the subtraction constants in dispersion relations and renormalization constants is clarified in a simple model.

I. INTRODUCTION

IN a series of papers[1–3] the formulation of field theories based on the generalized unitarity condition and the parametric dispersion relations has been developed. In particular in reference 2 the connection between the renormalizable Lagrangian theory and the present approach was established in the perturbation theory. One of the main results obtained in that paper was that there is a one-to-one correspondence between an interaction term in the Lagrangian and a subtraction in the parametric dispersion relation for the corresponding Green's function. Roughly speaking the coupling constants of elementary interactions were introduced into our framework theory as the subtraction constants in the parametric dispersion relations.

The subtractions in the dispersion relations serve to introduce interactions into the theory in the lowest order perturbation theory. As the order of approximation proceeds, the subtractions in turn serve to eliminate divergences from the dispersion integrals. If the number of subtractions assumed in the beginning is not sufficient to eliminate divergences from the dispersion integrals in the higher orders, this is an indication that the theory does not possess any convergent solution in the perturbation theory, and the theory is called unrenormalizable.

It is important and interesting to clarify the criterion for the renormalizability of a theory in our scheme. Our result naturally agrees with that of the conventional Feynman-Dyson theory, but it is perhaps worthwhile to mention that the renormalizability condition can be derived without reference to Feynman diagrams.

In Sec. II the generalized unitarity condition and the parametric dispersion relations are briefly recapitulated. Then the renormalization of the meson-nucleon interaction is discussed in Sec. III. The renormalization of quantum electrodynamics will be carried out in a special gauge. This problem is not so simple as one anticipates, and it will be discussed in Sec. IV putting a special emphasis on the characteristic features of quantum electrodynamics.

Our formulation is in some sense rather abstract, and for this reason the physical meaning of subtractions in the parametric dispersion relations is investigated in Sec. V. It will be shown that the subtractions in the dispersion relations really replace the renormalization procedures.

II. GENERALIZED UNITARY CONDITION AND PARAMETRIC DISPERSION RELATIONS

Our arguments on renormalization are based on the generalized unitarity condition and parametric dispersion relations. Although these subjects have been repeatedly discussed in references 1–3, we shall briefly recapitulate them for the sake of completeness.

Take for simplicity the neutral scalar field $\varphi(x)$, and define the τ functions by

$$\tau(x_1\cdots x_n) = (-i)^n K_{x_1}\cdots K_{x_n}\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle, \quad (2.1)$$

where K_x denotes the Klein-Gordon operator and $|0\rangle$ the vacuum state. Then the generalized unitarity condition is given by a set of coupled equations of the form

$$\tau(x_1\cdots x_n)+\tau^*(x_1\cdots x_n) + \sum_{\text{comb}}{}' \sum_{l=0}^{\infty}\frac{i^l}{l!}\int (du)(dv)\tau(x_1'\cdots x_k'u_1\cdots u_l) \times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l) \times\tau^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)=0, \quad (2.2)$$

where $(du)=d^4u_1\cdots d^4u_l$, $(dv)=d^4v_1\cdots d^4v_l$, and τ^* is the complex conjugate of τ. $\sum'$ means to take the sum over all possible combinations which divide $x_1, \cdots, x_n$ into two groups, one entering into τ and the other into τ^*, excluding $k=0$ and $k=n$. The contraction function $\Delta^{(+)}$ is defined by

$$i\Delta^{(+)}(u-v)=\sum_p\langle 0|\varphi(u)|p\rangle\langle p|\varphi(v)|0\rangle \quad (2.3)$$

$$=\frac{1}{(2\pi)^3}\int d^4p\, e^{ip(u-v)}\theta(p_0)\delta(p^2+m^2), \quad (2.4)$$

where the summation is taken over all single-particle

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[1] K. Nishijima, Phys. Rev. **114**, 485 (1960).
[2] M. Muraskin and K. Nishijima, Phys. Rev. **122**, 331 (1961).
[3] K. Nishijima, Phys. Rev. **122**, 248 (1961).

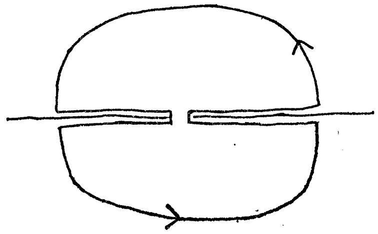

FIG. 1. Path of integration in the complex ξ plane used in the derivation of Eq. (2.8) from Cauchy's formula. The analytic continuation of $\mathcal{G}(\xi)$ from the upper half-plane to the lower half-plane is made through an infinitesimal gap between the cuts along the positive and negative real axes.

states with energy-momentum p, and $\theta(p_0)$ is the ordinary step function defined by $\theta(p_0)=1$ for $p_0>0$, and $=0$ for $p_0<0$.

The Feynman diagrams contributing to a τ function are in general disconnected, and we shall call the sum of contributions from connected Feynman diagrams the ρ function. The relationship between τ and ρ functions is given by a recursion formula

$$\tau(xx_1\cdots x_n)=\rho(x_1\cdots x_n)+\sum_{\text{comb}}\rho(xx_1'\cdots x_k)\tau(x_{k+1}'\cdots x_n'), \quad (2.5)$$

where the summation is extended over all possible ways to divide $x_1\cdots x_n$ into two groups, one entering into ρ and the other into τ, excluding $k=n$.

The Fourier transform of ρ is introduced by

$$\rho(x_1\cdots x_n)=\frac{-i}{(2\pi)^{4(n-1)}}\int d^4p_1\cdots d^4p_n \times\delta(p_1+\cdots+p_n)e^{i(p_1x_1+\cdots+p_nx_n)}\mathcal{G}(p_1\cdots p_n). \quad (2.6)$$

$\mathcal{G}$ is a function of scalar products of p's and will be denoted by $\mathcal{G}(p_\alpha p_\beta)$ hereafter. Then $\mathcal{G}$ satisfies, irrespective of the number of variables, the dispersion relation

$$\mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{\mathrm{P}}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'-\xi}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'), \quad (2.7)$$

where ξ is a common scaling parameter to be multiplied into all the scalar products. When a subtraction is needed we differentiate both sides of Eq. (2.7) with respect to ξ. In Eq. (2.7) all the scalar products $p_\alpha p_\beta$ are assumed to be real, but then $\mathcal{G}(p_\alpha p_\beta\cdot\xi)$ can be defined for complex values of ξ with the help of the dispersion relation (2.7), i.e.,

$$\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{1}{\pi}\left[\int_0^\infty\frac{d\xi'}{\xi'-\xi-i\epsilon}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi')-\int_{-\infty}^0\frac{d\xi'}{\xi'-\xi+i\epsilon}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi')\right]. \quad (2.8)$$

It is clear from Eq. (2.8) that $\mathcal{G}(p_\alpha p_\beta\cdot\xi)$ is the boundary value of an analytic function of ξ with cuts along the positive and negative real axes:

$$\begin{aligned}\mathcal{G}(\xi)&=\lim_{\epsilon\to0}\mathcal{G}(\xi+i\epsilon)\quad\text{for}\quad\xi>0,\\&=\lim_{\epsilon\to0}\mathcal{G}(\xi-i\epsilon)\quad\text{for}\quad\xi<0,\end{aligned} \quad (2.9)$$

and

$$\mathrm{Im}\mathcal{G}(0)=0. \quad (2.10)$$

Equation (2.8) is nothing but Cauchy's formula when the path of integration is taken as in Fig. 1.

III. RENORMALIZATION OF MESON-NUCLEON INTERACTION

In reference 2 the power counting theorem in the simplest scalar theory was proved, and in this paper this theorem will be generalized so as to cover more interesting cases.

First the essential ideas involved in the previous paper will be recapitulated. The most important point in the discussion of renormalization of Green's functions is the recognition that divergences occur only in the dispersive parts but never in the absorptive parts. This fact facilitates the separation of divergences. In the perturbation theory we can proceed to higher orders step by step by combining the generalized unitarity condition and the parametric dispersion relations. The former is used to calculate the absorptive part of a Green's function from the lower order Green's functions, and the latter enables us to calculate the dispersive part of that Green's function from its absorptive part. Clearly the application of the unitarity condition in the calculation of the absorptive part of a higher order Green's function does not cause any divergences, but the dispersion integral used to calculate the dispersive part from the absorptive part does not necessarily converge. Thus we have to introduce subtracted dispersion relations in some cases. We have already learned in reference 2 that subtractions in the parametric dispersion relations for Green's functions introduce interactions into the perturbation theory. If more and more subtractions are needed to eliminate divergences from the dispersion integrals as the order of approximation proceeds, i.e., if an unlimited number of subtractions are needed in higher orders, the theory is not renormalizable. Therefore, when we introduce subtractions, and consequently interactions, there must be a guarantee that the originally assumed subtractions are sufficient to make all the dispersion integrals converge in all orders of perturbation theory. This guarantee, called the renormalizability condition, is the subject of this section.

In order to solve the renormalization problem we have to postulate certain high-energy behavior of the $\mathcal{G}$ functions. In connection with the parametric dispersion relations we have to know the behavior of

$\mathcal{G}(p_\alpha p_\beta\cdot\xi)$ for large values of ξ, and we introduce the following postulate:

Postulate I. For large values of ξ the asymptotic behavior of $\mathcal{G}(p_\alpha p_\beta\cdot\xi)$ is governed by a power law:

$$\mathcal{G}(p_\alpha p_\beta\cdot\xi)\sim\xi^{\alpha/2}, \tag{3.1}$$

provided that all the scalar products are real. More precisely this equation must be understood as that the proportionality factor may depend on p's but the power $\alpha/2$ should be independent of p's in almost all configurations of p's.

This postulate is satisfied in the conventional perturbation theory.[4] Furthermore it should be mentioned that functions like $\ln\xi$ are counted as ξ^0, or more precisely Eq. (3.1) states that for an arbitrary given positive ϵ the following equations hold:

$$\lim_{\xi\to\infty}\frac{\mathcal{G}(p_\alpha p_\beta\cdot\xi)}{\xi^{\frac{1}{2}\alpha+\epsilon}}=0,\quad \lim_{\xi\to\infty}\frac{\mathcal{G}(p_\alpha p_\beta\cdot\xi)}{\xi^{\frac{1}{2}\alpha-\epsilon}}=\infty. \tag{3.1a}$$

The power $\alpha/2$ of course depends on the number of variables and species of fields entering into the definition of the $\mathcal{G}$ function under consideration.

In what follows we shall write the power law in a more intuitive but not rigorous manner, i.e.,

$$\mathcal{G}(p_\alpha p_\beta)\sim p^\alpha. \tag{3.2}$$

In this formula we do not distinguish between different p's since the purpose of the power law consists in the discussion of the subtractions in the parametric dispersion relations.

For simplicity we shall start from the discussion of the neutral scalar theory. The power α then depends on the number of variables n and will be denoted by $c(n)$.

We now study the necessary condition for the renormalizability of the theory, and for this purpose we start from the generalized unitarity condition (2.2) since the ordinary unitarity condition, which does not allow the scale transformation used in the parametric dispersion relation (2.7), does not meet our present need. Let us retain in (2.2) only those terms that correspond to connected Feynman diagrams; then as has been mentioned already in reference 2, the connected part of $\tau+\tau^*$ is $\rho+\rho^*$ and its Fourier transform is the absorptive part of $\mathrm{Im}\mathcal{G}$.

The connected part of the last term in (2.2) is obtained by expanding τ's into ρ's and picking up only connected terms in the sense of reference 1. In this way we find the generalized unitarity condition for the connected ρ functions. This new form of the generalized unitarity condition reads:

$$\rho(x_1\cdots x_n)+\rho^*(x_1\cdots x_n)+\sum_{\text{comb}}{}'\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)$$
$$\times\rho(x_1'\cdots x_k'u_1\cdots u_l)\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)$$
$$\times\rho^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)+(\text{trilinear and further multilinear forms in }\rho\text{ and }\rho^*)=0. \tag{3.3}$$

Now let us examine the high-energy behavior of the Fourier transform of Eq. (3.3). The power of $\mathrm{Im}\mathcal{G}^{(n)}$ which is the Fourier transform of $\mathrm{Re}\rho(x_1\cdots x_n)$ cannot exceed $c(n)$, the power of $\mathcal{G}^{(n)}$. Next we have to examine the power of the nonlinear part of Eq. (3.3) and compare with $c(n)$. This is, however, not an easy task unless we introduce another postulate which is less valid than the power law.

Postulate II. The power of the nonlinear part in the unitarity equation (3.3) is given by the highest one of the powers of all the terms in the nonlinear part.

This postulate is certainly restrictive and probably valid only in the perturbation theory as illustrated by the counter example

$$1+a\ln p+(a^2/2!)(\ln p)^2+\cdots=\exp(a\ln p)=p^a.$$

The power of this series is given by a, whereas the power of each term in this series is zero. Once this postulate is taken for granted the derivation of the renormalizability condition is straightforward. First it can be concluded that the highest power in the bilinear part of Eq. (3.3) should not exceed $c(n)$, and this condition can be written down explicitly.

$$c(n)+4(n-1)\geq\max[c(k+l)+c(n-k+l)+4(n+2l-2)]-6l. \tag{3.4}$$

The term $4(n-1)$ on the left-hand side as well as $4(n+2l-2)$ on the right-hand side stem from the definition of $\mathcal{G}^{(n)}$ from $\rho(x_1\cdots x_n)$, Eq. (2.6). "max" means to pick up the highest power from the sum over various values of the k's and l's in the bilinear part of Eq. (3.3). $6l$ on the right-hand side comes from (du), (dv), and $\Delta^{(+)}(u-v)$.

There is one important observation concerning the structure of Eq. (3.3), namely the two-point ρ function never appears in the nonlinear part of Eq. (3.3). Therefore we study the inequality (3.4) subject to the conditions $k+l>2$, $n-k+l>2$. The special case $n=2$ will be discussed after all others are settled since the two-point ρ function is completely decoupled from all others. This situation is characteristic of the present approach and has already been mentioned in reference 2.

Another important observation is that if we take trilinear or further multilinear terms we get another inequality involving three or more c's on the right-hand side but this new inequality is always satisfied provided that (3.4) is satisfied. In other words, the inequality

[4] The validity of the power law seems to be wider than that of the perturbation theory. For instance, this law is obeyed by certain Green's functions calculated in the ladder approximation. See in this connection the following articles: S. F. Edward, Phys. Rev. **90**, 284 (1953); P. Federbush, M. L. Goldberger, and S. B. Treiman, *ibid.* **112**, 642 (1958).

(3.4) already represents the necessary condition for the renormalizability.

Having established the mathematical method of discussing this problem we shall now switch from the simple scalar theory to the more interesting problem of the meson-nucleon interaction. We assume that the meson is a spinless particle and the nucleon a Dirac particle. Then the most general τ function in this case is defined by

$$(-i)^a K_{x_1}\cdots K_{x_a} D_{y_1}\cdots D_{y_b}\tilde{D}_{z_1}\cdots D_{z_b} \times\langle 0|T[\varphi(x_1)\cdots\varphi(x_a)\psi(y_1)\cdots\psi(y_b) \times\bar{\psi}(z_1)\cdots\bar{\psi}(z_b)]|0\rangle, \quad (3.5)$$

where $D_x=\gamma\partial_x+M$, $\tilde{D}_x=\gamma^T\partial_x-M$, and M is the nucleon mass. From the conservation of nucleon number, this τ function must involve the same number of ψ's and $\bar{\psi}$'s.

Furthermore the contraction functions for the nucleon field are given by

$$\sum\langle 0|\psi(x)|p\rangle\langle p|\bar{\psi}(y)|0\rangle=-iS^{(+)}(x-y) =-i(\gamma\partial_x-M)\Delta^{(+)}(x-y,M),$$

for one-nucleon intermediate states;

$$\sum\langle 0|\bar{\psi}(x)|p\rangle\langle p|\psi(y)|0\rangle=-i\tilde{S}^{(+)}(x-y) =-(\gamma^T\partial_x+M)\Delta^{(+)}(x-y,M), \quad (3.6)$$

for one-antinucleon intermediate states.

Let us denote the $\mathcal{G}$ function corresponding to the τ function (3.5) by $\mathcal{G}^{(n,m)}$, where $n=a$ and $m=2b$. We write the power law for $\mathcal{G}^{(n,m)}$ as

$$\mathcal{G}^{(n,m)}\sim p^{c(n,m)}, \quad \text{as} \quad p\to\infty; \quad (3.7)$$

then the inequality (3.4) for the scalar theory is generalized to[5]

$$c(n,m)+4(n+m-1)\geq\max[c(k+l,k'+l') +c(n-k+l,m-k'+l')+4(n+m+2l+2l'-2) -6(l+l')+l'], \quad (3.8)$$

subject to the restrictions

$$k+l+k'+l'>2, \quad n-k+l+m-k'+l'>2. \quad (3.9)$$

The last term l' on the right-hand side of (3.8) results from the substitution of $S^{(+)}$ or $\tilde{S}^{(+)}$ for $\Delta^{(+)}$. The restrictions (3.9) are due to the absence of the two-point $\mathcal{G}$ functions in the unitarity condition. Put now

$$d(n,m)=c(n,m)+n+\tfrac{3}{2}m-4; \quad (3.10)$$

then Eq. (3.8) is simplified and is given by

$$d(n,m)\geq\max[d(k+l,k'+l') +d(n-k+l,m-k'+l')]. \quad (3.11)$$

If we further define

$$d(n)=\max d(k,n-k),$$

then this $d(n)$ satisfies

$$d(n)\geq\max[d(k+l)+d(n-k+l)], \quad (3.11\text{a})$$

again subject to $k+l>2$, and $n-k+l>2$. From (3.11a) we can conclude for $n>2$ that all $d(n)$'s are nonpositive,

$$d(n)\leq 0, \quad (3.12)$$

or, in terms of $c(n,m)$,

$$c(n,m)\leq 4-n-\tfrac{3}{2}m. \quad (3.13)$$

So far the arguments are based solely on the generalized unitarity condition, and (3.13) is a necessary condition for the satisfaction of the unitarity.[6] The power equation (or inequality) (3.11) possesses generally many solutions if no other condition is posed.

In order to narrow the solutions of the power equation we have to introduce the parametric dispersion relations. We shall first list those combinations of n and m for which the upper limit of $c(n,m)$ can be nonnegative and consequently $\mathcal{G}^{(n,m)}$ might imply subtractions.

(i) $n=3$, $m=0$: upper limit for $c(n,m)=1$;

(ii) $n=4$, $m=0$: upper limit for $c(n,m)=0$;

(iii) $n=1$, $m=2$: upper limit for $c(n,m)=0$.

The above list shows that the generalized unitarity condition implies that the $\mathcal{G}$ functions other than those listed above should satisfy unsubtracted parametric dispersion relations and that for the $\mathcal{G}$ functions listed above one subtraction suffices to make the dispersion integral converge. In what follows, the significance of this result will be studied in the language of the Lagrangian theory.

First one subtraction for $\mathcal{G}^{(3,0)}$ or $\mathcal{G}^{(4,0)}$ introduces in the corresponding Lagrangian theory the interaction φ^3 or φ^4, but the subtraction for $\mathcal{G}^{(1,2)}$ requires a detailed elucidation as to the nature of the interaction. For the sake of definiteness let us assume that the meson is pseudoscalar; then $\mathcal{G}^{(1,2)}$ will be decomposed, as discussed in Sec. IV of reference 2, into a sum of invariants:

$$\mathcal{G}(q,p,p')=i\gamma_5[\mathcal{G}_P(q,p,p')+iq\gamma\cdot\mathcal{G}_A(q,p,p') +\sigma_{\mu\nu}p_\mu p_\nu'\mathcal{G}_T(q,p,p')], \quad (3.14)$$

where $\mathcal{G}(q,p,p')$ is the Fourier transform (2.6) of

$$\rho(xyz)=(-i)K_xD_y\tilde{D}_z\langle 0|T[\varphi(x)\psi(y)\bar{\psi}(z)]|0\rangle. \quad (3.15)$$

We already know in reference 2 that one subtraction for $\mathcal{G}_P$ introduces the pseudoscalar coupling and that for $\mathcal{G}_A$, the axial vector coupling. On the other hand, we also know from the above discussion that the power of $\mathcal{G}$ for large values of q, p, and p' is at highest zero. This means that we can introduce a subtracted dis-

[5] The power of each term in the bilinear part of the unitarity condition is not necessarily given by the right-hand side of (3.8). This is not a contradiction to postulate II, but due to the over-estimation of the power of some terms in the bilinear part. For instance, for n odd and $m=0$ the Green's function $\mathcal{G}^{(\mathrm{odd},0)}$ must have an even power of p, and an odd power of p cannot appear because of the Lorentz invariance. Therefore if we write the right-hand side of (3.8) simply $\max A$, then for n odd and $m=0$ the right-hand side must be replaced by $\max 2[A/2]$, where $[A/2]=A/2$ for even A, $=(A-1)/2$ for odd A.

[6] This condition corresponds to Dyson's Eq. (55) of his paper on renormalization; F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

persion relation only for $\mathcal{G}_P$ and that $\mathcal{G}_A$ and $\mathcal{G}_T$ should satisfy unsubtracted dispersion relations. Hence the unitarity condition allows us to introduce only the pseudoscalar coupling in the perturbation theory.

Once we introduce one subtraction for a certain $\mathcal{G}$ function, then in the lowest order perturbation theory its power is zero, and this gives a restriction or a kind of boundary condition to solve the power Eq. (3.11). If this restriction can be incorporated into the power equation, then the theory may be called renormalizable provided that the two postulates are satisfied. In what follows, we shall study various combinations of subtractions (or interactions) case by case.

Pseudoscalar Meson Theory

In the pseudoscalar meson theory $\mathcal{G}^{(3,0)}$ is absent because of parity conservation and we shall assume subtracted dispersion relations for $\mathcal{G}^{(1,2)}$—or more precisely for $\mathcal{G}_P$ in (3.14)—and for $\mathcal{G}^{(4,0)}$. This implies that in the lowest order $\mathcal{G}^{(1,2)}$ and $\mathcal{G}^{(4,0)}$ are constants and hence

$$c(4,0)\geq 0, \quad c(1,2)\geq 0,$$
or
$$d(4,0)\geq 0, \quad d(1,2)\geq 0. \tag{3.16}$$

Then the power equation for $d(1,2)$ and $d(4,0)$ is given by

$$d(n,m)\geq \max[d(k+l,\, k'+l') + d(n-k+l,\, m-k'+l'),\, 0]. \tag{3.17}$$

All other d's satisfy Eq. (3.11). In these equations we naturally retain d's only for those values of n and m for which $\mathcal{G}^{(n,m)}\neq 0$.

Equation (3.17) already involves a kind of boundary condition (3.16) and determines the solution of the coupled power Eqs. (3.11) and (3.17) uniquely, i.e., the only solution is given by

$$d(n,m)=0, \quad \text{or} \quad c(n,m)=4-n-\tfrac{3}{2}m. \tag{3.18}$$

It is clear that even in higher orders the originally assumed subtractions are sufficient to make all the dispersion integrals converge. Thus this theory may be called renormalizable. One has to notice, however, that the above arguments do not guarantee the existence of the solution in the perturbation theory since the whole arguments are based on the assumed existence of the solution satisfying the two postulates.

The solution (3.18) also determines the powers of the absorptive parts of two-point Green's functions. They are given, respectively, by

$$c(2,0)=2, \quad c(0,2)=1. \tag{3.19}$$

These powers are consistent with the Kallén-Lehmann representation[7,8] as has been discussed in reference 2. This problem will be discussed in further detail in Sec. V.

Pure Meson Theory

For simplicity we shall discuss here a neutral scalar theory interacting with itself. Then the power equation is given by Eq. (3.4) or in terms of $d(n)=c(n)+n-4$ by

$$d(n)\geq \max[d(k+l)+d(n-k+l)]. \tag{3.20}$$

(a) One subtraction for $n=3$.

If we assume one subtraction for $\mathcal{G}^{(3,0)}$ and no subtraction for all others, the power equations are given by

$$d(n)\geq \max[d(k+l)+d(n-k+l),\, -1] \tag{3.21}$$

for $n=3$ since $c(3)=0$ implies $d(3)=-1$, and by Eq. (3.20) for all $n>3$. Then the only solution is given by

$$d(n)=2-n, \quad \text{or} \quad c(n)=6-2n. \tag{3.22}$$

This is renormalizable in our terminology.

(b) One subtraction for $n=4$.

In this case all $\mathcal{G}$'s with n odd vanish, and as one can easily check the only solution is given by

$$d(n)=0, \quad \text{or} \quad c(n)=4-n. \tag{3.23}$$

(c) One subtraction for $n=3$ and $n=4$.

The only solution in this case is given by

$$\begin{aligned} d(n)&=0 && \text{for even } n \\ &=-1 && \text{for odd } n. \end{aligned} \tag{3.24}$$

Scalar Meson Theory

Finally we shall discuss the scalar meson field interacting with the nucleon field. As the most general case we assume subtractions for $\mathcal{G}^{(1,2)}$, $\mathcal{G}^{(3,0)}$, and $\mathcal{G}^{(4,0)}$ corresponding to the interactions $\bar{\psi}\psi\varphi$, φ^3, and φ^4 in the Lagrangian theory. The power equations are given in this case by

$$d(n,m)\geq \max[d(k+l,\, k'+l') + d(n-k+l,\, m-k'+l'),\, d_0(n,m)], \tag{3.25}$$

where

$$\begin{aligned} d_0(n,m)&=\ \ 0 && \text{for } n=1,\, m=2 \\ &=\ \ 0 && \text{for } n=4,\, m=0, \\ &=-1 && \text{for } n=3,\, m=0 \\ &=-\infty && \text{for all others,} \end{aligned} \tag{3.26}$$

expressing the boundary conditions built in by means of the subtractions.

If one tries to solve the above power equation one finds that Eq. (3.25) possesses no solution at all, but this is not a serious difficulty because Eq. (3.25) is not the correct power equation. The correct equation can be given if one remembers the remark in reference 5, i.e., for $n=$odd, $m=0$ the right-hand side is overestimated and one has to modify the right-hand side following the prescription in reference 5. Then we find

[7] G. Källén, Helv. Phys. Acta. **25**, 417 (1952); **26**, 755 (1953).
[8] H. Lehmann, Nuovo cimento **11**, 342 (1954).

that the solution is given by

$$d(n,m)=-1, \quad \text{for } n \text{ odd, } m=0$$
$$= \ 0 \quad \text{otherwise.} \tag{3.27}$$

In all cases we have discussed, the solutions of the power equations suggest that the inequality signs in power equations can be replaced by equality signs. This means that in all examples that we discussed the absorptive part of a $\mathcal{G}$ function must have the same power as that of the whole $\mathcal{G}$ function if there could exist solutions at all satisfying the both postulates.[8a]

When a theory is not renormalizable in the sense of this paper it does not necessarily mean the absence of the solution. For instance, take the vector coupling of a neutral scalar field to the nucleon field

$$if\bar{\psi}\gamma_\mu\psi(\partial\varphi/\partial x_\mu); \tag{3.28}$$

then this theory is unrenormalizable in our terminology, but on the other hand the exact solution of this theory can be found by means of a canonical transformation.[9] The exact solution indicates that the power law is not obeyed in this example, nor is it obeyed the Källén-Lehmann representation for the two-point nucleon propagation function. In quantum electrodynamics a similar situation seems to be present as we shall see in the next section.

IV. RENORMALIZATION OF QUANTUM ELECTRODYNAMICS

In discussing the renormalization of quantum electrodynamics, the problem of gauge is a very serious object insofar as one tries to accommodate the power law that was fully utilized in Sec. III.

In previous papers[1,3] quantum electrodynamics was discussed in a special gauge which we shall refer to as Källén gauge.[7,10] This gauge was so chosen as to make the equal-time commutators between potentials vanish, i.e.,

$$[A_\mu(x),A_\nu(y)]=0 \quad \text{for} \quad x_0=y_0. \tag{4.1}$$

In this gauge a set of equations called W-T equations has been derived to characterize the gauge-invariant interactions:

$$\Box_x\frac{\partial}{\partial x_\mu}T[A_\mu(x)A_\nu(x')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]$$
$$=\left[\sum_a e_a\delta(x-x_a)+i\frac{\partial}{\partial x_\mu}\frac{\delta}{\delta A_\mu(x)}\right]$$
$$\times T[A_\nu(x')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]. \tag{4.2}$$

It is perhaps worth mentioning that Eq. (4.2) is valid only in the Källén gauge. As a special case of (4.2) we shall write down an equation which is important in the discussion of the renormalization of quantum electrodynamics, i.e.,

$$\frac{\partial}{\partial x_\lambda}\rho_{\lambda\mu\nu\sigma}(xyzw)=0, \tag{4.3}$$

where $\rho_{\lambda\mu\nu\sigma}$ is the generalization of the ρ function defined in Sec. II for the scalar case, i.e., it is given by

$$\rho_{\lambda\mu\nu\sigma}(xyzw)=(-i)^4\Box_x\Box_y\Box_z\Box_w$$
$$\times\langle 0|T[A_\lambda(x)A_\mu(y)A_\nu(z)A_\sigma(w)]|0\rangle_{\text{conn.}} \tag{4.4}$$

Another important characteristic of this gauge is that the contraction function is given by

$$(\delta_{\mu\nu}+2M\partial^2/\partial x_\mu\partial x_\nu)D^{(+)}(x-y), \tag{4.5}$$

where $D^{(+)}$ is the $\Delta^{(+)}$ function for zero mass, and M is a certain finite constant expressible in terms of the polarization operator.[10] It is just this contraction function that prevents us from applying the power law in the discussion of the renormalization of quantum electrodynamics. If we set up a set of power equations in this gauge, no solution can be found. This does not mean the absence of the solution in this gauge, but it just indicates the failure of the power law in this gauge. This is a situation similar to the one discussed at the end of the previous section.

In order to discuss the renormalization problem along the line presented in Sec. III, it is necessary to find the right gauge in which the power law is obeyed. The gauge transformation properties of Green's functions have been discussed by several authors,[11–13] and under gauge transformations Green's functions are multiplied by exponential functions of some singular gauge functions. This clearly suggests that if the power law is obeyed in a special gauge, this property will be lost after a gauge transformation. So what one has to try is to look for a suitable gauge transformation which carries the Källén gauge into the special gauge in which the power law is obeyed.

The appropriate gauge transformation which meets our requirement was given by Rollnik *et al.*[14] If we denote the renormalized field operators in the Källén gauge by $\tilde{A}$ and $\tilde{\psi}$, and the operators in the special gauge by A and ψ, they are connected by the gauge transformation

$$\tilde{A}_\mu=A_\mu+M\partial^2A_\nu/\partial x_\mu\partial x_\nu,$$
$$\tilde{\psi}=\exp(ieM\partial A_\nu/\partial x_\nu)\psi. \tag{4.6}$$

We shall call this special gauge the Dyson gauge.[15] In this gauge the equal-time commutators between po-

[8a] See note added in proof.

[9] S. Okubo, Progr. Theoret. Phys. (Kyoto) **11**, 80 (1954).

[10] G. Källén, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1958), Vol. 5.

[11] S. Okubo, Nuovo cimento **15**, 949 (1960).

[12] B. Zumino, J. Math. Phys. **1**, 1 (1960).

[13] I. Bialynicki-Birula, Nuovo cimento **17**, 951 (1960).

[14] H. Rollnik, B. Stech, and E. Nunnemann, Z. Physik **159**, 482 (1960).

[15] This is exactly the gauge used by Dyson in his proof of the renormalizability of quantum electrodynamics, reference 6.

tentials do not vanish, but this is not important in the present discussion. Furthermore, the form of Eq. (4.2) is modified in general. Nevertheless one can show that the Eq. (4.3) is still valid in the Dyson gauge.[16] But among other things it is most important that the contraction function is given by

$$\delta_{\mu\nu}D^{(+)}(x-y). \tag{4.7}$$

The absence of the derivative term in the contraction function enables us to use the power law again.

If we define τ function by (3.5) with $\varphi(x)$ and K_x replaced by $A_\mu(x)$ and $\Box_x$, we can assume the power law for the $\mathcal{G}$ function which is the Fourier transform of the connected part of the τ function (i.e., the ρ function):

$$\mathcal{G}^{(n,m)}\sim p^{c(n,m)}, \tag{4.8}$$

where n is the number of potentials entering into τ and m the number of spinor operators. First from the charge conjugation invariance it follows that

$$\mathcal{G}^{(n,m)}=0, \quad \text{for } n \text{ odd}, m=0. \tag{4.9}$$

From the construction of the power equations it is clear that the structure of the power equations and consequently the solution thereof for quantum electrodynamics are identical with those for the pseudoscalar meson theory.[17] Thus quantum electrodynamics is renormalizable.

Having established the renormalizability of quantum electrodynamics, we shall examine which $\mathcal{G}$ functions need subtractions. From the solution of the pseudoscalar meson theory we can pick up $\mathcal{G}$ functions which might need subtractions; namely, the nonnegative c's are given by

$$c(1,2)=c(4,0)=0. \tag{4.10}$$

We first study the $\mathcal{G}$ function corresponding to the case $n=1$, $m=2$, i.e., the Fourier transform of the expression

$$\mathcal{G}_\mu(xyz)=(-i)\Box_x D_y\tilde{D}_z\langle 0|T[A_\mu(x)\psi(y)\bar{\psi}(z)]|0\rangle. \tag{4.11}$$

This $\mathcal{G}$ function which is denoted by $\mathcal{G}_\mu(k,p,p')$ is decomposed into a sum of invariants,

$$\mathcal{G}_\mu(k,p,p')=-i\gamma_\mu\mathcal{G}_D(k,p,p')+\sigma_{\mu\nu}k_\nu\mathcal{G}_P(k,p,p'), \tag{4.12}$$

with $k+p+p'=0$. Equation (4.10) tells us that the asymptotic behavior of $\mathcal{G}_\mu$ is given by

$$\mathcal{G}_\mu\sim p^0,$$

and consequently

$$\mathcal{G}_D\sim p^0, \quad \mathcal{G}_P\sim p^{-1}. \tag{4.13}$$

Therefore only $\mathcal{G}_D$ needs a subtraction. As discussed in references 1 and 3, a subtraction for $\mathcal{G}_D$ introduces the Dirac interaction $-ie\bar{\psi}\gamma_\mu\psi\cdot A_\mu$ and that for $\mathcal{G}_P$ the Pauli interaction $i\lambda\bar{\psi}\sigma_{\mu\nu}\psi\cdot F_{\mu\nu}$, and the renormalizability condition excludes the latter.

Next we examine the case $n=4$, $m=0$. The Fourier transform of the ρ function (4.4) will be denoted by

$$\mathcal{G}_{\lambda\mu\nu\sigma}(k,k',k'',k'''). \tag{4.14}$$

(1) The asymptotic behavior for large values of k's is supposed to be expressed by

$$\mathcal{G}\sim k^0. \tag{4.15}$$

(2) This $\mathcal{G}$ is completely symmetric in the variables $(k\lambda)$, $(k'\mu)$, $(k''\nu)$, and $(k'''\sigma)$. This symmetry property poses the so-called crossing relations.
(3) Equations (4.3), which is valid in both Källén and Dyson gauges, implies

$$k_\lambda\mathcal{G}_{\lambda\mu\nu\sigma}(k,k',k'',k''')=0. \tag{4.16}$$

In order to study this problem further, one has to decompose this $\mathcal{G}$ function into a sum of invariants:

$$\mathcal{G}_{\lambda\mu\nu\sigma}=\sum_S a_{\lambda\mu\nu\sigma}{}^{(S)}\mathcal{G}_S, \tag{4.17}$$

where a denotes an invariant, and the coefficient $\mathcal{G}_S$ is a function of the scalar products of k's alone. Typical examples of a's are given by

$$\delta_{\lambda\mu}\delta_{\nu\sigma}, \quad \delta_{\lambda\mu}k_\nu k_\sigma, \quad k_\lambda k_\mu k_\nu k_\sigma, \quad \text{etc.} \tag{4.18}$$

Since we can substitute either one of k, k', k'', and k''' for the k's in (4.18), there are indeed many independent invariants, and we shall not try to exhaust them. Let us denote the coefficients of the invariants (4.18) respectively by $\mathcal{G}^{(0)}$, $\mathcal{G}^{(2)}$, and $\mathcal{G}^{(4)}$, or more precisely there are many coefficients that are indiscriminately denoted by the same notations $\mathcal{G}^{(0)}$, $\mathcal{G}^{(2)}$, and $\mathcal{G}^{(4)}$. Then it is clear that the asymptotic behaviors of these coefficients are given by

$$\mathcal{G}^{(0)}\sim k^0, \quad \mathcal{G}^{(2)}\sim k^{-2}, \quad \mathcal{G}^{(4)}\sim k^{-4}, \tag{4.19}$$

and only the first one, $\mathcal{G}^{(0)}$, needs to be studied. Now if we use Eq. (4.16), one easily finds that $\mathcal{G}^{(0)}$ can be expressed in terms of $\mathcal{G}^{(2)}$, and, very roughly speaking, $\mathcal{G}^{(0)}$ is obtained from $\mathcal{G}^{(2)}$ by multiplying scalar products of k's by $\mathcal{G}^{(2)}$. This shows that $\mathcal{G}^{(0)}=0$ if all the scalar products of k's are equal to zero, or[18]

$$\mathcal{G}^{(0)}(k_\alpha k_\beta\cdot\xi)=0 \quad \text{if} \quad \xi=0. \tag{4.20}$$

Thus, integrating the subtracted dispersion relation,

$$\frac{\partial}{\partial\xi}\operatorname{Re}\mathcal{G}^{(0)}(k_\alpha k_\beta\cdot\xi)$$

$$=\frac{\mathrm{P}}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{(\xi'-\xi)^2}\operatorname{Im}\mathcal{G}^{(0)}(k_\alpha k_\beta\cdot\xi'), \tag{4.21}$$

[16] M. Muraskin (unpublished). This can be shown by using the technique developed in reference 13.

[17] In this connection it is important to notice the absence of $\mathcal{G}^{(3,0)}$ for parity conservation in the pseudoscalar meson theory and for charge conjugation invariance in electrodynamics.

[18] In what follows, $k_\alpha k_\beta$ denotes a scalar product of any two of k, k', k'', and k'''.

with the boundary condition (4.20), we get

$$\mathrm{Re}\mathcal{G}^{(0)}(k_\alpha k_\beta\cdot\xi)=\frac{\xi}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'(\xi'-\xi)}\mathrm{Im}\mathcal{G}^{(0)}(k_\alpha k_\beta\cdot\xi'). \quad (4.22)$$

This shows that although one needs a subtraction for $\mathcal{G}^{(0)}$'s one cannot introduce any arbitrary constant, and that the quadrilinear interaction $A_\mu^2\cdot A_\nu^2$ is not present in the corresponding Lagrangian theory.[19]

V. SUBTRACTIONS AND RENORMALIZATION

In the foregoing sections it was shown that both spinless meson theory and quantum electrodynamics are renormalizable, but in order to understand the connection between the present dispersion approach and the conventional renormalization procedure it is necessary to investigate the relation between the subtraction constants in the dispersion theory and the renormalization constants in the conventional theory. For this purpose we shall take a simple model defined by the Lagrangian

$$L=-\left(\frac{\partial\Phi^*}{\partial x_\lambda}\frac{\partial\Phi}{\partial x_\lambda}+M_0^2\Phi^*\Phi\right)-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_\lambda}\frac{\partial\varphi}{\partial x_\lambda}+m_0^2\varphi^2\right)-g_0\Phi^*\Phi\varphi. \quad (5.1)$$

Then the unrenormalized field equations are given by

$$(\Box-M_0^2)\Phi=g_0\Phi\varphi,\quad (\Box-M_0^2)\Phi^*=g_0\Phi^*\varphi,$$

and

$$(\Box-m_0^2)\varphi=g_0{:}\Phi^*\Phi{:}. \quad (5.2)$$

The field operator and mass renormalizations are given by

$$\Phi_{\mathrm{ren}}=Z_2^{-\frac{1}{2}}\Phi_{\mathrm{unren}},\quad \varphi_{\mathrm{ren}}=Z_3^{-\frac{1}{2}}\varphi_{\mathrm{unren}}, \quad (5.3)$$

and

$$M^2=M_0^2+\delta M^2,\quad m^2=m_0^2+\delta m^2.$$

From now on we use only renormalized field operators. The two-point $\mathcal{G}$ functions are defined by

$$(-i)^2K_x{}^MK_y{}^M\langle 0|T[\Phi(x)\Phi^*(y)]|0\rangle=\frac{-i}{(2\pi)^4}\int d^4p\,e^{ip(x-y)}\mathcal{G}_2(p^2), \quad (5.4)$$

$$(-i)^2K_x{}^mK_y{}^m\langle 0|T[\varphi(x)\varphi(y)]|0\rangle=\frac{-i}{(2\pi)^4}\int d^4p\,e^{ip(x-y)}\mathcal{G}_3(p^2).$$

Then the $\mathcal{G}$'s are given in terms of the Lehmann weight function[8] σ by

$$\mathcal{G}_2(p^2)=-\left[p^2+M^2+(p^2+M^2)^2\int\frac{\sigma_2(\kappa^2)d\kappa^2}{p^2+\kappa^2-i\epsilon}\right],$$
$$\mathcal{G}_3(p^2)=-\left[p^2+m^2+(p^2+m^2)^2\int\frac{\sigma_3(\kappa^2)d\kappa^2}{p^2+\kappa^2-i\epsilon}\right]. \quad (5.5)$$

These two functions satisfy dispersion relations with two subtractions, and in order to calculate the subtraction constants we shall first refer to Lehmann's relations[8]:

$$Z_2^{-1}=1+\int\sigma_2(\kappa^2)d\kappa^2,\quad Z_3^{-1}=1+\int\sigma_3(\kappa^2)d\kappa^2, \quad (5.6)$$

and

$$\delta M^2=-Z_2\int(\kappa^2-M^2)\sigma_2(\kappa^2)d\kappa^2<0,$$
$$\delta m^2=-Z_3\int(\kappa^2-m^2)\sigma_3(\kappa^2)d\kappa^2<0. \quad (5.7)$$

The absorptive part of the $\mathcal{G}_2$ function is given by

$$\mathrm{Im}\mathcal{G}_2(p^2)=-\pi(p^2+M^2)^2\sigma_2(-p^2), \quad (5.8)$$

where

$$\sigma_2(-p^2)=0\quad\text{if}\quad -p^2<0. \quad (5.9)$$

Let us first apply the unsubtracted dispersion relation to $\mathrm{Im}\mathcal{G}_2$; then with the help of Eq. (5.9) one finds

$$\text{“}\mathrm{Re}\mathcal{G}_2(p^2)\text{”}=\frac{1}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'-1}\mathrm{Im}\mathcal{G}_2(p^2\xi')=-\int_0^\infty\frac{(x-M^2)^2}{x+p^2}\sigma_2(x)dx. \quad (5.10)$$

On the other hand, the subtracted dispersion relation or Eq. (5.5) gives

$$\mathrm{Re}\mathcal{G}_2(p^2)=-\left[p^2+M^2+(p^2+M^2)^2\int_0^\infty\frac{\sigma_2(x)dx}{p^2+x}\right]. \quad (5.11)$$

If we take the difference between (5.10) and (5.11) we can learn what is subtracted in the dispersion approach.

$$\text{“}\mathrm{Re}\mathcal{G}_2(p^2)\text{”}-\mathrm{Re}\mathcal{G}_2(p^2)=(p^2+M^2)\left[1+\int\sigma_2(x)dx\right]+\int(M^2-x)\sigma_2(x)dx$$
$$=(p^2+M^2)Z_2^{-1}+\delta M^2\,Z_2^{-1}. \quad (5.12)$$

Hence it can be concluded that the two subtractions for the two-point $\mathcal{G}$ functions are nothing but the mass and Z renormalizations. Since Z_2^{-1} never vanishes,

[19] This statement can be verified by using the argument presented in Sec. 4 of reference 2 and Eq. (4.22); i.e., the vanishing of $\mathrm{Im}\,\mathcal{G}^{(0)}$ implies the vanishing of $\mathrm{Re}\,\mathcal{G}^{(0)}$.

we always need two subtractions in the parametric dispersion relation.

Next we study the three-point $\mathcal{G}$ function. From the result of the previous section we take it for granted that $\mathcal{G}$ obeys the power law

$$\mathcal{G}(p_\alpha p_\beta)\sim p^0. \tag{5.13}$$

From the unrenormalized field Eq. (5.2), the equation for the *unrenormalized* vertex function is given by

$$\begin{aligned}(\Box_x-m^2+\delta m^2)\langle 0|T[\varphi(x)\Phi(y)\Phi^*(z)]|0\rangle\\ =g_0\langle 0|T[\Phi(y)\Phi^*(x)]|0\rangle\langle 0|T[\Phi(x)\Phi^*(z)]|0\rangle\\ +g_0\langle 0|T[:\Phi^*(x)\Phi(x):,\Phi(y),\Phi^*(z)]|0\rangle_{\text{conn}}.\end{aligned} \tag{5.14}$$

In the momentum representation this equation can be written in terms of the *renormalized* $\mathcal{G}$ functions as

$$\begin{aligned}&\left(1-\frac{\delta m^2}{k^2+m^2-i\epsilon}\right)\mathcal{G}(k,p,p')\\ &=g_0Z_2Z_3^{-\frac{1}{2}}\left(1+(p^2+M^2)\int\frac{\sigma_2(x)dx}{p^2+x-i\epsilon}\right)\\ &\times\left(1+(p'^2+M^2)\int\frac{\sigma_2(x)dx}{p^2+x-i\epsilon}\right)+\frac{ig_0Z_2Z_3^{-\frac{1}{2}}}{(2\pi)^4}\\ &\times\int d^4q\frac{\mathcal{G}(q,\,k-q,\,p,\,p')}{[q^2+M^2-i\epsilon][(k-q)^2+M^2-i\epsilon]}.\end{aligned} \tag{5.15}$$

Now put $p^2=-M^2\xi$, $p'^2=-M^2\xi$, $k^2=-m^2\xi$ and take the limit $\xi\rightarrow\infty$. Since the renormalization constants are treated as if they were finite, it may be reasonable to assume

$$\lim_{\xi\to\infty}\frac{\delta m^2}{-m^2\xi+m^2}=0. \tag{5.16}$$

The last term on the right-hand side of Eq. (5.15) is convergent except for the multiplicative constant factor, and with reference to the result of the last section this term is supposed to decrease as ξ^{-1} for large values of ξ. Now define $\mathcal{G}(\xi)$ by

$$\mathcal{G}(\xi)=\mathcal{G}(k^2=-m^2\xi,\,p^2=-M^2\xi,\,p'^2=-M^2\xi), \tag{5.17}$$

and take the limit $\xi\rightarrow\infty$; then we get

$$\mathcal{G}(\infty)=g_0Z_2Z_3^{-\frac{1}{2}}Z_2^{-2}. \tag{5.18}$$

The last factor Z_2^{-2} results from the limiting value

$$\lim_{p^2\to\infty}\left(1+(p^2+M^2)\int\frac{\sigma_2(x)dx}{p^2+x-i\epsilon}\right)=Z_2^{-1}. \tag{5.19}$$

Hence our result is given by

$$\mathcal{G}(\infty)=g_0Z_2^{-1}Z_3^{-\frac{1}{2}}>g_0. \tag{5.20}$$

We can also express this result in terms of the renormalized coupling constant g, i.e.,

$$\mathcal{G}(1)=g=g_0Z_1^{-1}Z_2Z_3^{\frac{1}{2}}, \tag{5.21}$$

and one gets

$$\mathcal{G}(\infty)=gZ_1Z_2^{-2}Z_3^{-1}. \tag{5.22}$$

Z_1 is the renormalization constant for the amputated vertex[20] and may be defined by

$$\mathcal{G}(\infty)/\mathcal{G}(1)=Z_1Z_2^{-2}Z_3^{-1}. \tag{5.23}$$

The subtracted dispersion relation yields[21]

$$\mathrm{Re}\mathcal{G}(\xi)=g+\frac{\xi-1}{\pi}\int_0^\infty\frac{\mathrm{Im}\mathcal{G}(\xi')d\xi'}{(\xi'-\xi)(\xi'-1)}, \tag{5.24}$$

and in particular for $\xi=\infty$ we get

$$\mathcal{G}(\infty)=g+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\mathcal{G}(\xi')d\xi'}{1-\xi'}, \tag{5.25}$$

or

$$\mathrm{Re}\mathcal{G}(\xi)=\mathcal{G}(\infty)+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\mathcal{G}(\xi')}{\xi'-\xi}d\xi', \tag{5.26}$$

and the subtraction constant $\mathcal{G}(\infty)$ is given by (5.22). Generalizing Eq. (5.26) we find

$$\begin{aligned}\mathrm{Re}\mathcal{G}(p_\alpha p_\beta)=\mathcal{G}(\infty)+\frac{\mathrm{P}}{\pi}\int_0^\infty\frac{\mathrm{Im}\mathcal{G}(\xi')d\xi'}{\xi'}\\ +\frac{\mathrm{P}}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'(\xi'-1)}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi').\end{aligned} \tag{5.27}$$

This shows that also in the case of three-point $\mathcal{G}$ functions the subtraction constant is a kind of renormalization constant. In the present model the subtraction constant $gZ_1Z_2^{-2}Z_3^{-1}$ is finite in all orders of the perturbation theory.

Note added in proof. In order to show that the perturbation theory really reproduces the right powers of the Green's functions deduced in Sec. III one has to prove that the absorptive part of a $\mathcal{G}$ function has the same power as that of the whole $\mathcal{G}$ function. For this purpose it is useful to use the dispersion relation

$$\mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{P}{\xi^n}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'-\xi}\xi'^n\,\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'),$$

where n is a certain positive integer. This equation is true in all orders of perturbation theory, provided that the dispersion integral converges for the given positive integer n. If one takes this modified dispersion relation for granted, one can regard the subtraction condition discussed in the text as the necessary and sufficient condition for the renormalizability of a theory.

[20] If $\Gamma(\xi)$ denotes the amputated vertex defined similarly to (5.17), Z_1 is given by $\Gamma(\infty)/\Gamma(1)=Z_1$. Notice here that both $\mathcal{G}(\xi)$ and $\Gamma(\xi)$ are real for $\xi=1$ and $\xi=\infty$.

[21] Notice that $\mathrm{Im}\,\mathcal{G}(\xi)=0$, for $\xi<0$.