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"Gauge-Invariant" Variables in General Relativity*

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Einstein's field equations for the gravitational field possess solutions having a large variety of topological properties; among them there are solutions whose curvature goes asymptotically to zero at spatial infinity. If we restrict ourselves to solutions that are asymptotically Minkowskian, then it is tempting to try to divide the effects of curvilinear coordinate transformations into those that correspond to a Lorentz transformation and those that represent "gauge-type" effects. In fact a number of authors have followed a variety of approaches toward a reformulation of general relativity that would make the theory resemble, to some extent, a conventional Lorentz-covariant field theory. In this paper we analyze the group-theoretical aspects of such schemes. Making a definite assumption concerning the group of curvilinear transformations that will preserve the asymptotic Minkowski character of the metric field, we come to the conclusion that the reduction to a Lorentz-covariant theory is in fact impossible. The course of the analysis suggests, however, that this negative result depends on the initial group of transformations adopted; it is conceivable that a slightly different invariance group would be compatible with a special-relativistic formulation of the theory.

I. INTRODUCTION

THE physical justification for the covariance of Einstein's theory of gravitation is represented by the principle of equivalence, according to which the accelerations caused by inertial and by gravitational fields at one world point are experimentally indistinguishable. The principle of equivalence is local; as soon as we examine the inhomogeneity of the acceleration fields, the presence of a gravitational field may be established. Accordingly, the principle of equivalence gives us no guidance for setting global requirements for the metric field. In particular, there is no contradiction between the principle of equivalence and the conjecture that gravitational fields should approach zero strength at sufficient distance from their sources.

In cosmological investigations one naturally considers primarily metric fields, which, far from dropping off at large distances, are homogeneous and isotropic. But if we are concerned with the possible applications of general relativity to localized physical systems, we are inclined to view the gravitational field as similar to the electromagnetic field. The nonlinear aspects of the gravitational field appear relatively unimportant in view of the fact that the gravitational potentials associated with an elementary particle (such as the electron) are of the order of 10^{-20} (in dimensionless units) at distances of the order of the classical electron radius. Hence at atomic distances, at any rate, and beyond, the gravitational potentials cannot deviate widely from the solutions of the "linearized" field equations, i.e., from the wave equations for massless bosons of spin 2. Whether quantization of the theory will preserve the meaning of such order-of-magnitude arguments remains, of course, to be seen. In the meantime, it is eminently reasonable to render the formulation of the gravitational dynamics as intuitive as possible, in terms of field theories with which we are familiar.

In the linearized theory, the coordinate transformations are replaced in their role as the invariance group of the theory by a group that consists of the inhomogeneous Lorentz transformations supplemented by transformations of the form

$$\gamma_{\mu\nu}{}'=\gamma_{\mu\nu}+\xi_{\mu,\nu}+\xi_{\nu,\mu}-\eta_{\mu\nu}\xi^{\rho}{}_{,\rho}. \tag{1}$$

In this expression the four variables ξ_μ are arbitrary functions of the four coordinates. Replacing a solution of the field equations $\gamma_{\mu\nu}$ by a field $\gamma_{\mu\nu}{}'$ in accordance with Eq. (1) leads to another solution, which is considered physically equivalent to the original solution. In fact the transitions (1) resemble in many respects the gauge transformations of electrodynamics; in accordance with well-established usage we shall call them "gauge-like" transformations. It is well known that there are no "gauge-invariant" first-order derivatives of the potentials $\gamma_{\mu\nu}$; the lowest "gauge-invariant" differential forms are of the second differential order,

$$\begin{aligned} \mathrm{P}_{\iota\kappa\lambda\mu}&=\tfrac{1}{2}(h_{\iota\lambda,\kappa\mu}+h_{\kappa\mu,\iota\lambda}-h_{\kappa\lambda,\iota\mu}-h_{\iota\mu,\kappa\lambda}),\\ h_{\iota\kappa}&=\tfrac{1}{2}\eta_{\iota\kappa}\gamma^{\rho}{}_{\rho}-\gamma_{\iota\kappa}, \end{aligned} \tag{2}$$

and linear combinations of the $\mathrm{P}_{\iota\kappa\lambda\mu}$. Hence statements about solutions that involve only the expressions (2), including their derivatives, are "gauge-invariant". With respect to Lorentz transformations, the fields $\gamma_{\mu\nu}$, $h_{\mu\nu}$, and $\mathrm{P}_{\iota\kappa\lambda\mu}$ all transform as components of a tensor field.

It is tempting to reformulate the general theory of relativity along the lines suggested by its linearized modification. It is not surprising that such a program will not succeed if the theory is deemed to consist wholly of the field equations, without additional boundary conditions, for in that case the gravitational fields to be considered are not bound to be "weak" in

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any sense, and the linearized theory provides no guidance to intuition. But it has been widely conjectured that with appropriate boundary conditions imposed on the solutions the full theory could be cast into a form in which the totality of the remaining coordinate transformations could be separated into inhomogeneous Lorentz transformations and gauge-like transformations. This point of view has been presented most forcefully by Arnowitt *et al.*[1] and by Fock,[2] but it is also implicit in some of the work by Dirac.[3] In this paper we shall examine whether such a classification of coordinate transformations is in fact possible. The answer to this question probably depends on the precise boundary conditions assumed to be preserved at spatial infinity; in this paper we shall investigate one set of conditions. We shall comment in the concluding section on the import of the assumptions made.

II. OBSERVABLES AND GAUGE-INVARIANT VARIABLES

One method of casting the general theory of relativity into a form in which all statements assume a significance independent of the choice of coordinates is to reformulate the theory in terms of *observables*. By definition, observables are quantities that are individually independent of the choice of coordinate system but whose values depend on properties of a particular solution of the field equations of general relativity.[4] They may be constructed by means of intrinsic coordinates, that is to say, a system of coordinates defined uniquely in terms of geometric properties of a Riemannian manifold that satisfies the gravitational field equations. If the definition of intrinsic coordinates is based on purely local properties,[5] then they do not depend on asymptotic boundary conditions. On the other hand, intrinsic coordinates, locally defined or otherwise, do not exist in manifolds that admit motions, that is to say manifolds that contain Killing vector fields. Kerr has shown that in the absence of Killing fields there always exist locally defined scalar fields that may be used as a complete set of intrinsic coordinates.[6]

Intrinsic coordinates are uniquely determined by the conditions imposed on them; in other words, such conditions will not be compatible with any further coordinate transformations. Thus intrinsic coordinates are naturally associated with the construction of observables, by definition quantities that will not change their values under any coordinate transformations. By contrast, we are now searching for quantities that are "gauge-invariant" but not Lorentz-invariant. If they are to be obtained also by means of restrictions on the choice of coordinates, then these coordinate conditions must be constructed so that they admit precisely the Lorentz transformation, or a group of transformations isomorphic with the (inhomogeneous) Lorentz group; this is the point of view advanced by Fock.[2]

Whether quantities of this type are obtained by means of coordinate conditions or by a more general procedure is, in a sense, a question of technology. Let us first formalize the method of coordinate conditions. We might start with a metric $g(x)$ that merely satisfies the asymptotic conditions; we assume now the existence of a certain procedure, or algorithm, (such as the integration of a set of differential equations with sufficient initial and boundary conditions) that leads from this field to a transformed field $f(x)$ which satisfies the coordinate conditions. We shall denote this algorithm by A, so that $f(x)=Ag(x)$. A symbolizes both the procedure that furnishes us with the desired coordinate transformation, and with the performance of that transformation.

Suppose now that we had first gone over from $g(x)$, by means of a coordinate transformation C, to a new field $\bar{g}(x)$, so that $\bar{g}$ is also a field that satisfies the asymptotic flatness requirements. Then application of the procedure A on $\bar{g}(x)$ should have led us to a field $\bar{f}(x)$ that satisfies the coordinate conditions as well. By assumption, f and $\bar{f}$ are now related to each other by an (inhomogeneous) Lorentz transformation L, so that we may write

$$\begin{aligned}\bar{f}(x)&=ACg(x)=LAg(x),\\ AC&=LA.\end{aligned}\tag{3}$$

The last symbolic equation characterizes the reduction of the group of general coordinate transformations that are consistent with asymptotic flatness to the inhomogeneous Lorentz group by means of coordinate conditions. This reduction preserves the group properties, so that we have actually generated a homomorphism between C and L. In the nature of things, the reduction procedure A is not reversible, that is to say, there is no procedure A^{-1}.

Having gone through this formalization we discover in retrospect that we could also have interpreted A in a more general sense than the one in which we introduced it here. Suppose that A simply represents an algorithm by which we assign to the metric field $g(x)$ a new set of functions, or functionals, of $g(x)$, which are not necessarily coordinate transforms of $g(x)$ and that these functionals $f(x)$ remain fixed functions of the coordinates under coordinate transformations that go over into the identity transformation at infinity ("gauge-type" transformations). Then the reduction relationship (3) still applies, and our previous inter-

[1] R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **121**, 1556 (1961), where references to earlier papers will be found.
[2] V. Fock, *The Theory of Space, Time and Gravitation,* translated by N. Kemmer (Pergamon Press, New York and London, 1959).
[3] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).
[4] P. G. Bergmann, S. Helv. Phys. Acta **4**, 79 (1956).
[5] A. B. Komar, Phys. Rev. **111**, 1182 (1958).
[6] R. P. Kerr (to be published).

pretation of A and $f(x)$ represented merely a special type of transition to "gauge-invariant" variables.

With this more general formulation we shall now ask whether "gauge-invariant" variables do in fact exist. This question is related to the structure of the group of coordinate transformations C, namely the question whether there exists a homomorphism between C and the inhomogeneous Lorentz group L. Of course, if it does, then the members of C that in L are to be represented by the identity transformation form an invariant subgroup of C, and L is the corresponding factor group. Hence we must search for the invariant subgroup of "gauge-type" transformations which leads to the desired homomorphism.

We assume that we have constructed a field of gauge-invariant variables, F. If we denote the group of coordinate transformations that preserve our asymptotic boundary conditions by the symbol C, then the transformed field which will be obtained under a transformation c (which belongs to the group C) is obtained from F by means of a prescribed procedure, whose result we shall denote by the symbol cF. Again there is no presumption that the procedure c is linear. However, it is clear that if the product of two transformations c_1 and c_2 is c_3, $c_3=c_2c_1$, then we must also have

$$c_3F=c_2c_1F. \tag{4}$$

Any transformation law is a realization (or representation) of the group of coordinate transformations C.

Let us now consider a gauge-type transformation g. By assumption we have $gF=F$, regardless of the initial choice of a particular field F. Clearly, the gauge-type transformations form a subgroup of C, which we shall identify by the symbol G. We shall now form the similarity transform of a gauge-type transformation g, under any transformation cgc^{-1}. We have:

$$\begin{aligned} g'F&=cgc^{-1}F=cc^{-1}F=F,\\ g'&=cgc^{-1}. \end{aligned} \tag{5}$$

Clearly, the transformation g' is also a gauge-type transformation, it belongs to the subgroup G, so that, in terms of group symbolism, we have

$$cGc^{-1}=G. \tag{6}$$

This equality is the definition of an invariant subgroup, a group that is mapped on itself by all similarity transformations in C.

By assumption, the factor group

$$L=C/G \tag{7}$$

is the inhomogeneous Lorentz transformation. Our task is, thus, given the group C, to find an invariant subgroup G such that the factor group is L. If no such invariant subgroup exists, then there are no gauge-invariant variables within the meaning of our definition.

Before we turn to the actual investigation of the structure of the chosen group C, we shall make one additional remark. We shall find in the next section an invariant subgroup of C whose factor group is the homogeneous Lorentz group. Fields that transform in accordance with a law that is wholly determined by the matrix of coefficients of the homogeneous Lorentz transformations are constants of the motion. Such fields cannot reveal the dynamics of the theory in the sense in which this term is usually, and intuitively, understood.

III. GROUP C AND ONE INVARIANT SUBGROUP

Given a "permissible" coordinate system, we shall introduce the following new coordinates σ, u, θ, and ϕ:

$$\begin{aligned} &x^0=\sigma\sinh u, && \sigma^2=(x^1)^2+(x^2)^2+(x^3)^2-(x^0)^2\\ &x^1=\sigma\cosh u\sin\theta\cos\phi, && \quad =r^2-t^2,\\ &x^2=\sigma\cosh u\sin\theta\sin\phi, && u=\tfrac{1}{2}\ln[(r+t)/(r-t)],\\ &x^3=\sigma\cosh u\cos\theta, && 0\leq\sigma\leq\infty,\\ & && -\infty\leq u\leq\infty. \end{aligned} \tag{8}$$

These new, quasi-spherical, coordinates span a portion of the Riemannian manifold which for sufficiently large values of σ represents the exterior of the light cone. We shall call any curve for which the variables u, θ, ϕ approach constant limiting values for infinite positive σ a space-like direction.

We shall now require that in any coordinate transformation belonging to C the new coordinates x' possess derivatives with respect to the original coordinates x which asymptotically satisfy the conditions

$$\partial x^{\rho\prime}/\partial x^\mu=\gamma^\rho{}_\mu+O(1/\sigma) \tag{9}$$

in any space-like direction. The $\gamma^\rho{}_\mu$ are to be sixteen constants forming a Lorentz matrix. Integrating the expression (9) we can verify that in a space-like direction the limit of (σ'/σ) goes to unity; hence the transformations (9) form a group, and this group will be our group C.

Parenthetically, it might be remarked that this definition of C makes no explicit reference to the metric field, or to its transformation properties. Although it might be possible to define a transformation group in such a manner that reference to the fields to be transformed is not completely eliminated, such references tend to complicate the proof of group property. For instance, the transformations that lead from one isothermal (harmonic) coordinate system to another isothermal coordinate system do not form a group. This is because the components of the field enter into the defining differential equation for isothermal coordinates explicitly. After one such transformation the components of the metric field have changed; hence a particular transformation that would

have been isothermal to begin with will no longer have this property if preceded by another isothermal transformation.

We return now to our group C and define a subgroup J as the set of all those transformations satisfying the requirement (9) for which $\gamma_\mu{}^\rho=\delta_\mu{}^\rho$. It is easy to show that J is an invariant subgroup, and that the factor group, $C/J=L_h$, is the homogeneous Lorentz group.

Variables that are invariant under J but not under C will have transformation laws into which only the sixteen numerics $\gamma_\mu{}^\rho$ enter. They might, for instance, be free-vectors or free-tensors.

IV. SECOND SUBGROUP OF C

The subgroup J includes the translations, and that is the reason that the factor group is confined to the homogeneous Lorentz transformations. Any invariant subgroup whose factor group contains the translations as well must be smaller than J, and that is why we shall look for a group that is an invariant subgroup of J, and hence of G as well. We shall define the subgroup I as the group of all those coordinate transformations which, in addition to obeying Eq. (9), satisfy the requirement that in any space-like direction

$$x^{\rho\prime}=x^\rho+O(1/\sigma). \tag{10}$$

This new subgroup consists of all those members of C which asymptotically go over into the identity transformation. Its factor group, $C_a=C/I$, represents a classification of all transformations of C into equivalence classes characterized by asymptotic behavior. The subscript a is to call attention to the importance of the asymptotic behavior.

The inhomogeneous Lorentz transformations, or rather the transformations that are asymptotically Lorentzian, are contained within C_a. It remains to be seen whether we can construct an invariant subgroup of C_a such that its factor group is the inhomogeneous Lorentz transformation. In view of the fact that the homogeneous Lorentz transformations form a factor group of the inhomogeneous Lorentz group (the translations being the invariant subgroup), we can reduce our task by excluding from C_a first the homogeneous Lorentz transformations. We do this by forming the factor group $J_a=J/I$, which obeys the equation

$$C_a/J_a=C/J=L_h. \tag{11}$$

J_a represents a classification of all transformations J (which are those transformations in which new and old coordinate axes are asymptotically parallel) in terms of their asymptotic properties.

The group J_a is commutative. Hence every subgroup is an invariant subgroup, and the whole group may be considered the direct product of any subgroup and the corresponding factor group. Because of this property of J_a, we shall attempt to establish the translations as a factor group in J_a. If this attempt is successful, then the next step would be to see whether the factor group J_a/T (T standing for the translations) is an invariant subgroup of C_a.

Let us consider the possible asymptotic behavior of the difference between old and new coordinates in transformations that belong to J. If $\bar{f}$ is such a difference, we can easily verify that in order to satisfy this requirement the derivatives with respect to our quasi-spherical coordinates (8) must obey the requirements

$$\partial\bar{f}/\partial\sigma=O(1/\sigma);\quad \partial\bar{f}/\partial u,\ \partial\bar{f}/\partial\theta,\ \partial\bar{f}/\partial\phi=O(\sigma^0). \tag{12}$$

It follows that the difference between the values of $\bar{f}$ at two points with the same σ and two sets of values of (u,θ,ϕ) has an upper bound $B(u_1,\theta_1,\phi_1;\,u_2,\theta_2,\phi_2)$ for large values of σ, but that $\bar{f}$, independently of the direction, may be a function of the order $O(\ln\sigma)$. If we characterize an individual member of J_a by the asymptotic behavior of the four functions $\bar{f}^\rho$, we have, then

$$\lim_{\sigma\to\infty} f^\rho=g^\rho=e^\rho(\sigma)+k^\rho(u,\theta,\phi),\qquad e^\rho=O(\ln\sigma). \tag{13}$$

The product of two members of J_a is represented by the sums of their respective g^ρ. The translations are characterized by constant values of all four g^ρ.

A group J_a/T would exist if there were a unique manner in which we could split every set of g^ρ into a set of constants and a remainder. The prescription would have to be linear. Such a prescription could be developed for the k^ρ alone, but not for the e^ρ. Hence our search has failed, and we must conclude that the inhomogeneous Lorentz group is not a factor group of C.

Our result might not hold if we assume analyticity in $1/\sigma$ for all coordinate transformations considered. In that case the term e^ρ would actually be proportional to $\ln\sigma$ and could be split off from the remainder. In that case, one can separate a constant out from k^ρ by taking its mean over the Lorentz sphere, identifying this mean with the translation vector, and assigning the remainder (whose mean vanishes) to the gauge-type transformations. Even then our task is not over. The mean over the Lorentz sphere is not a simple average, as the area of that sphere is infinite. One might say that almost all of the Lorentz sphere is adjacent to the light cone, and the average is defined only if the limit of the k_ρ exists for infinite values of u. But even if we restrict ourselves to transformations in which this limit exists, the averaging procedure outlined is not invariant under the transformations L_h, and thus the program breaks down at the last stage, where the invariance of the subgroup is tested under the full group C_a. Our end result is, then that, with or without assumptions concerning analyticity, no homomorphism exists between the full Lorentz group and the group C.

V. CONCLUDING REMARKS

The strategy of this paper has been to examine the structure of the transformation group, in preference to the examination of the transformation laws of specific functionals. Where this approach is successful, it leads to results of considerable generality, with a relatively slight expenditure of computational effort.

From the course of the argument it is obvious that the negative result depends on the structure of the initially chosen group C. Our construction was based on the assumption that one is interested in situations with nonvanishing rest mass and that one is, accordingly interested in the transformations that preserve the asymptotic flatness of order $O(1/r)$ of the metric field. If we were to concern ourselves with the set of solutions of the field equations whose metric field at spatial infinity goes as $1/r^2$, then we should find that the inhomogeneous Lorentz group is available as a factor group. One might well search for other assumptions concerning the metric field, and hence different appropriate transformation groups that admit a variety of homomorphisms.

Another argument to circumvent the results of the group-theoretical analysis is the following: Granted that there is no homomorphism between the group C and the Lorentz group L, might it not be reasonable to represent the invariant contents of the general theory of relativity in a formulation that is Lorentz-covariant but which is unrelated to the original invariance group of the theory? From a purely formal point of view such a presentation is entirely possible, but only on condition that the coordinates of the Lorentz-covariant formulation are unrelated to those of the original formulation. Not only would the coordinates not coincide at spatial infinity, there would be no rule by which a particular Lorentz frame would be related to a particular curvilinear frame. It is the assumption of the existence of such a rule that forms the point of departure for this paper.

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