THE
PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 124, NO. 2 OCTOBER 15, 1961

Conservation Laws and Correlation Functions

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(Received May 15, 1961)

In describing transport phenomena, it is vital to build the conservation laws of number, energy, momentum, and angular momentum into the structure of the approximation used to determine the thermodynamic many-particle Green's functions. A method for generating conserving approximations has been developed. This method is based on a consideration, at finite temperature, of the equations of motion obeyed by the one-particle propagator G, defined in the presence of a nonlocal external scalar field U. Approximations for $G(U)$ are obtained by replacing the $G_2(U)$ which appears in these equations by various functionals of $G(U)$. If the approximation for $G_2(U)$ satisfies certain simple symmetry conditions, then the $G(U)$ thus defined obeys all the conservation laws. Furthermore, the two-particle correlation function, generated as $(\delta G/\delta U)_{U=0} \equiv \pm L$, in terms of which all linear transport can be described, will obey all the conservation laws as well as several essential sum rules, such as the longitudinal f-sum rule.

Several examples of conserving approximations are described. The Hartree approximation, $G_2(U)=G(U)G(U)$, generates the random-phase approximation for L. The Hartree-Fock approximation for $G(U)$ leads to a natural generalization of the random-phase approximation in which hole-particle ladder diagrams are summed. Another conserving approximation for $G(U)$ is obtained by expanding the self-energy to first order in the many-particle scattering matrix $T(U)$. This T is obtained by summing ladder diagrams in which the sides of the ladder are composed of $G(U)$'s. The resulting L equation, which involves coefficients proportional to $|T|^2$, is analogous to the linearized version of the usual Boltzmann equation. Finally, in order to obtain a description of collisions in a plasma, the self-energy is expanded to first order in a dynamically shielded potential, $V_s(U)$. This potential is obtained by summing bubbles composed of two $G(U)$'s. The resulting L equation is similar in structure to a Boltzmann equation in which the collision cross section is proportional to $|V_s|^2$.

I. INTRODUCTION

A MAJOR difficulty in developing a quantum theory of transport is that most approximations lead to transport equations that do not include the conservation laws. This has the consequence that one is unable, from these equations, to describe correctly even the qualitative features of the transport processes of the system. For example, the Bardeen-Cooper-Schrieffer–Bogoliubov evaluation[1] of the two-particle correlation function for a superconductor does not agree with the differential charge conservation law. As a result, the longitudinal collective oscillation discussed by Anderson[2] fails to appear in their density-density correlation function. Conversely, one may derive the existence of sound waves in a gas by applying the conservation laws of number, momentum, and energy to a system in local thermodynamic equilibrium.[3] These and many other examples point to the necessity for building the conservation laws into any description of transport processes in a many-particle system. It is the purpose of this paper to develop an approximation method that yields transport equations which automatically include the conservation laws of particle number N, momentum $\mathbf{P}$, angular momentum $\mathbf{L}$, and energy H.

In order to observe a transport process, one applies an external disturbance to a system initially in thermodynamic equilibrium. Quantum mechanically, one can describe the responses to an external disturbance by computing the changes, induced by the disturbance, in the expectation values of the densities of conserved operators. The conservation laws are then requirements on the allowable changes in these expectation values. For example, the conservation law for momentum is the condition that the time derivative of the expectation value of the total momentum must be just the total force applied by the external disturbance.

The expectation values of the conserved operators can all be expressed in terms of the one-particle Green's function,[4] $G(1,1')$, defined in the presence of the

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957). N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, Inc., New York, 1959).

[2] P. W. Anderson, Phys. Rev. **112**, 1900 (1958); Y. Nambu, *ibid.* **117**, 648 (1960).

[3] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Massachusetts, 1959).

[4] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

external disturbance. The linear response of G to the disturbance may then be used to compute the linear transport properties of the system. In the next section of this paper we describe how one may appropriately define G at finite temperature in the presence of a general external disturbance. We then devote the third section to developing an approximation procedure that leads to G's that satisfy all the conservation laws. The linear responses to the external disturbance, which are generated by these approximate one-particle propagators, also include all the conservation laws as well as several important sum rules. Finally, we give several examples of conserving approximations for G and the linear response function, and we discuss some of their possible applications to transport.

II. BOUNDARY CONDITIONS AND EXTERNAL DISTURBANCES

Before introducing an external disturbance, we shall review the time boundary conditions obeyed by the equilibrium many-particle Green's functions. Since the form of the boundary conditions is identical for all the many-particle Green's functions, we shall consider only the one-particle function. This is defined by the grand canonical ensemble average:

$$G(1-1')=-i\,\mathrm{Tr}\{e^{-\beta(H-\mu N)}T[\psi(1)\psi^\dagger(1')]\}/\mathrm{Tr}(e^{-\beta(H-\mu N)})$$
$$=-i\langle T[\psi(1)\psi^\dagger(1')]\rangle. \quad (1)$$

Here, T represents the Wick time-ordered product, β is the inverse temperature, μ is the chemical potential, and the operators are in the Heisenberg representation. For the remainder of this section we shall absorb the μN into H.

The trace structure of (1) implies that G obeys a simple periodicity condition along the imaginary time axis. To see this we first extend the domain of definition of G to complex values of its time arguments, using the relation

$$\psi(\mathbf{r},t)=e^{iHt}\psi(\mathbf{r},0)e^{-iHt}. \quad (2)$$

In particular, if we consider the time variables to be pure imaginary numbers, then G is well defined as long as

$$0\leqslant it\leqslant\beta, \quad 0\leqslant it'\leqslant\beta. \quad (3)$$

One cannot show that the trace representation converges on the entire imaginary axis. To complete the extension of G to this imaginary time interval, we redefine the time ordering for imaginary time by[5]

$$T[\psi(1)\psi^\dagger(1')]=\psi(1)\psi^\dagger(1'), \qquad it>it'$$
$$=\pm\psi^\dagger(1')\psi(1), \quad it'>it.$$

To exhibit the periodicity boundary condition we use (1) and (2) to write

$$G(\mathbf{r}_1,-i\beta;\mathbf{r}_1',t_1')$$
$$=-i\,\mathrm{Tr}[e^{-\beta H}(e^{\beta H}\psi(\mathbf{r}_1,0)e^{-\beta H})\psi^\dagger(\mathbf{r}_1',t_1')]/\mathrm{Tr}(e^{-\beta H}).$$

Rearranging this expression by using the cyclic property of the trace, we find

$$G(\mathbf{r}_1,-i\beta;\mathbf{r}_1',t_1')$$
$$=-i\,\mathrm{Tr}[e^{-\beta H}\psi^\dagger(\mathbf{r}_1',t_1')\psi(\mathbf{r}_1,0)]/\mathrm{Tr}(e^{-\beta H}).$$

Therefore, the boundary condition is seen to be

$$G(\mathbf{r}_1,-i\beta;\mathbf{r}_1',t_1')=\pm G(\mathbf{r}_1,0;\mathbf{r}_1',t_1'), \quad 0<it_1'<\beta. \quad (4)$$

This same boundary condition is obeyed by all the time arguments of the multiparticle Green's functions. These boundary conditions enable us to integrate approximate equations of motion and thereby determine the Green's functions for imaginary time.

The functions for real time are then determined by the following analytic continuation. In the real time domain, G is composed of two continuous functions, one for $t>t'$, and the other for $t<t'$. Similarly, G defined on the imaginary time axis is also composed of two continuous functions, one for $it>it'$, and the other for $it<it'$. These functions are simply related; in fact, the $t>t'$ function is just the analytic continuation of the $it>it'$ function, and similarly for the $t<t'$ and $it<it'$ functions. Hence if we determine G on the imaginary time axis, it is fully determined on the real time axis.[6]

Thus the method of determining the equilibrium Green's functions from equations of motion is based on an examination of these functions when their time arguments lie in the interval $0\leqslant it\leqslant\beta$. However, we are now interested in determining the one-particle Green's function in the presence of an external disturbance. We must see, therefore, how the boundary conditions may be stated in the presence of the disturbance.

Let us consider, in particular, a disturbance represented by a term $U(r_1t_1)n(r_1t_1)$ added to the Hamiltonian density. U is an arbitrary external scalar potential and n is the number density operator. In the presence of the disturbance, the one-particle Green's function is changed from its equilibrium value. In order to define $G(U)$, we need to state the boundary condition that it obeys.

The most natural way of defining the boundary condition is to demand that $G(U)$ describe the causal response to the disturbance. That is, we demand that before the disturbance is turned on, $G(1,1';U)$ be the equilibrium function defined by (1). Then the effect of the disturbance can be represented by modifying Eq. (2) to include the effect of U, i.e., we can take

$$\psi(\mathbf{r},t)=\mathfrak{U}^{-1}(t)\psi(\mathbf{r},-\infty)\mathfrak{U}(t), \quad (5)$$

[5] Throughout this paper, the upper sign refers to Bose-Einstein statistics, and the lower to Fermi-Dirac statistics.

[6] N. D. Mermin and G. Baym, J. Math. Phys. **2**, 236 (1961).

where $\mathfrak{U}$ is given by

$$\mathfrak{U}(t)=T\left[\exp\left\{-i\int_{-\infty}^{t}dt'\left(H(t')+\int d\mathbf{r}\,U(\mathbf{r},t')n(\mathbf{r},t')\right]\right)\right\}. \quad (6)$$

The first-order change of ψ due to U is

$$\delta\psi(r_1,t_1)=i\int_{-\infty}^{t_1}dt_2d\mathbf{r}_2[n(2),\psi(1)]U(2). \quad (7)$$

By applying (7) we can see that the change in the density of particles induced by U is

$$\delta\langle n(1)\rangle_c=\pm i\delta G(1,1^+)_c$$
$$=-i\int_{-\infty}^{t_1}dt_2dr_2\,\mathrm{Tr}\{e^{-\beta H}[n(1),n(2)]\}\times U(2)/\mathrm{Tr}(e^{-\beta H}), \quad (8)$$

where we use the subscript c to indicate the causal response. Picking out the linear response of $\langle n\rangle$ to U, we find

$$\delta\langle n(1)\rangle_c/\delta U(2)=-i\langle[n(1),n(2)]\rangle_{U=0},\quad t_1>t_2;$$
$$=0,\quad t_1\leqslant t_2. \quad (9)$$

This causal response represents the actual response of the physical system, initially in thermal equilibrium, to the applied disturbance. However, the boundary condition which requires the system to be initially in thermal equilibrium leads to a $G(U)$ which does not satisfy the periodicity condition (4). But it is this condition which enables us to integrate Green's function equations of motion. Hence it is inconvenient to determine directly the causal response of G to U.

Let us define a Green's function for imaginary time, in the presence of an external potential that acts in the imaginary time interval $0\leqslant it<\beta$.

Writing the operators in the interaction representation for the external disturbance, we define

$$G(1,1';U)=-i\,\mathrm{Tr}\{e^{-\beta H}T[S\psi(1)\psi^\dagger(1')]\}/\mathrm{Tr}(e^{-\beta H}T[S]), \quad (10)$$

where H is the unperturbed Hamiltonian. Here S, defined by

$$S=\exp\left(-i\int_0^{-i\beta}d\mathbf{r}_2dt_2\,n(\mathbf{r}_2,t_2)U(\mathbf{r}_2,t_2)\right), \quad (10')$$

is the S matrix in the interaction representation. The time dependence of ψ is still given by (2). By working in the interaction representation we exhibit all the U dependence of G.

Notice that we can again use the cyclic property of the trace to prove that $G(U)$ defined by (10) obeys the periodicity condition (4). Thus this $G(U)$ may be determined conveniently.

Since $G(U)$ is defined in the imaginary time domain, it has no very direct physical interpretation. However, once $G(U)$ has been determined [with the aid of boundary condition (4)], we may extract from it physical information about the transport processes of the system. In particular, we may consider the linear variation of G with U:

$$G(1,1';U)=G(1-1')_{U=0}+(-i)^2\int_0^{-i\beta}dt_2d\mathbf{r}_2\{\langle T[\psi(1)\psi^\dagger(1')n(2)]\rangle_{U=0}-\langle T[\psi(1)\psi^\dagger(1')]\rangle_{U=0}\langle n(2)\rangle_{U=0}\}U(2).$$

The coefficient of the linear variation we call the variational derivative of G with respect to U. Thus for all times in the $[0,-i\beta]$ interval,

$$\left.\frac{\delta G(1,1';U)}{\delta U(2)}\right|_{U=0}=(-i)^2\{\langle T[\psi(1)\psi^\dagger(1')n(2)]\rangle-\langle T[\psi(1)\psi^\dagger(1')]\rangle\langle n(2)\rangle\}$$
$$=\pm[G_2(12,1'2^+)-G(1-1')G(2-2^+)]_{U=0}. \quad (11)$$

This response function is different from that defined by (9), since the boundary condition is different. However, once we have analytically continued G_2-GG to real times we can readily determine the causal response. For example, it may be verified that for real times such that $t_1>t_2$,

$$\langle[n(1),n(2)]\rangle=-2i\,\mathrm{Im}[G_2(12,1^+2^+)-G(1-1^+)G(2-2^+)]. \quad (12)$$

We shall find it convenient to introduce the notation

$$L(12,1'2')=[G_2(12,1'2')-G(1-1')G(2-2')]_{U=0}. \quad (13)$$

We can generate $L(12,1'2^+)$ as $\pm[\delta G(1,1')/\delta U(2)]_{U=0}$.

To discuss coupling of external disturbances to the momentum, energy, or angular momentum densities, as well as to the number density, it is convenient to define G in the presence of an external disturbance that is nonlocal in space and time. This G will then be a generator for the four-point function $L(12,1'2')$. We define such a G by Eq. (10), where, now, S is given by

$$S=\exp\left[-i\int_0^{-i\beta}dt_2dt_2'd\mathbf{r}_2d\mathbf{r}_2'\,\psi^\dagger(\mathbf{r}_2,t_2)\times U(\mathbf{r}_2t_2,\mathbf{r}_2't_2')\psi(\mathbf{r}_2',t_2')\right]. \quad (14)$$

We can see that $U(2,2')$ represents a disturbance in which a particle is removed from the system at 2 and added at $2'$. [When $U(2,2')=U(2)\delta(2-2')$, S clearly reduces to (10').] With this extended definition of S,

G still obeys the canonical periodicity boundary condition (4).

The coefficient of the term in G linear in U is

$$\left.\frac{\delta G(1,1';U)}{\delta U(2',2)}\right|_{U=0}=\pm L(12,1'2'). \tag{15}$$

Thus, $G(U)$ serves as a generator for the two-particle correlation function L. When continued to real values of its time arguments, L describes all the linear transport properties of the system.[7]

III. CONSERVING APPROXIMATIONS

Let us consider a many-particle system which is governed by the Hamiltonian,

$$H(t_1)=\frac{1}{2m}\int d\mathbf{r}_1\,\nabla\psi^\dagger(1)\cdot\nabla\psi(1)$$
$$+\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_3 dt_3\,\psi^\dagger(1)\psi^\dagger(3)V(1-3)\psi(3)\psi(1), \tag{16}$$

where V is the local central potential,

$$V(1-3)=\delta(t_1-t_3)v(|\mathbf{r}_1-\mathbf{r}_3|). \tag{17}$$

Then the total number of particles, total momentum, total angular momentum, and total energy are all conserved when there is no external disturbance. With this Hamiltonian, the exact $G(U)$ obeys the equation of motion,

$$\int G_0^{-1}(1,\bar{1})G(\bar{1},1';U)$$
$$=\delta(1-1')+\int U(1,\bar{1})G(\bar{1},1';U)$$
$$\pm i\int V(1-\bar{1})G_2(1\bar{1}^-,1'\bar{1}^+;U), \tag{18a}$$

as well as the adjoint equation,

$$\int G(1,\bar{1};U)G_0^{-1}(\bar{1},1')$$
$$=\delta(1-1')+\int G(1,\bar{1};U)U(\bar{1},1')$$
$$\pm i\int G_2(1\bar{1}^-,1'\bar{1}^+;U)V(\bar{1}-1'). \tag{18b}$$

Here, $G_0^{-1}(1,1')$ is the operator defined by

$$G_0^{-1}(1,1')=(i\partial/\partial t_1+\nabla_1^2/2m)\delta(1-1')$$
$$=(-i\partial/\partial t_1'+\nabla_1'^2/2m)\delta(1-1'). \tag{19}$$

[7] R. Kubo [Can. J. Phys. 34, 1274 (1956)] and Martin and Schwinger, reference 4, discuss how the dynamic linear electromagnetic transport properties may be described in terms of the two-particle correlation function. D. E. McCumber [Suppl. Nuovo cimento 17, 8 (1960)] discusses sound propagation in terms of L. Also, H. Mori, Phys. Rev. 111, 694 (1958); *ibid.* 112, 829 (1958); M. S. Green, J. Chem. Phys. 22, 398 (1954).

In Eqs. (18) and subsequent equations we shall use the convention that variables with a bar over them (e.g., $\bar{1}$) are variables of integration. The integrals run over all space and, in order to include the boundary condition, over imaginary times from 0 to $-i\beta$. The notation $\bar{1}^\pm$ indicates the ordering of the field operators in G_2 at equal time.

It is, in general, impossible to solve (18) exactly, and hence one must make some form of approximation. We shall substitute various approximations for $G_2(U)$, expressed in terms of $G(U)$ and V, into Eq. (18). The Hartree-Fock approximation, for example, is given by

$$G_2(13,1'3';U)=G(1,1';U)G(3,3';U)$$
$$\pm G(1,3';U)G(3,1';U). \tag{20}$$

The solution to Eq. (18a) with an approximate $G_2(U)$ such as (20) defines $G(U)$ completely. Alternatively, however, we could have substituted the form for G_2 into the adjoint equation (18b). In general, there is no reason why these two procedures should lead to the same determination of $G(U)$. However, for most G_2 approximations that are used in many-particle physics, these two procedures do indeed lead to the same $G(U)$.

We will show that whenever the following conditions are satisfied:

For a given approximate $G_2(U)$, the approximate

$$G(U) \text{ satisfies both (18a) and (18b),} \tag{A}$$

The $G(U)$ chosen satisfies the symmetry condition

$$G_2(13,1^+3^+;U)=G_2(31,3^+1^+;U), \tag{B}$$

then the approximate $G(U)$ satisfies all the conservation laws.

The number conservation law for the approximate $G(U)$ follows from statement (A) alone. Subtracting (18b) from (18a) we find

$$(\partial/\partial t_1+\partial/\partial t_1')iG(1,1';U)$$
$$+[(\nabla_1+\nabla_1')\cdot(\nabla_1-\nabla_1')/2im]iG(1,1';U)$$
$$=\int[U(1,\bar{1})G(\bar{1},1';U)-G(1,\bar{1};U)U(\bar{1},1')]$$
$$\pm i\int[V(1-\bar{1})-V(1'-\bar{1})]G_2(1\bar{1}^-,1'\bar{1}^+;U). \tag{21}$$

When we set $1'=1^+$ in (21) we find

$$(\partial/\partial t_1)iG(1,1^+;U)+\nabla_1\cdot[[(\nabla_1-\nabla_1')/2im]$$
$$\times iG(1,1';U)]_{1'=1^+}$$
$$=\pm(\partial\langle n(1)\rangle_U/\partial t+\nabla_1\cdot\langle\mathbf{j}(1)\rangle_U)$$
$$=\int[U(1,\bar{1})G(\bar{1},1^+)-G(1,\bar{1})U(\bar{1},1^+)]. \tag{22}$$

Equation (22) is an exact statement of the number conservation law in the presence of an external dis-

turbance which adds and removes particles from the system. If $U(2,2')\to\delta(2-2')U(2)$, i.e., if the disturbance can be represented in the Hamiltonian by a coupling to the density, then (22) becomes

$$\partial\langle n\rangle_U/\partial t+\nabla\cdot\langle\mathbf{j}\rangle_U=0. \tag{23}$$

The right side vanishes, since such a disturbance does not add or subtract particles. We see then, that whenever the approximation for $G(U)$ satisfies condition (A), the approximate $G(U)$ *exactly* satisfies the number conservation law, (22).

Equation (22) is a statement of the number conservation law in differential form. We can write it in such a simple form only because we can give the particle current density as

$$(1/2im)[(\nabla_1-\nabla_1')\psi^\dagger(1')\psi(1)]_{1'=1}.$$

We cannot evaluate the energy and momentum current explicitly[8] and therefore we cannot state the differential forms of these conservation laws. Instead, we must state these laws in their integral form.

To compute the time derivative of the total momentum in our approximation we again consider Eq. (21). We apply $-\frac{1}{2}i(\nabla_1-\nabla_1')$ to (21), set $1'=1^+$ and integrate over all r_1. Then (21) becomes

$$\frac{d}{dt_1}\int d\mathbf{r}_1\left[\frac{\nabla_1-\nabla_1'}{2i}iG(1,1';U)\right]_{1'=1^+}=\pm\frac{d}{dt_1}\langle\mathbf{P}(t_1)\rangle_U$$

$$=\int d\mathbf{r}_1\int d\bar{3}\left\{\frac{\nabla_1-\nabla_1'}{2i}[U(1,\bar{3})G(\bar{3},1';U)-G(1,\bar{3};U)U(\bar{3},1')]\right\}_{1'=1^+}$$

$$\pm i\int d\mathbf{r}_1\int d\bar{3}\left\{\left(\frac{\nabla_1-\nabla_1'}{2i}[V(1-\bar{3})-V(1'-\bar{3})]\right)\times G_2(1\bar{3}^-,1'\bar{3}^+;U)\right\}_{1'=1^+}. \tag{24}$$

The G_2 term in this equation vanishes, since this term may also be written as

$$\pm\int d\mathbf{r}_1\int d\mathbf{r}_3\,\nabla_1\mathcal{V}(|\mathbf{r}_1-\mathbf{r}_3|)G_2(13,1^+3^+)\,|_{t_3=t_1}.$$

From symmetry condition (B) we see that the integral changes sign under the interchange of $\mathbf{r}_1$ and $\mathbf{r}_3$, and hence this term vanishes. Thus our approximate $G(U)$ satisfies the momentum conservation law,

$$\frac{d}{dt_1}\langle\mathbf{P}(t_1)\rangle_U=\pm\int d\mathbf{r}_1\int d\bar{3}\left\{\frac{\nabla_1-\nabla_1'}{2i}[U(1,\bar{3})G(\bar{3},1')-G(1,\bar{3})U(\bar{3},1')]\right\}_{1'=1^+}. \tag{25}$$

[8] Martin and Schwinger[4] determine the momentum current approximately, and show that it is local in time but not in space.

When $U(2,2')\to U(2)\delta(2-2')$, the right side of (25) becomes

$$-\int d\mathbf{r}_1\,[\nabla U(1)]\langle n(1)\rangle_U.$$

In this case Eq. (25) says that the time derivative of the total momentum equals the total applied force.

An almost identical argument may be used to compute the time derivative of the total angular momentum from the approximate $G(U)$. The result is

$$\frac{d}{dt_1}\int d\mathbf{r}_1\,\mathbf{r}_1\times\left(\frac{\nabla_1-\nabla_1'}{2i}iG(1,1';U)\right)_{1'=1^+}$$

$$=\pm\frac{d}{dt_1}\langle\mathbf{L}(t_1)\rangle_U$$

$$=\int d\mathbf{r}_1\int d\bar{2}\,\mathbf{r}_1\times\left\{\frac{\nabla_1-\nabla_1'}{2i}[U(1,\bar{2})G(\bar{2},1')-G(1,\bar{2})U(\bar{2},1')]\right\}_{1'=1^+}. \tag{26}$$

When U becomes local, Eq. (26) says that the time derivative of L is equal to $-\int d\mathbf{r}_1[\mathbf{r}_1\times\nabla U(1)]\langle n(1)\rangle_U$, the total torque on the system. Thus when conditions (A) and (B) are satisfied, the approximation conserves angular momentum as well.

The discussion of energy conservation is slightly more complex, but it requires no additional assumptions. The expectation value of the Hamiltonian in the presence of the external field is

$$\langle H(t_1)\rangle_U=\pm\int d\mathbf{r}_1\left\{\frac{1}{4}\left(i\frac{\partial}{\partial t_1}-i\frac{\partial}{\partial t_1'}+\frac{\nabla_1\cdot\nabla_1'}{m}\right)\times iG(1,1';U)\right\}_{1'=1^+}\mp\frac{i}{4}\int d\mathbf{r}_1\int d\bar{2}\times[U(1,\bar{2})G(\bar{2},1^+)+G(1,2)U(2,1^+)]. \tag{27}$$

This may be verified by using the equations of motion, (18a) and (18b). In the Appendix we show, by using only conditions (A) and (B), that

$$\frac{d}{dt_1}\langle H(t_1)\rangle_U=\mp\int d\mathbf{r}_1\int d\bar{2}\left[U(1,\bar{2})i\frac{\partial}{\partial t_1}G(\bar{2},1)+i\frac{\partial}{\partial t_1}G(1,\bar{2})U(\bar{2},1)\right]. \tag{28}$$

If U is local, the right side of (28) becomes $-\int d\mathbf{r}_1\times U(1)(\partial/\partial t_1)\langle\rho(1)\rangle_U=\int d\mathbf{r}_1(-\nabla U(1))\cdot\langle\mathbf{j}(1)\rangle_U$. Thus, the approximation is energy conserving since the time derivative of the internal energy is equal to the power fed into the system.

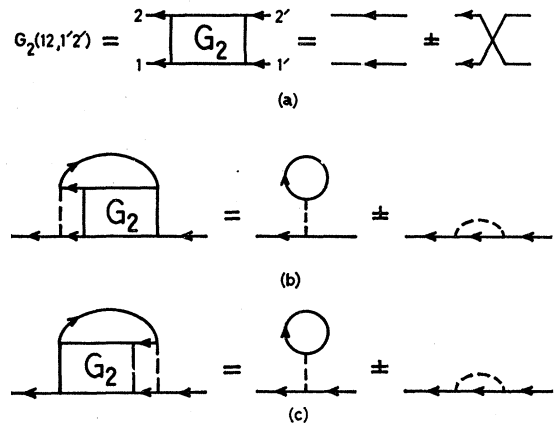

FIG. 1. Diagrammatic statement of conditions (A) and (B). In Fig. 1(a), we have indicated the Hartree-Fock approximation for $G_2(U)$. The solid lines represent $G(U)$. Condition (B) is simply the statement that the picture for G_2 with $1'=1^+$ and $2'=2^+$ must look the same when it is turned upside down. To see the graphical form of condition (A) in the Hartree-Fock approximation, we form, in (b) and (c), the right and left sides, respectively, of Eq. (29). The dashed lines represent V. Since diagrams (b) and (c) are identical for the Hartree-Fock approximation, it must satisfy (A).

We may conclude, therefore, that whenever conditions (A) and (B) are satisfied, the approximation leads to a fully conserving $G(U)$.

Condition (B) may always be verified by simply examining the form of the approximation for $G_2(U)$. It is convenient to cast condition (A) into a form in which it too may be verified by inspection. Condition (A) requires that (18a) and (18b) be consistent with one another. However, we notice that we can form the matrix product $G(G_0^{-1}-U)G$ from each of these equations. We demand that these two evaluations be identical and hence find the identity

$$\int G(1,\bar{1})V(\bar{1}-\bar{3})G_2(\bar{1}\bar{3}^-,1'\bar{3}^+) = \int G_2(1\bar{3}^-,\bar{1}\bar{3}^+)V(\bar{3}-\bar{1})G(\bar{1},1'). \quad \text{(A,29)}$$

This equation is equivalent to condition (A). Equation (29), as a condition on G_2, is easily verified by inspection for a choice of approximation. In Fig. 1 it is shown how this can be done diagrammatically.

In order to write Eqs. (18) in a more convenient form we define the self-energy function, Σ, by

$$\Sigma(1,1')=\pm i\int V(1-\bar{3})G_2(1\bar{3}^-,\bar{2}\bar{3}^+)G^{-1}(\bar{2},1'), \quad (30)$$

where G^{-1} is the matrix inverse of G. From (29) we see that in addition

$$\Sigma(1,1')=\pm i\int G^{-1}(1,\bar{2})G_2(\bar{2}\bar{3}^-,1'\bar{3}^+)V(\bar{3}-1'). \quad (31)$$

The Green's function equation is then

$$G^{-1}(1,1')=G_0^{-1}(1,1')-U(1,1')-\Sigma(1,1'). \quad (32)$$

The self-energy function, like G_2, is a functional of G and therefore U. As an example, in the Hartree-Fock approximation, (20), Σ is given by

$$\Sigma(1,1')=\pm i\delta(1-1')\int d\bar{3}\ V(1-\bar{3})G(\bar{3},\bar{3}^+;U) + iV(1-1')G(1,1';U). \quad (33)$$

Once we have a determination of $G(U)$, we can find the linear response to the external disturbance by picking out the linear coefficient of U in $G(U)$. This linear response is defined by Eqs. (11) and (13) to be

$$L(12,1'2')=\pm[\delta G(1,1')/\delta U(2',2)]_{U=0}. \quad (34)$$

The conservation laws for $G(U)$ imply conservation laws for L. If we take the variational derivative of the exact number conservation law (22), and then set $U=0$, we find the number conservation law obeyed by L:

$$-\{(\partial/\partial t_1)L(12,12') + \nabla_1\cdot[((\nabla_1-\nabla_{1'})/2im)L(12,1'2')]_{1'=1}\} = \pm iG(2-2')[\delta(1-2')-\delta(2-1)]. \quad (35)$$

It is instructive to consider the results of applying $(2im)^{-1}(\nabla_2-\nabla_{2'})$ to (35) and then setting $2'=2^+$. Equation (35) then becomes

$$(\partial/\partial t_1)\langle T[n(1)\mathbf{j}(2)]\rangle+\nabla_1\cdot\langle T[\mathbf{j}(1)\mathbf{j}(2)]\rangle = i\delta(t_1-t_2)\nabla_1\delta(\mathbf{r}_1-\mathbf{r}_2)\langle n\rangle/m. \quad (36)$$

Equation (36) indicates that the approximate L not only satisfies the density conservation law in the form

$$\langle T[(\partial n(1)/\partial t_1+\nabla_1\cdot\mathbf{j}(1))\mathbf{j}(2)]\rangle=0, \quad (36a)$$

but that L has preserved the information that the expectation value of the equal-time density-current commutator is given by

$$\langle[n(\mathbf{r}_1),\mathbf{j}(\mathbf{r}_2)]\rangle=i\nabla_1\delta(\mathbf{r}_1-\mathbf{r}_2)\langle n\rangle/m. \quad (36b)$$

Since the terms on the right side of (35) come from the discontinuous time derivative of the time-ordered product on the left, these terms in general are statements of the expectation values of commutators of n with the field operators ψ and $\psi^\dagger$.

In the description of electrical transport phenomena, L may be used to determine the dynamic longitudinal conductivity.[7] Then the commutation relation (36b) is just the sum rule on the conductivity.[9]

The integral conservation laws for $G(U)$, namely (25), (26), and (28), also imply integral conservation

[9] This sum rule is also known as the "longitudinal f-sum rule." It is quite important in the discussion of the superconductor. See Anderson, reference 2. In the theory of the inelastic scattering of slow neutrons, this sum rule is known as Placzek's sum rule [G. Placzek, Phys. Rev. 86, 377 (1952)].

laws for L. For momentum,

$$\left(\frac{\delta}{\delta U(2',2)}\frac{d}{dt_1}\langle \mathbf{P}(t_1)\rangle_U\right)_{U=0}$$
$$=i\frac{d}{dt_1}\int d\mathbf{r}_1\left(\frac{\nabla_1-\nabla_1'}{2i}L(12,1'2')\right)_{1'=1^+}$$
$$=\pm[\delta(t_1-t_2')-\delta(t_2-t_1)]\frac{\nabla_2-\nabla_2'}{2i}G(2-2'); \quad (37a)$$

for angular momentum,

$$\left(\frac{\delta}{\delta U(2'2)}\frac{d}{dt_1}\langle \mathbf{L}(t_1)\rangle_U\right)_{U=0}$$
$$=i\frac{d}{dt_1}\int d\mathbf{r}_1\,\mathbf{r}_1\times\left[\frac{\nabla_1-\nabla_1'}{2i}L(12,1'2')\right]_{1'=1^+}$$
$$=\pm[\delta(t_1-t_2')-\delta(t_2-t_1')]\frac{\mathbf{r}_2+\mathbf{r}_2'}{2}$$
$$\times\frac{\nabla_2-\nabla_2'}{2i}G(2-2'); \quad (37b)$$

and for energy,

$$\left(\frac{\delta}{\delta U(2'2)}\frac{d}{dt_1}\langle H(t_1)\rangle_U\right)_{U=0}$$
$$=\frac{1}{4}\frac{d}{dt_1}\Big\{\int d\mathbf{r}_1\Big[\left(i\frac{\partial}{\partial t_1}-i\frac{\partial}{\partial t_1'}+\frac{\nabla_1\cdot\nabla_1'}{m}\right)$$
$$\times L(12,1'2')\Big]_{1'=1^+}\mp[\delta(t_1-t_2')+\delta(t_2-t_1)]$$
$$\times G(2-2')\Big\}$$
$$=\mp[\delta(t_1-t_2)-\delta(t_1-t_2')]i(\partial/\partial t_2)G(2-2'). \quad (37c)$$

It is important to realize that Eqs. (35) and (37), all consequences of the conservation laws, are obeyed by both the exact L and G, and our approximate L's and G's. As with Eq. (35), the left sides of Eq. (37) are matrix elements of the operator conservation laws, and the right sides, which are statements of commutation relations, arise from the time derivative of the discontinuous time-ordered products on the left.

In principle, then, we can obtain a conserving L by first solving for a conserving $G(U)$. We can then generate L as $\pm(\delta G/\delta U)_{U=0}$, the coefficient of U in the term in $G(U)$ that is linear in U. In practice, we can almost never solve the equation of motion for G in the presence of an arbitrary disturbance. Fortunately, a much simpler calculational procedure is available. From the relation $GG^{-1}=1$ we notice that

$$\frac{\delta G(1,1')}{\delta U(2',2)}=-\int d\bar{3}d\bar{4}\,G(1,\bar{3})\frac{\delta G^{-1}(\bar{3},\bar{4})}{\delta U(2',2)}G(\bar{4},1'). \quad (38)$$

Calculating $\delta G^{-1}/\delta U$ from Eq. (32), we find

$$\frac{\delta G(1,1')}{\delta U(2',2)}=\int d\bar{3}d\bar{4}\,G(1,\bar{3})\Big[\delta(\bar{3},2')\delta(4,2)+\frac{\delta\Sigma(\bar{3},\bar{4})}{\delta U(2',2)}\Big]G(\bar{4},1')$$

or

$$L(12,1'2')=\pm G(1,2')G(2,1')\pm\int d\bar{3}d\bar{4}\,G(1,\bar{3})\frac{\delta\Sigma(\bar{3},\bar{4})}{\delta U(2',2)}G(\bar{4},1'). \quad (39)$$

The self-energy depends on U only through its dependence on $G(U)$. Since we know Σ as a functional of G, we can then compute $\delta\Sigma/\delta U$ by the chain-rule for differentiation, i.e.,

$$\left.\frac{\delta\Sigma(\bar{3},\bar{4})}{\delta U(2',2)}\right|_{U=0}=\int d\bar{5}d\bar{6}\left(\frac{\delta\Sigma(\bar{3},\bar{4};G)}{\delta G(\bar{6},\bar{5})}\right)_{U=0}\times\left(\frac{\delta G(\bar{6},\bar{5})}{\delta U(2',2)}\right)_{U=0}. \quad (40)$$

$\delta\Sigma/\delta G$ is to be understood as differentiating the functional form of Σ. We define an effective two-particle interaction, Ξ, by

$$\Xi(35,46)=[\delta\Sigma(3,4)/\delta G(6,5)]_{U=0}. \quad (41)$$

For example, in the Hartree-Fock approximation

$$\Xi(35,46)=\frac{\delta}{\delta G(6,5)}\Big[\pm i\delta(3-4)\int d\bar{2}\;V(3-\bar{2})G(\bar{2},\bar{2}^+)+iV(3-4)G(3,4)\Big]_{U=0}$$
$$=\pm i\delta(3-4)\delta(5-6)V(3-5)+i\delta(3-6)\delta(5-4)V(3-5). \quad (42)$$

When we combine Eqs. (40), (39), and (41) we arrive at the following integral equation for L:

$$L(12,1'2')=\pm G(1-2')G(2-1')+\int G(1-\bar{3})G(\bar{4}-1')\Xi(\bar{3}\bar{5},\bar{4}\bar{6})L(\bar{6}2,\bar{5}2') \quad (43)$$

The two-particle correlation function, L, has been constructed in such a manner that $L(12,1'2')$ obeys the conservation laws for the conserved operators constructed from the 1 and 1′ variables. In the discussion of transport, it is important to have L also be conserving in the 2 and 2′ variables. For example, in using L to describe electrical transport, the number conservation law in the 2 and 2′ variables is necessary for

gauge invariance,[10] while in the 1 and 1′ variables, it guarantees that the currents and charges induced by the external field obey the continuity equation. We shall therefore demand that the approximate L satisfy

$$L(12,1'2')=L(21,2'1'). \tag{44}$$

Then L will certainly have to be conserving in the variables 2 and 2′. To ensure this symmetry, we impose just one more condition on our approximation, namely:

$$\Xi(35,46)=\Xi(53,64). \tag{C}$$

Clearly the Hartree-Fock approximation satisfies this condition.

[Actually the conservation laws in 2 and 2′ are much more easily established than in the above manner. They follow directly from the local conservation laws at each vertex. For example, any expansion of $\Sigma(U)$ in $G(U)$ and $G_0(U)$ leads to gauge invariance as long as one line enters and one line emerges from every vertex. Therefore it would appear that condition (C) is not independent of conditions (A) and (B). This will be discussed in a future publication.]

We should point out that in our derivation of (43) we required that the G's which appear in this equation satisfy

$$G^{-1}(1-1')=G_0^{-1}(1-1')-\Sigma(1-1').$$

If, by inserting a better evaluation of G, we attempted to "improve" Eq. (43), then it would no longer satisfy the conservation laws. Since these laws play an essential role in transport phenomena, there would be no improvement at all!

More generally, the conservation laws for L, which contain discontinuity terms proportional to G, link the approximations one may use to determine G and L. If the conservation laws are to be satisfied, the L approximation uniquely defines G. One of the main virtues of the procedure described here is the fact that it automatically builds in the close connection between G and L.

This link may be stated more physically. The approximation for $G(1-1')$ is defined by the kind of correlations which are considered to be important in the equilibrium situation. Since L describes small deviations from equilibrium, the same kind of correlations which are important in equilibrium must also determine the behavior of L.

Let us briefly review our method for constructing a conserving approximation for the two-particle correlation function L. We start by picking an approximate form for $G_2(U)$ as a functional of $G(U)$, which we then substitute into the equation of motion for $G(U)$. In order that this approximation lead to a conserving G and L, we demand that the approximate G_2 satisfy the symmetry requirements (A), (B), and (C). These three requirements can be verified by merely examining the structure of G_2. If they are met, then we can derive an approximate integral equation, whose solution is a fully conserving L, by taking the first variational derivative with respect to U of the equation of motion for $G(U)$.

Since $G_2=L+GG$, we start from a nonconserving approximation for G_2 and end up with a fully conserving G_2.

IV. EXAMPLES OF CONSERVING APPROXIMATIONS

The examples of conserving approximations that we shall describe in this paper are all nonperturbative in the sense that all the G's and L's will include terms of arbitrarily high order in V. We consider only nonperturbative examples because we wish to discuss applications to transport phenomena, which are by nature nonperturbative. In fact, all the transport coefficients, such as the sound-wave damping, diverge as V^{-2} in the $V\rightarrow 0$ limit.

A. The Hartree and Hartree-Fock Approximations

The Hartree approximation, which describes the motion of the particles in the system as that of free particles in a self-consistent potential field, is the simplest nonperturbative approximation for G. We get it by taking

$$G_2(13^-,1'3^+;U)=G(1,1';U)G(3,3^+;U) \tag{45}$$

in Eq. (18). The self-energy function Σ, which becomes

$$\Sigma(1,1';U)=\pm i\delta(1-1')\int V(1-\bar{3})G(\bar{3},\bar{3}^+;U)$$

$$=\delta(1-1')\int d\mathbf{r}_3\, v(|\mathbf{r}_1-\mathbf{r}_3|)\langle n(\mathbf{r}_3,t_1)\rangle_U, \tag{46}$$

is just the average potential at $\mathbf{r}_1$, t_1 produced by the particles in the system. Hence

$$G^{-1}(1,1';U)=G_0^{-1}(1,1')-U(1,1')\mp i\delta(1-1')$$

$$\times\int V(1-\bar{3})G(\bar{3},\bar{3}^+;U), \tag{47}$$

and we see that the self-consistent field is

$$U(1,1')+\delta(1-1')\int d\mathbf{r}_3\, v(|\mathbf{r}_1-\mathbf{r}_3|)\langle n(\mathbf{r}_3,t_1)\rangle_U.$$

The variational derivative of (47) yields the following

[10] L. P. Kadanoff and P. C. Martin, Phys. Rev. (to be published), show that number conservation in 2 and 2′, together with the sum rule on the longitudinal conductivity, is equivalent to gauge invariance. Since Eq. (35) together with (44) imply these two conditions, our approximation method necessarily leads to a gauge-invariant description.

equation for L:

$$L(12,1'2')=\pm G(1-2')G(2-1')$$
$$\pm i\int G(1-\bar{3})G(\bar{3}-1')V(\bar{3}-\bar{4})L(\bar{4}2,\bar{4}2'). \quad (48)$$

Since

$$\Xi(34,3'4')=\pm i\delta(3-3')\delta(4-4')V(3-4),$$

it is clear that conditions (A), (B), and (C) are satisfied, and L is fully conserving.

Equation (48) and Eq. (47) with U turned off, constitute a Green's function statement of the random-phase approximation (RPA).[11] We see that the RPA is contained in the Hartree approximation for $G(U)$. The solution to (48) is

$$L(12,1'2')=\pm G(1-2')G(2-1')+i\int G(1-\bar{3})$$
$$\times G(\bar{3}-1')V_s(\bar{3}-\bar{4})G(\bar{4}-2')G(2-\bar{4}), \quad (49)$$

where the "shielded interaction," V_s, is given as the solution to

$$V_s(1-1')=V(1-1')$$
$$\pm i\int V(1-\bar{1})G(\bar{1}-\bar{1}')G(\bar{1}'-\bar{1})V_s(\bar{1}'-1'). \quad (50)$$

Equation (49) is of the form of the first-order iterative solution to (48) with V replaced by V_s. The shielded interaction is a sum of bubble diagrams (Fig. 2) in which each bubble is composed of just two Hartree

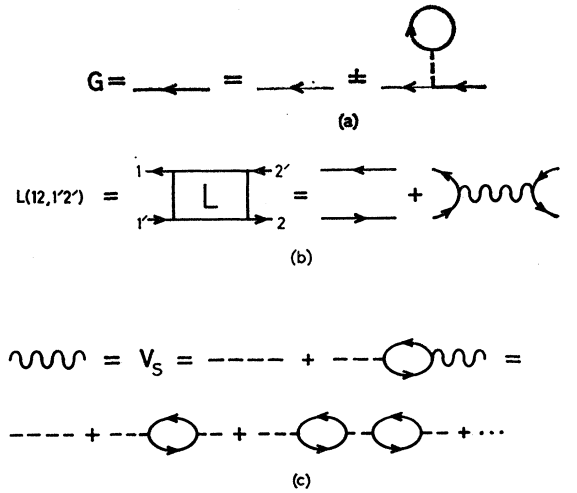

FIG. 2. Hartree approximation. We use a light line to represent G_0 and a heavy line to represent the approximate one-particle Green's function. (a) defines the usual Hartree approximation for G. (b) is a statement of Eq. (49) in terms of V_s, which is illustrated in (c).

[11] P. Nozières and D. Pines, Phys. Rev. **109**, 1062 (1958), where earlier references are given.

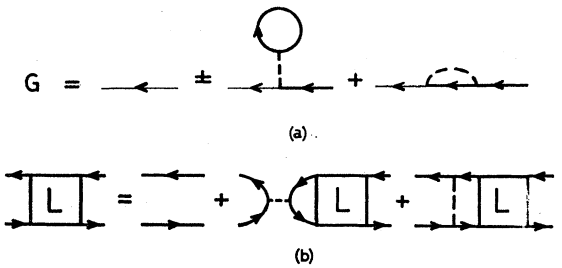

FIG. 3. The Hartree-Fock approximation.

Green's functions. In an electron gas V_s contains the plasma oscillation as poles in its Fourier transform.

The Hartree-Fock approximation, as given by Eqs. (20), (33), and (42) in the previous section, also describes the particles as moving in an average potential field. However, this field now includes exchange effects. The Green's function obeys

$$G^{-1}(1,1')=G_0^{-1}(1,1')-U(1,1')$$
$$\mp i\delta(1-1')\int V(1-\bar{3})G(\bar{3},\bar{3}^+;U)$$
$$-iV(1-1')G(1,1';U), \quad (51)$$

where the last term on the right represents the exchange effects. The resulting equation for L,

$$L(12,1'2')=\pm G(1-2')G(2-1')$$
$$\pm i\int G(1-\bar{3})G(\bar{3}-1')V(\bar{3}-\bar{4})L(\bar{4}2,\bar{4}2')$$
$$+i\int G(1-\bar{3})G(\bar{4}-1')V(\bar{3}-\bar{4})L(\bar{3}2,\bar{4}2'), \quad (52)$$

is equivalent to the generalization of the RPA to include exchange.[12] The solution to (52) is also a sum of bubble diagrams (Fig. 3) where now each bubble is essentially composed of a hole-particle scattering matrix made of Hartree-Fock Green's functions.

B. The T Approximation

Both the Hartree and the Hartree-Fock approximations leave out the detailed correlations produced by interparticle collisions. Thus they lead to the same description of transport phenomena with the long-wavelength limit as would be obtained from solving a collisionless Boltzmann equation, with and without exchange forces, respectively. These approximations may be used to describe phenomena, like plasma oscillations[11] and zero sound,[12,13] for which collisions are not important.

[12] G. E. Brown and D. J. Thouless, Suppl. Physica **26**, S145 (1960).

[13] K. Gottfried and L. Pičman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **32**, No. 13 (1960).

However, transport processes like ordinary sound and heat conduction only take place when there are collisions capable of restoring local equilibrium. The importance of the collision term is seen in the Boltzmann-equation description of these processes. We shall now describe a conserving approximation that includes collision effects. As a starting point we use the $G_2(U)$ which is obtained as a solution to the Bethe-Goldstone[14] equation,

$$G_2(13,1'3';U)=G(1,1';U)G(3,3';U)$$
$$\pm G(1,3';U)G(3,1';U)+i\int G_2(13,\bar{1}\bar{3};U)$$
$$\times V(\bar{1}-\bar{3})G(\bar{1},1';U)G(\bar{3},3';U). \quad (53)$$

This G_2 is a sum of ladder diagrams, in which each line represents the propagator $G(U)$. The Bethe-Goldstone approximation is conveniently described in terms of the many-particle scattering matrix $\langle 13|T[G(U)]|1'3'\rangle$ that satisfies

$$\langle 13|T|1'3'\rangle=[\delta(1-1')\delta(3-3')$$
$$\pm\delta(1-3')\delta(3-1')]V(1'-3')+i\int\langle 13|T|\bar{1}\bar{3}\rangle$$
$$\times G(\bar{1},1';U)G(\bar{3},3';U)V(1'-3'). \quad (54)$$

Here T represents the scattering amplitude for a process in which two-particles in the medium, initially at the points $1'$ and $3'$, collide and are later found at the points 1 and 3. In the low density limit, $(T)_{U=0}$ reduces to the scattering amplitude of conventional scattering theory.[15]

Now $T[G(U)]$ appears in the equation of motion for $G(U)$ through the relation

$$V(1-3)G_2(13,1'3')$$
$$=\int\langle 13|T[G(U)]|\bar{1}\bar{3}\rangle G(\bar{1},1';U)G(\bar{3},3';U). \quad (55)$$

The self-energy function is therefore

$$\Sigma(1,1';G(U))=\pm i\int d\bar{3}d\bar{4}\langle 1\bar{3}|T|1'\bar{4}\rangle G(\bar{4},\bar{3}^+). \quad (56)$$

Equation (56), called the "T approximation," has been used by Brueckner and others[16] to describe the equilibrium propagator G in nuclear matter. It is also a useful approximation for classical gases.

[14] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1957).

[15] N. M. Hugenholtz, in *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1960), Vol. II.

[16] The relation between the Brueckner approach, the Bethe-Goldstone equation, and graphical perturbation theory is described by Hugenholtz.[15]

The reader may easily verify, using Fig. 3, that conditions (A), (B), and (C) are satisfied, so that the T approximation leads to a fully conserving $G(U)$.

To derive the equation of motion for L we must construct Ξ. Now from Eq. (56)

$$\Xi(12,1'2')$$
$$=[\delta\Sigma(1,1';G)/\delta G(2',2)]_{U=0}$$
$$=\pm i\Big(\frac{\delta}{\delta G(2',2)}\int\langle 1\bar{3}|T[G(U)]|1'\bar{4}\rangle$$
$$\times G(\bar{4},\bar{3}^+;U)\Big)_{U=0}$$
$$=\pm i\langle 12|T|1'2'\rangle_{U=0}$$
$$\pm i\Big[\int\frac{\delta\langle 1\bar{3}|T[G(U)]|1'\bar{4}\rangle}{\delta G(2',2)}\Big]_{U=0}G(\bar{4}-\bar{3}). \quad (57)$$

We may then use Eq. (54) to evaluate $\delta T/\delta G$ as

$$\frac{\delta\langle 13|T|1'4\rangle}{\delta G(2',2)}-i\int\frac{\delta\langle 13|T|\bar{1}\bar{3}\rangle}{\delta G(2',2)}G(\bar{1},1')G(\bar{3},3')V(1'-3')$$
$$=\pm i\int\langle 13|T|\bar{1}2'\rangle G(\bar{1},1')V(1'-4)\delta(2-4)$$
$$\pm i\int\langle 13|T|2'\bar{3}\rangle G(\bar{3},4)V(1'-4)\delta(2-1').$$

Again using Eq. (54), we solve for $\delta T/\delta G$ as

$$\frac{\delta\langle 13|T|1'4\rangle}{\delta G(2',2)}=i\int\langle 13|T|\bar{5}2'\rangle G(\bar{5},\bar{6})\langle\bar{6}2|T|1'4\rangle. \quad (58)$$

Therefore

$$\Xi(12,1'2')=\pm i\langle 12|T|1'2'\rangle$$
$$\mp\int\langle 1\bar{3}|T|2'\bar{4}\rangle G(\bar{4}-\bar{5})G(\bar{6}-\bar{3})\langle 2\bar{5}|T|1'\bar{6}\rangle, \quad (59)$$

where the T's are taken at $U=0$. The equation for L is thus (see Fig. 4):

$$L(12,1'2')=\pm G(1-2')G(2-1')$$
$$\pm i\int G(1-\bar{3})G(\bar{5}-1')\langle\bar{3}\bar{4}|T|\bar{5}\bar{6}\rangle L(\bar{6}2,\bar{4}2')$$
$$\mp\int G(1-\bar{3})G(\bar{3}'-1')\langle\bar{3}\bar{5}|T|\bar{4}'\bar{6}'\rangle G(\bar{6}'-\bar{6})$$
$$\times G(\bar{5}'-\bar{5})\langle\bar{4}\bar{6}|T|\bar{3}'\bar{5}'\rangle L(\bar{4}'2,\bar{4}2'). \quad (60)$$

Notice that $\Xi(12,1'2')=\Xi(21,2'1')$ so that the L derived from Eq. (60) is fully conserving (see Fig. 4).

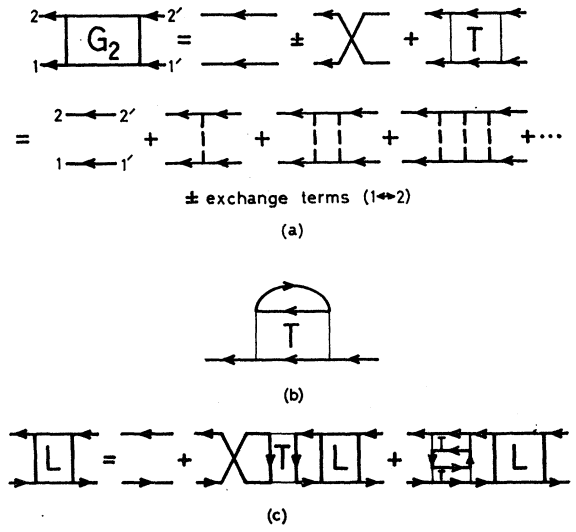

FIG. 4. The T approximation. (a) indicates the ladder structure of the solution of the Bethe-Goldstone equation for G_2. Condition (B) is clear from this figure. (b) is the form that both sides of condition (A) assume. (c) shows Eq. (60) in diagrammatic form.

In this way we derive an approximation for L that includes collisions, while maintaining all the conservation laws. A slightly modified version of this approximation has been used by Gottfried and Pičman[13] to discuss the damping of zero sound. In a future publication, we intend to show how, in the long-wavelength limit, a linearized form of the ordinary Boltzmann equation may be derived from this equation for L. This Boltzmann equation will contain a collision term in which the collision cross section is proportional to $|T|^2$.

C. The Shielded Interaction Approximation

The T approximation gives a useful description of systems in which the force range is much smaller than the interparticle distance. However, when a long-range force, such as the Coulomb force, is present, there are important polarization effects, and we must use a different approximation when including collisions.

The random-phase approximation, Eqs. (49) and (50), is a convenient starting point for the discussion of such systems, since the shielded interaction given by Eq. (50) provides a very simple description of how the long-range forces are dynamically shielded by the particles in the system. We therefore take a $G_2(U)$ which is given, in form, by the solution to the RPA:

$$G_2(13,1'3';U)=G(1,1';U)G(3,3';U)\pm G(1,3';U)\times G(3,1';U)+i\int G(1,\bar{1};U)G(\bar{1},1';U)\times V_s(\bar{1},\bar{2})G(\bar{2},3';U)G(3,\bar{2};U), \quad (61)$$

where

$$V_s(1,2)=V(1-2)\pm i\int V(1-\bar{1})G(\bar{1},\bar{2};U)G(\bar{2},\bar{1};U)V_s(\bar{2},2). \quad (62)$$

Unlike the RPA in Eqs. (49) and (50), the Green's functions in Eqs. (61) and (62) are $G(U)$'s and not just the Hartree Green's functions.

We then find that Σ is given by

$$\Sigma(1,1';G)=\pm i\delta(1-1')\int V(1-\bar{3})G(\bar{3},\bar{3}^+)-iV_s(1,1')G(1,1'). \quad (63)$$

The one-particle propagator thus defined,

$$G^{-1}(1,1')=G_0^{-1}(1,1')-U(1,1')\mp i\delta(1-1')\int V(1-\bar{3})G(\bar{3},\bar{3}^+)+iV_s(1,1'), \quad (64)$$

is very useful for the description of a high-density electron gas.

Equations (62) and (63) can easily be extended to a two-component system. Then, these equations can be used to describe a gas of ions and electrons. If the mass of the ion is much larger than that of the electron, V_s contains not only the plasmon poles but also photon-like sound-wave poles. Thus this approximation is a useful starting point for a two-component plasma model of a metal.

The equation for L, from which we may find linear transport properties, is found from Eqs. (62) and (63). From (63),

$$\Xi(12,1'2')=\pm i\delta(1-1')\delta(2-2')V(1-2)+iV_s(1-1')\delta(1-2')\delta(2-1')+iG(1-1')\left(\frac{\delta V_s(1,1')}{\delta G(2',2)}\right)_{U=0}. \quad (65)$$

However, Eq. (62) implies that

$$\left(\frac{\delta V_s(1,1')}{\delta G(2',2)}\right)_{U=0}=\pm i[V_s(1-2')V_s(2-1')+V_s(1-2)V_s(2'-1')]G(2-2'). \quad (66)$$

Hence Ξ becomes

$$\Xi(12,1'2')=\pm i\delta(1-1')\delta(2-2')V(1-2)+iV_s(1-1')\delta(1-2')\delta(2-1')\mp G(1-1')G(2-2')[V_s(1-2')V_s(2-1')+V_s(1-2)V_s(2'-1')] \quad (67)$$

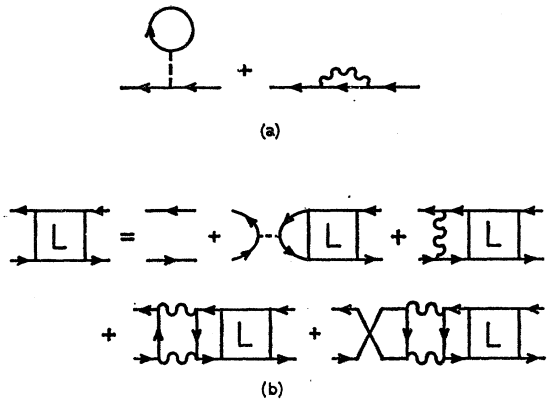

FIG. 5. The V_s approximation. (a) is the form that both sides of condition (A) assume. (b) is Eq. (68) in diagrammatic form.

and L obeys

$$L(12,1'2')=\pm G(1-2')G(2-1')$$
$$\pm i\int G(1-\bar{3})G(\bar{3}-1')V(\bar{3}-\bar{4})L(\bar{4}2,\bar{4}2')$$
$$+i\int G(1-\bar{3})G(\bar{4}-1')V_s(\bar{3}-\bar{4})L(\bar{3}2,\bar{4}2')$$
$$\mp\int G(1-\bar{3})G(\bar{3}'-1')G(\bar{3}-\bar{3}')G(\bar{4}-\bar{4}')$$
$$\times[V_s(\bar{3}-\bar{4}')V_s(\bar{4}-\bar{3}')+V_s(\bar{3}-\bar{4})V_s(\bar{4}'-\bar{3}')]$$
$$\times L(\bar{4}'2,\bar{4}2'). \quad (68)$$

It is again easily verified that conditions (A), (B), and (C) are satisfied (see Fig. 5) and therefore this L, a function of the shielded interaction, is fully conserving.

In a future publication we shall indicate how Eq. (68) for L may be converted, in the long-wavelength limit, into a Boltzmann equation in which the collision terms involve a collision cross section $\sim|V_s|^2$. This Boltzmann equation may be applied to a description of transport in a two-component plasma model of a metal.

D. Other Approximations

The four approximations (Hartree, Hartree-Fock, T, and V_s) that we have described by no means exhaust the list of conserving approximations obtainable by the techniques outlined above. On the contrary, one can construct an almost unlimited number of such approximations. For example, by extending the Gor'kov[17] equations for a superconductor to include an external field, one obtains a very economical derivation of the Anderson collective oscillations.[18] A formally rather similar technique may be applied to the description of the transport properties of a hard-sphere Bose gas. Also, other conserving approximations can be obtained by expanding $G_2(U)$ in a series in $G(U)$ and V.

All of the approximations mentioned in this paper lead to integral equations for L. In future publications, we shall describe how the transport properties of the system of interest can be determined from these integral equations.

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[18] V. Ambegaokar and L. P. Kadanoff (to be published).

ACKNOWLEDGMENTS

We are indebted to Professor P. C. Martin for discussions that established the forms of the conservation laws obeyed by L. We should also like to express our gratitude to Professor Niels Bohr and the members of the Institute for Theoretical Physics for the hospitality afforded us during the course of this work.

APPENDIX. PROOF OF ENERGY CONSERVATION

To show that our approximation is energy conserving, we first prove an identity which must be satisfied by the approximate $G_2(U)$, if $G(U)$ is to be number conserving. We multiply Eq. (18a) on the right by G_0^{-1}, and Eq. (18b) on the left by G_0^{-1}. Then, subtracting the second resulting equation from the first and setting $1'=1^+$, we find

$$\pm i\int d\bar{3}d\bar{4}\ V(1-\bar{3})[G_0^{-1}(1,\bar{4})G_2(\bar{4}\bar{3}^-,1'\bar{3}^+)$$
$$-G_2(1\bar{3}^-,\bar{4}\bar{3}^+)G_0^{-1}(\bar{4},1')]_{1'=1^+}$$
$$=\int d\bar{2}d\bar{4}[U(1,\bar{2})G(\bar{2},\bar{4})G_0^{-1}(\bar{4},1')$$
$$-G_0^{-1}(1,\bar{4})G(\bar{4},\bar{2})U(\bar{2},1')]. \quad (A.1)$$

This is the identity that we shall use in the calculation of the time derivative of Eq. (27), the expectation value of the energy. From (27),

$$\pm\frac{d}{dt_1}\langle H(t_1)\rangle_U$$
$$=-\frac{i}{4}\frac{d}{dt_1}\int d\mathbf{r}_1 d\mathbf{r}_2[U(1,2)G(2,1)+G(1,2)U(2,1)]$$
$$+\int d\mathbf{r}_1\,\tfrac{1}{4}[(i\partial/\partial t_1-i\partial/\partial t_1'+\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_1'/m)$$
$$\times(i\partial/\partial t_1+i\partial/\partial t_1')G(1,1';U)]_{1'=1^+}. \quad (A.2)$$

We can evaluate $(i\partial/\partial t_1+i\partial/\partial t_1')G(1,1';U)$ from Eq. (21). The term $(\boldsymbol{\nabla}_1+\boldsymbol{\nabla}_1')\cdot(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_1')G(1,1';U)$ does not contribute to the $\mathbf{r}_1$ integral, since it is proportional

to a total divergence. Hence

$$\pm\frac{d}{dt_1}\langle H(t_1)\rangle_U$$
$$=\pm i\int d\mathbf{r}_1\Big\{\tfrac{1}{4}(i\partial/\partial t_1-i\partial/\partial t_1'+\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_1'/m)$$
$$\times\int d\bar{3}[V(1-\bar{3})-V(1'-\bar{3})]G_2(1\bar{3}^-,1'\bar{3}^+)\Big\}_{1'=1^+}$$
$$+\frac{1}{2}\int d\mathbf{r}_1\int d\bar{2}\{[-i\partial/\partial t_1'+\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_1'/2m]$$
$$\times U(1,\bar{2})G(\bar{2},1')+[-i\partial/\partial t_1-\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_1'/2m]$$
$$\times G(1,\bar{2})U(\bar{2},1')\}_{1'=1^+}. \quad \text{(A.3)}$$

The G_2 term in (A.3) we rewrite as

$$\pm\frac{i}{2}\int d\mathbf{r}_1\int d\bar{3}\; V(1-\bar{3})i(\partial/\partial t_3-\partial/\partial t_1)G_2(13,1^+3^+)$$
$$\pm\frac{i}{2}\int d\mathbf{r}_1\int d\bar{3}\; V(1-\bar{3})\{[i\partial/\partial t_1+i\partial/\partial t_1'$$
$$+(\boldsymbol{\nabla}_1+\boldsymbol{\nabla}_1')\cdot(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_1')/2m]$$
$$\times G_2(1\bar{3}^-,1'\bar{3}^+)\}_{1'=1^+}. \quad \text{(A.4)}$$

By use of the identity (A.1), expression (A.4) becomes

$$\pm\frac{i}{2}\int d\mathbf{r}_1 d\mathbf{r}_3\, v(|\mathbf{r}_1-\mathbf{r}_3|)$$
$$\times i[(\partial/\partial t_3-\partial/\partial t_1)G_2(13,1^+3^+)]_{t_1=t_3}$$
$$+\frac{1}{2}\int d\bar{2}d\bar{3}[U(1,\bar{2})G(\bar{2},\bar{3})G_0^{-1}(\bar{3},1')$$
$$-G_0^{-1}(1,\bar{3})G(\bar{3},\bar{2})U(\bar{2},1')]_{1'=1^+}. \quad \text{(A.5)}$$

The first term in (A.5) vanishes from the assumed symmetry of $G_2(13,1^+3^+)$ under the interchange of 1 and 3. Collecting our results, we find

$$\frac{d}{dt_1}\langle H(t_1)\rangle_U=\mp\int d\mathbf{r}_1\int d\bar{2}\Big[U(1,\bar{2})i\frac{\partial}{\partial t_1}G(\bar{2},1)$$
$$+i\frac{\partial}{\partial t_1}G(1,\bar{2})U(\bar{2}1)\Big]. \quad \text{(28)}$$

This is the energy conservation law that we set out to prove.