Transport Equations for Plasmas in Strong External Fields*

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(Received June 7, 1961)

In an intense field plasma particles are much more strongly coupled to the field to each other, the motion of each particle depending on the strength and direction of the field rather than on its individual interactions with other particles. Pair and higher correlations become unimportant, and singlet superposition for the n-particle distribution function provides an excellent approximation. The singlet distribution function is then given by a Vlasov-type transport equation. This conjecture is proven correct on expanding the reduced Liouville equation in powers of a dimensionless *field parameter*, $\lambda=e/E\alpha^2$ (with α interpreted as an average approach distance), which expresses the ratio of the intensity of interparticle interactions to that of particle-field interactions; singlet superposition is then exact through terms linear in λ. A general closed hierarchy valid through any given power of λ is derived: solution exact through λ^m retains correlations through order m. Thus we see that, as the field becomes weaker, successively higher order correlations become important. Examples of the range of validity of the new treatment are given, and the lack of justification for the use of Boltzmann or Fokker-Planck type equations for microwave plasma diagnostics is briefly discussed, considering the low intensity of microwave beams, and thus the probable importance of pair and higher correlations.

TRANSPORT equations for plasmas are usually of the Boltzmann, Fokker-Planck, or Vlasov type. None of these equations has clearly defined limits of validity in terms of simple physical magnitudes; in particular, none of them is considered valid in the presence of strong fields which cause severe nonlinear effects. Such effects, however, are important in many physical situations such as the propagation of high-intensity microwaves in plasmas, microwave plasma diagnostics, etc. In this note we wish to outline the derivation of a hierarchy of closed transport equations in which each successive set yields an approximation valid through successively higher powers of a dimensionless parameter which expresses the strength of interparticle forces relative to the external field force, thus providing a description of precisely those situations in which the existing transport equations are inapplicable.

In very strong external electric fields we would expect the motion of each charged particle to be relatively independent of the *individual* motions of other particles, because each particle is much more strongly coupled to the field than to the remaining particles. In terms of statistical distributions this means that pair and higher correlations become unimportant, and singlet superposition of many-particle distribution functions should provide a good approximation. Therefore, the resulting equation would be expected to be of the nonlinear Vlasov type.[1] As we shall see, even higher approximations are readily obtained.

We consider a system of N particles of s kinds, N_i particles of kind i, in a volume V, each carrying a charge z_ie. The external field force on particle i at $\mathbf{r}_i$ is given by $Eez_i\boldsymbol{\varepsilon}(\mathbf{r}_i)$, E being the field intensity and $\boldsymbol{\varepsilon}(\mathbf{r}_i)$ the dimensionless field vector of order of magnitude unity. All forces are expressed in *units* of Ee; the Coulombic force between particle 1 and any of the remaining $N-1$ particles is then written as

$$\frac{z_1z_ie^2}{Ee}\frac{\mathbf{r}_{1i}}{(r_{1i})^3}=z_1\lambda_i\mathbf{F}(\mathbf{r}_{1i});\quad \lambda_i=\frac{z_ie}{E\alpha^2},\quad \mathbf{F}(\mathbf{r}_{1i})=\frac{\mathbf{r}_{1i}\alpha^2}{(r_{1i})^3},\qquad(1)$$

where α is a length parameter to be discussed later. As long as particles 1 and i do not come closer than α, we have $|\mathbf{F}(\mathbf{r}_{1i})|\leq 1$, and the dimensionless *field parameter* λ_i expresses correctly the ratio of the force exerted on particle 1 by i to the force exerted on 1 by the external field. This latter in units of Ee is written $z_1\boldsymbol{\varepsilon}(\mathbf{r}_1)$.

The reduced distribution function of the set $\mathbf{n}$ of n particles is denoted by

$$\phi(1,2,\cdots n)=\phi^{(n)}(\mathbf{r}_i,\cdots\mathbf{r}_n,\mathbf{p}_1,\cdots\mathbf{p}_n,t)=\phi(\mathbf{n});$$

all distribution functions are *normalized to unity.* With these definitions the exact reduced Liouville equation (the first equation of the BBGKY hierarchy[2]) for the distribution function $\phi(1)$ is

$$\frac{\partial}{\partial t}\phi(1)+\frac{\mathbf{p}_1}{m_1}\cdot\nabla_{\mathbf{r}1}\phi(1)+z_1\boldsymbol{\varepsilon}(\mathbf{r}_1)\cdot\nabla_{\mathbf{p}1}\phi(1)+z_1\sum_{i=2}^{N}\lambda_i\int\mathbf{F}(\mathbf{r}_{1i})\cdot\nabla_{\mathbf{p}1}\phi(1i)d\mathbf{r}_id\mathbf{p}_i=0.\qquad(2)$$

In Eq. (2) we assign a different field parameter to each particle in order to facilitate further analysis. Ulti-

* This work was supported by an Office of Aerospace Research contract monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

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[1] A. A. Vlasov, J. Phys. (U.S.S.R.) **9**, 25 (1945). The term "nonlinear" here means nonlinearized with respect to departures from equilibrium.

[2] N. N. Bogolubov, J. Phys. (U.S.S.R.) **10**, 256, 265 (1946); M. Born and H. S. Green, Proc. Roy. Soc. (London) **A118**, 10 (1946); J. G. Kirkwood, J. Chem. Phys. **14**, 180 (1946); J. Yvon, *Actualities Scientifiques et Industrielles* (Hermann & Cie, Paris, 1935). We refer to the transport equations in the theory developed by these authors as the "BBGKY hierarchy."

mately, the development will be in powers of the single parameter $\lambda = e/E\alpha^2$. Such a development necessitates a consistent ordering of the field parameters since the force between particles i and j could be written as $z_i\lambda_j\mathbf{F}(\mathbf{r}_{ij})$ or as $z_j\lambda_i\mathbf{F}(\mathbf{r}_{ij})$. Our principle of ordering stems from the fact that we need only the singlet and pair distribution functions in order to evaluate transport properties. In the evaluation of $\phi(1)$, particle 1 is regarded as moving in the field of 2, 3, $\cdots$, N, and thus only the $N-1$ parameters $\lambda_2, \cdots, \lambda_N$ are used. These are the field parameters of the particles whose coordinates and momenta are integrated out in Eq. (2). An extension of this principle leads to writing the general equation[2] for $\phi(\mathbf{n})$, $n \geq 2$, as follows:

$$\frac{\partial}{\partial t}\phi(\mathbf{n}) + \sum_{i=1}^{n}\left[\frac{\mathbf{p}_i}{m_i}\cdot\nabla_{\mathbf{r}_i}\phi(\mathbf{n}) + z_i\boldsymbol{\varepsilon}(\mathbf{r}_i)\cdot\nabla_{\mathbf{p}_i}\phi(\mathbf{n}) + z_i\sum_{j=n+1}^{N}\lambda_j\int\mathbf{F}(\mathbf{r}_{ij})\cdot\nabla_{\mathbf{p}_i}\phi(\mathbf{n}j)d\mathbf{r}_jd\mathbf{p}_j\right] + \left[z_1\sum_{i=2}^{n}\lambda_i\mathbf{F}(\mathbf{r}_{1i})\cdot(\nabla_{\mathbf{p}_1}-\nabla_{\mathbf{p}_i}) + z_2\sum_{i=3}^{n}\lambda_i\mathbf{F}(\mathbf{r}_{2i})\cdot(\nabla_{\mathbf{p}_2}-\nabla_{\mathbf{p}_i}) + \cdots + z_{n-1}\lambda_n\mathbf{F}(\mathbf{r}_{n-1,n})\cdot(\nabla_{\mathbf{p}_{n-1}}-\nabla_{\mathbf{p}_n})\right]\phi(\mathbf{n}) = 0. \quad (3)$$

The ordering principle is apparent here: particle 1 moves in the field of 2, $\cdots N$, particle 2 in the field of 3, $\cdots N$, and so on. If the field is sufficiently strong it will be sufficient to evaluate $\phi(1)$ through *first order* in $\lambda_2, \cdots\lambda_N$. Then, from Eq. (2), it follows that $\phi(12)$, $\phi(13)\cdots$ should be of *zeroth order* in $\lambda_2, \lambda_3, \cdots$; this, in turn, means that the triplet and higher distribution functions, $\phi(\mathbf{n})$, Eq. (3), should be evaluated to *zeroth order* in the field parameters of the sets $\mathbf{n}$. The resulting equation for the first approximation $\phi_1(\mathbf{n})$ is

$$\frac{\partial}{\partial t}\phi_1(\mathbf{n}) + \sum_{i=1}^{n}\left[\frac{\mathbf{p}_i}{m_i}\cdot\nabla_{\mathbf{r}_i}\phi_1(\mathbf{n}) + z_i\boldsymbol{\varepsilon}(\mathbf{r}_i)\cdot\nabla_{\mathbf{p}_i}\phi_1(\mathbf{n}) + z_i\sum_{j=n+1}^{N}\lambda_j\int\mathbf{F}(\mathbf{r}_{ij})\cdot\nabla_{\mathbf{p}_i}\phi_1(\mathbf{n}j)d\mathbf{p}_id\mathbf{r}_j\right] = 0. \quad (4)$$

A self-consistent solution of Eq. (4) is given by

$$\phi_1(\mathbf{n}) = \phi_1(1)\phi_1(2)\cdots\phi_1(n),$$

$$\frac{\partial}{\partial t}\phi_1(1) + \frac{\mathbf{p}_1}{m_1}\cdot\nabla_{\mathbf{r}_1}\phi_1(1) + z_1\left[\boldsymbol{\varepsilon}(\mathbf{r}_1) + \sum_{j=2}^{N}\lambda_j\int\mathbf{F}(\mathbf{r}_{1j})\phi_1(j)d\mathbf{p}_jd\mathbf{r}_j\right]\cdot\nabla_{\mathbf{p}_1}\phi_1(1) = 0. \quad (5)$$

Reverting to the usual normalization,

$$f(\mathbf{n}) = \prod_{j=1}^{s}[N_j!/(N_j-n_j)!]\phi(\mathbf{n}),$$

and taking the customary limits $N\to\infty$, $V\to\infty$, N_i/V constant, we obtain, as our first approximation to $f(\mathbf{n})$,

$$f_1(\mathbf{n}) = f_1(1)\cdot f_1(2)\cdots f_1(n),$$

$$\frac{\partial}{\partial t}f_1(1) + \frac{\mathbf{p}_1}{m_1}\cdot\nabla_{\mathbf{r}_1}f_1(1) + z_1\left[\boldsymbol{\varepsilon}(\mathbf{r}_1) + \lambda\sum_{k=1}^{s}z_k\int\mathbf{F}(\mathbf{r}_{1k})f(k)d\mathbf{p}_kd\mathbf{r}_k\right]\cdot\nabla_{\mathbf{p}_1}f(1) = 0, \quad (6)$$

$$\lambda = e/E\alpha^2.$$

Equation (6) is just the nonlinear Vlasov equation.[1] It will provide a good approximation to $f(1)$ and to $f(\mathbf{n})$ (as product of singlet distributions) when the field is sufficiently strong (λ small). Thus we see that our initial conjecture concerning the decoupling of correlations in strong fields is indeed correct.

From our derivation it is apparent that the solution of Eq. (5) are *multilinear* in λ_i's, containing terms of the type λ_2, $\lambda_2\lambda_3$, $\lambda_2\lambda_3\lambda_4\cdots$, but excluding terms λ_2^2, $\lambda_2\lambda_3^2$, $\lambda_3^2\lambda_4$, etc. Thus, when we convert to the single parameter λ in Eq. (6), we will also have *some*, but not *all*, terms of order higher than first in λ. Thus, to make the solution consistent and exact through $O(\lambda)$, it should be carefully linearized.

Entirely analogous reasoning leads to a general solution of the problem of evaluating $f(1)$ exactly through *any given power* of λ. Denoting a solution through $O(\lambda^\nu)$ by $f_\nu(\mathbf{n})$, we obtain the following result. The distribution functions $f_\nu(\mathbf{n})$ of less than ν particles are given by appropriate equations of the BBGKY hierarchy [Eq. (3) with $z_i\lambda$ instead of λ_i and $f(\mathbf{n})$, normalized to $N!/(N-n)!$, instead of $\phi(\mathbf{n})$ normalized to 1], while the distribution function of ν particles is given by the approximate equation

$$\frac{\partial}{\partial t}f_\nu(\boldsymbol{\nu}) + \sum_{i=1}^{\nu}\left\{\frac{\mathbf{p}_i}{m_i}\cdot\nabla_{\mathbf{r}_i}f_\nu(\boldsymbol{\nu}) + z_i\left[\boldsymbol{\varepsilon}(\mathbf{r}_i) + \lambda\sum_{j\neq i}^{\nu}z_j\mathbf{F}(\mathbf{r}_{ij}) + \lambda\sum_{k=1}^{s}z_k\int\mathbf{F}(\mathbf{r}_{ik})f(k)d\mathbf{p}_kd\mathbf{r}_k\right]\cdot\nabla_{\mathbf{p}_i}f_\nu(\boldsymbol{\nu})\right\} = 0. \quad (7)$$

For example, in order to obtain $f_2(1)$ (singlet distribution exact through λ^2), we have two coupled equations

$$\frac{\partial}{\partial t}f_2(1) + \frac{\mathbf{p}_i}{m_i}\cdot\nabla_{\mathbf{r}_1}f_2(1) + z_i\left[\boldsymbol{\varepsilon}(\mathbf{r}_i)\cdot\nabla_{\mathbf{p}_1}f_2(1) + \lambda\sum_{k=1}^{s}z_k\int\mathbf{F}(\mathbf{r}_{1k})\cdot\nabla_{\mathbf{p}_1}f_2(1k)d\mathbf{p}_kd\mathbf{r}_k\right] = 0,$$

with $f_2(12)$ given by

$$\frac{\partial}{\partial t}f_2(12)+\sum_{i=1}^{2}\left\{\frac{\mathbf{p}_i}{m_i}\cdot\nabla_{\mathbf{r}i}+z_i\left[\boldsymbol{\varepsilon}(\mathbf{r}_i)\right.\right.$$
$$\left.\left.+\lambda\sum_{k=1}^{s}z_k\int\mathbf{F}(\mathbf{r}_{ik})f_2(k)d\mathbf{p}_kd\mathbf{r}_k\right]\cdot\nabla_{\mathbf{p}_i}\right\}f_2(12)$$
$$+z_1z_2\lambda[\mathbf{F}(\mathbf{r}_{12})\cdot\nabla_{\mathbf{p}1}+\mathbf{F}(\mathbf{r}_{21})\cdot\nabla_{\mathbf{p}2}]\cdot f_2(12)=0.$$

In general, for $f(1)$ exact through $O(\lambda^\nu)$, we must solve ν coupled equations.[3] The physical interpretation is apparent. Terms of $O(\lambda^2)$ become important when the field is somewhat weaker. Then, however, pair correlations are still important, and just these correlations are retained in the approximation of Eqs. (8) and (9), where $f(12)$ is *not* (and cannot be) factorized. For successively weaker fields correspondingly higher order correlations are retained in the general approximation scheme, Eq. (7).

We conclude this note with a discussion of the probable range of validity of our approximation. While no exact value for the parameter α can be given, its interpretation as the average approach distance (i.e., two particles only very rarely come closer than α) leads us to identify it with the Debye length. This rough evaluation should be quite reasonable for plasmas of not too high density. Then, for example, for a two-component plasma with $N/V=10^{12}$, at $T=10^4$°K, we have $\lambda=0.6/E$, with E in volts per centimeter. Under these conditions λ will be very small for microwave intensity of 100 v/cm or more, and will be still smaller for higher temperatures and/or lower densities, when the Debye length is larger. Thus the *nonlinear* Vlasov Eq. (6) should provide an excellent approximation in these cases, in particular in problems of power transfer.

[3] Again, the solutions will contain some, but not all, terms of higher order, and these should be rejected from the final result.

Of course, modifications to include electron radiation and magnetic fields are necessary; these, however, are not difficult to develop.

Equations of the Boltzmann type, linearized with respect to departures from equilibrium, are commonly employed in conjunction with microwave diagnostics of hot plasmas.[4] In addition to the objections raised at the beginning of this note, it is doubtful that very hot plasmas are ever close to equilibrium (even in relatively weak fields). These objections carry even more weight in connection with plasma diagnostics where a reasonably firm theoretical basis is of paramount importance. It appears that the present development provides just such a basis for the proper choice of an approximate transport equation. For example, for a plasma at 10^7°K, density $N/V=10^{13}$, in a microwave beam of intensity 0.1 v/cm the field parameter λ is about 0.15. Thus it appears that our first approximation, the nonlinear Vlasov Eq. (6), may *not* be good enough in this case; however, our second approximation [Eqs. (8) and (9)] would be expected to be sufficiently accurate. On the other hand, our conclusion concerning the insufficiency of the first approximation, Eq. (6), casts some doubt on the validity of the use of any Boltzmann- or Fokker-Planck-type equation for plasma diagnostics since the latter involve approximations of the same type as the nonlinear Vlasov equation.

ACKNOWLEDGMENTS

I am greatly indebted to Irwin Oppenheim for numerous discussions and constructive criticism throughout the research which led to the present publication, and to N. G. Van Kampen and A. de-Shalit for stimulating comments.

[4] For references see, for example, *Symposium on Plasma Dynamics*, edited by F. Clauser (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960); S. Glasstone and R. H. Lovberg, *Controlled Thermonuclear Reactions* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960).