obtained from the equation

$$D=\sum_{i=0}^{0}\binom{n}{i}p_2{}^i(1-p_2)^{n-i}=(1-p_2)^n=\epsilon. \qquad (10)$$

The result is

$$p_2=1-\epsilon^{1/n}, \qquad (11)$$

and from (4) and (11) we find

$$A_2=(1/R)(\epsilon^{-1/n}-1). \qquad (12)$$

The 95% confidence interval for the X-particle flux relative to the μ-meson flux is obtained from (12) with $\epsilon=1-\beta=0.05$. For the Echo Lake run $n=432$ and $R=2.4$; hence $A_2\approx 0.3\%$. For the Princeton run $A_2\approx 0.2\%$ ($n=760$ and $R=2.0$).

CONCLUSION

On the basis of the results obtained at Echo Lake alone where the whole experimental arrangement was very similar to Alikhanian's, we find that there is less than 1% chance that Alikhanian's sample and our sample were drawn from the same population. If, furthermore, we include the similar results obtained at Princeton we can only arrive at the conclusion that Alikhanian's evidence that the $500m_e$ particle flux amounts to 0.5% of the μ-meson flux is not supported by the results of this experiment.

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Theory of Nuclear Matter*

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The methods of quantum statistics previously developed by the author are applied to the determination of properties of the ground state of nuclear matter. An expansion in powers of the pair-function of quantum statistics is made and expressions are derived for the momentum distribution, pair-correlation function, binding energy, and effective single-particle energies. The leading terms of these expressions can be interpreted in terms of an effective two-body interaction, and a model of nuclear matter which consists of interacting quasi-particles whose energies are the effective single-particle energies is thereby suggested. The theory is compared with Brueckner's theory and also with Landau's phenomenological theory of the Fermi liquid.

INTRODUCTION

THE nuclear many-body problem began in the nineteen-thirties with the observation that the binding energy per nucleon as measured for nuclei over the full range of the periodic table is roughly constant and of order -8 Mev. This property is in direct contrast with the behavior of a system with long range forces in which the energy increases with the square of the number of particles. It was therefore deduced, even before any scattering experiments were performed, that nuclear forces are short-ranged. With respect to the constant nuclear binding energies it was said that nuclear forces lead to "saturation." The nuclear many-body problem later took on an additional spect when careful measurements of nuclear radii showed that the central density of nuclei was also essentially constant for all nuclei with nucleon number. $A\gtrsim 30$. As a consequence of these observations, the concept of nuclear matter was introduced, in which both the Coulomb forces between protons as well as surface effects were assumed to be absent, since the additional complications presented by these effects are easily understood in a quantitative sense. In nuclear matter the number of neutrons equals the number of protons and these two different states are characterized by the two projections of the isotopic spin quantum number $I=\frac{1}{2}$.

With the concept of nuclear matter one is able to direct full attention towards understanding the phenomenon of the saturation of nuclear forces, and in fact one may consider the problem of infinite nuclear matter in this idealization. It is a solution of this problem to which the efforts of the present paper are directed.

An important measurable property of large nuclei is the momentum distribution $\langle n(k)\rangle$ of the nucleons. Although measurements of $\langle n(k)\rangle$ do not abound in the literature, it is known for light nuclei that there is a long tail in the distribution.[1] For very heavy nuclei, however, it is expected that the momentum distribution falls off rapidly for momentum values $k\gtrsim k_F$, where k_F is the maximum momentum of the corresponding ideal Fermi gas. This expectation is based upon the success

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[1] E. M. Henley, Phys. Rev. 85, 204 (1952); J. B. Cladis, W. N. Hess, and B. J. Moyer, Phys. Rev. 87, 425 (1952).

of the shell model, and it suggests that the momentum distribution in the ground state of nuclear matter may be approximated by an ideal Fermi distribution $\nu_F(k)$ in calculations of nuclear properties. The idea is, in fact, the basis[2] of the well-known Brueckner theory of nuclear matter in which it is *assumed* that a perturbation theoretic treatment of nuclear forces, starting from the ideal gas state, can lead to the correct expression for the binding energy per nucleon of nuclear matter. In any method based on this idea one only needs to calculate the "interaction energy," since the kinetic energy is given by $(3/5)E_F$, where $E_F=\hbar^2k_F^2/2M$ [see Eqs. (67) and (38)].

In the present investigation, it was felt that a thorough understanding of the momentum distribution should precede any attempt to calculate the ground-state energy of nuclear matter. Available for this purpose were the methods of quantum statistics in which distribution functions are readily related to the density matrix, which is then related to the grand partition function in the theory of the grand canonical ensemble. Thus, one is led to introduce the concept of a temperature T for nuclear matter in which the ground state is to be associated with the temperature $T=0$. Proceeding along these lines, a general prescription for calculating the momentum distribution of a very low temperature Fermi system was developed[3] by extending the quantum statistical methods of Lee and Yang.[4] It was shown that the momentum distribution and all intensive thermodynamic quantities can be calculated from expansions in which the weighting (or distribution) function in momentum state sums is the temperature-dependent function.

$$\nu'(k)=\frac{\exp\beta[\mathfrak{g}-\omega'(k)]}{1+\exp\beta[\mathfrak{g}-\omega'(k)]},\quad \beta=(kT)^{-1},$$

where

$$\omega'(k)=\hbar^2k^2/2M+\Delta(k,\beta),$$

and where $\mathfrak{g}$ is the thermodynamic potential of Gibbs. It was also shown that for a low-density Fermi gas

$$\rho=\Omega^{-1}\sum_\kappa \nu'(k),$$

where ρ is the particle density, and therefore that $\nu'(k)\rightarrow\nu_F(k)$ as $T\rightarrow 0$ is the free Fermi momentum distribution in this case. In Sec. I the general results of this investigation are reviewed and the procedure for applying these results to an actual Fermi system is outlined.

In Sec. II the leading contributions to $\langle n(k)\rangle$ and $\omega'(k)$ are both determined for nuclear matter in an expansion in powers of the "pair-function" (12) of quantum statistics. It is shown quite generally that $\omega'(k)$ is the energy-momentum relation for nucleons near the Fermi surface, and that up to and including two-pair terms $\nu'(k)$ is the free Fermi momentum distribution (the exact expression for $\langle n(k)\rangle$, Eq. (29), is rounded off from the free-particle form). It is also shown in Sec. II that in the low-temperature limit the pair-function expansion is equivalent to an expansion in powers of the function

$$\mathfrak{g}_1(k_1k_2|k_3k_4)$$

defined by Eq. (13). This function can be explicitly calculated in terms of the wave functions of the two-nucleon system, as is exhibited by Eqs. (10) and (11).

In Sec. III the expansion of the energy of nuclear matter to second order in $\mathfrak{g}_1(k_1k_2|k_3k_4)$ is derived. Then in Sec. V, Eqs. (73) and (74) it is shown that the function $\mathfrak{g}_1(k_1k_2|k_3k_4)$ can be exhibited as the matrix elements of a reaction matrix G_1 which is defined in terms of the two-body interaction V. This step facilitates comparison with the Brueckner method[5] which makes use of a different reaction matrix G_B. Thus, the difference between the present theory and the one developed by Brueckner lies in the use of different reaction matrices as expansion functions.

In Sec. IV an expression for the pair-correlation function in nuclear matter is derived, and the (leading) second order terms in $\mathfrak{g}_1$ are exhibited in Eq. (61). Although the first equation in this section can be readily understood by the casual reader, the rest of the section relies heavily on notation and concepts of II.

The purpose of deriving an expression for the pair-correlation function is so that one can unambiguously write down an approximate wave function for nuclear matter. This is done in Sec. VI, and it is shown there that the preceding results can all be interpreted in terms of a quasi-particle model of nuclear matter. These quasi-particles behave like free Fermions, with their energy-momentum relation given by $\omega'(k)$. The energy of nuclear matter, however, is calculated not by performing the sum $\sum_\kappa\omega'(k)\nu'(k)$, but by taking into account the fact that the quasi-particle interaction energies $\Delta(k)$ arise as a result of the mutual interaction between all the nucleons [Eqs. (81), (84), and (85)]. Thus, the theory is analogous to the generalized Hartree-Fock procedure developed from perturbation-theoretic treatments.

The present theory is also quite analogous to the Landau theory of the Fermi liquid[6] in that the relation between the binding energy and the quasi-particle energies is the same. An important distinction arises, however, when one considers the distribution functions. Whereas in the Landau theory the quasi-particle

[2] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955); H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

[3] F. Mohling, Phys. Rev. **122**, 1043 (1961), hereafter referred to as I; **122**, 1062 (1961), hereafter referred to as II.

[4] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1165 (1959); **117**, 22 (1960), hereafter referred to as LY IV.

[5] K. A. Brueckner, in *The Many-Body Problem* (John Wiley & Sons, New York, 1958), pp. 47–164.

[6] A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. XXII, p. 329.

momentum distribution is given by $\nu'(k)$, it is shown here that the quasi-particle momentum distribution is $\langle n(\mathbf{k})\rangle$ of Eq. (29) and not the "average function" $\nu'(k)$. The present treatment is also applicable to a large finite box of nuclear matter ($A\gg 1$) as well as to infinite nuclear matter.

The usual model for the two-nucleon interaction consists of a repulsive core outside of which are located the complex nuclear forces deduced from scattering experiments. When the core is assumed to be infinitely repulsive, then the model presents a special case to the theory and in the Appendix it is shown how the results of this paper may be generalized to include this mathematical idealization.

The question of convergence of an expansion of the physical quantities of nuclear matter in powers of $\mathfrak{g}_1(k_1k_2|k_3k_4)$ is reserved for a later paper. It is shown in Sec. II of this paper that the theory is restricted to the density region of nuclear matter for which

$$\mathfrak{g}<\tfrac{1}{2}\text{ (binding energy of deuteron)}.$$

This restriction permits a wide range of densities including the observed equilibrium density.

I. APPROACH OF QUANTUM STATISTICS

In quantum statistics we begin with a general expression for the grand partition function from the theory of the grand canonical ensemble:

$$\exp(\Omega f)=\sum_{N=0}^{\infty}[\exp(\beta\mathfrak{g})]^N\,\mathrm{Tr}_N[\exp(-\beta H^{(N)})],\quad (1)$$

where Ω is the volume of the system under consideration and $\beta=(kT)^{-1}$ refers to its temperature. The symbol Tr_N indicates that the trace of $\exp(-\beta H^{(N)})$ is to be taken over a complete set of symmetrized or antisymmetrized N-particle state vectors. Moreover, we assume in this paper that the N-particle Hamiltonian $H^{(N)}$ includes only two-particle interactions. The quantity f is called the grand potential and it is an intensive quantity. Finally, a quantity of great importance in the theory is the thermodynamic potential per particle $\mathfrak{g}$. It is defined by the thermodynamic relation

$$\begin{aligned}\mathfrak{g}&=\frac{\langle E\rangle}{\langle N\rangle}+\rho^{-1}\mathcal{P}-T\frac{\langle S\rangle}{\langle N\rangle}\\&=\left.\frac{\partial\langle E\rangle}{\partial\langle N\rangle}\right|_{\Omega,T},\end{aligned}\quad (2)$$

where $\langle E\rangle$, $\langle S\rangle$, and $\langle N\rangle$ are, respectively, the average energy, entropy, and number of particles of the system and $\rho=\langle N\rangle/\Omega$ is its density. The quantity $\mathcal{P}$ is the pressure. These thermodynamic quantities can all be calculated once the grand potential is known, by using prescriptions involving various partial derivatives of f. For example, the energy per particle is given by

$$\frac{\langle E\rangle}{\langle N\rangle}=\mathfrak{g}-\rho^{-1}\frac{\partial f}{\partial\beta}.\quad (3)$$

The distribution functions such as the momentum distribution $\langle n_k\rangle$ and the pair correlation function $\langle n_kn_{k'}\rangle$ can also be calculated from the grand potential.

Using the well-known method of Ursell,[7] it has been shown by Lee and Yang[4] that the grand potential can in general be expressed as a functional of the free particle quantity

$$\nu(k)=\frac{\exp\beta[\mathfrak{g}-\omega(k)]}{1-\epsilon\exp\beta[\mathfrak{g}-\omega(k)]},\quad \omega(k)=\frac{\hbar^2k^2}{2M},\quad (4)$$

where $\epsilon=+1$ for Bose statistics and $\epsilon=-1$ for Fermi statistics (we henceforth take $\epsilon=-1$). Now, the quantity $\nu(k)$ is the true momentum distribution for free particles. On the other hand, for a zero-temperature bound system at equilibrium ($\mathcal{P}=0$), the thermodynamic potential is equal to the binding energy [see Eq. (2)], and $\nu(k)$ vanishes identically. For such a system, and indeed for any low-temperature physical system, $\nu(k)$ is totally unrelated to the true momentum distribution. We therefore cannot expect to be able to calculate thermodynamic quantities very well as long as the grand potential is expressed as a functional of $\nu(k)$.

In an attempt to gain an understanding of the role of the momentum distribution in a low-temperature Bose system, Lee and Yang[4] were led to introduce a quantity $N(k)$ defined by

$$N(k)=\{\exp\beta[\mathfrak{g}-\omega(k)]\}[1+\epsilon\langle n(k)\rangle].\quad (5)$$

They then showed that the grand potential could be written as a functional of the "more-physical" quantity $N(k)$ instead of $\nu(k)$. Their work was extended by the present author[3] and it was shown that one can explicitly identify those terms in $N(k)$ which give large contributions for a very low-temperature Fermi system. Moreover, the terms identified were shown to be explicitly summed wherever they appear in $N(k)$ (and also in the grand potential) by means of the Λ transformation defined in II. Thus, after the Λ transformation, the grand potential is expressed as a functional of the quantity

$$N'(k)=\{\exp\beta[\mathfrak{g}-\omega'(k)]\}[1+\epsilon\langle n(k)\rangle],\quad (6)$$

where

$$\omega'(k)=\omega(k)-\epsilon\Delta(k,\beta).\quad (7)$$

We call $\omega'(k)$ the *effective single-particle energy* of the system corresponding to the momentum k. The prescription for calculating $\Delta(k,\beta)$ is given in II.

Before the Λ transformation, the grand potential is expressed entirely in terms of $N(k)$ and symmetrized matrix elements of the operator

[7] D. Ter Haar, *Elements of Statistical Mechanics* (Rinehart and Company, Inc., New York, 1954).

$$R(t_2,t_1)\equiv-\frac{\partial}{\partial t_1}\{\exp(t_2H_0{}^{(2)})\times\exp[-(t_2-t_1)H^{(2)}]\exp(-t_1H_0{}^{(2)})\},\quad(8)$$

where $H_0{}^{(2)}$ is the free, two-particle Hamiltonian. In II, Eq. (70), we have written these symmetrized matrix elements as

$$\begin{aligned}{}^{t_2}\begin{bmatrix}k_1k_2\\ k_3k_4\end{bmatrix}_{t_1}&=\langle k_1k_2|R(t_2,t_1)|k_3k_4\rangle\\&\quad+\epsilon\langle k_1k_2|R(t_2,t_1)|k_4k_3\rangle\\&=[\exp t_1(\omega_1+\omega_2-\omega_3-\omega_4)]f_1(k_1k_2|k_3k_4)\\&\quad+\sum_{k_5k_6}[\exp t_2(\omega_1+\omega_2-\omega_5-\omega_6)]\\&\quad\times[\exp t_1(\omega_5+\omega_6-\omega_3-\omega_4)]\qquad(9)\\&\quad\times f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right),\end{aligned}$$

where $k=(\mathbf{k},m,q)$ denotes momentum, ordinary spin, and isotopic spin state coordinates and P denotes the principal value. The summation $\sum_{k_5k_6}$ is over all states of the two-body Hamiltonian $H^{(2)}$, including any bound states. The particular form which this and subsequent expressions takes when the two-particle interaction includes an infinite repulsive core is discussed in the Appendix. The quantities f_1 and f_2 are well-defined functions which depend only on the two-body wave function corresponding to the interaction considered. We have shown in I that in general f_1 and f_2 can be expressed in terms of the two-body reaction matrices $\langle k|A^{(L)}|k_0\rangle$ each of which is equal to $\tan\delta_L(k_0)$ on the energy shell, where $\delta_L(k_0)$ is the phase shift of the Lth partial wave. We first exhibit the conservation of momentum and density of states factors in f_1 and f_2.

$$\begin{aligned}f_1(k_1k_2|k_3k_4)&=\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)[(2\pi)^3/\Omega]^2\tilde{f}_1(k_{12}|k_{34}),\\f_2(k_1k_2|k_5k_6|k_3k_4)&=\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_5-\mathbf{k}_6)[(2\pi)^3/\Omega]^3\tilde{f}_2(k_{12}|k_{56}|k_{34}).\end{aligned}\quad(10)$$

From I we then obtain explicit expressions for $\tilde{f}_1$ and $\tilde{f}_2$ for the case of a spin-independent interaction:

$$\tilde{f}_1(k_{12}|k_{34})=(2\pi^2)^{-1}(\hbar^2/M)\delta_{m_1m_3}\delta_{m_2m_4}\delta_{q_1q_3}\delta_{q_2q_4}\sum_{L=0}^{\infty}(2L+1)P_L(\hat{n}_{12}\cdot\hat{n}_{34})k_{12}{}^{-1}\langle k_{34}|A^{(L)}|k_{12}\rangle\cos^2\delta_L(k_{12})+(\text{antisymmetric expression})$$

$$\begin{aligned}\tilde{f}_2(k_{12}|k_{56}|k_{34})&=[(2\pi)^3/\Omega](2\pi^2)^{-2}(\hbar^2/M)^2\delta_{m_1m_3}\delta_{m_2m_4}\delta_{q_1q_3}\delta_{q_2q_4}\sum_{L=0}^{\infty}(2L+1)P_L(\hat{n}_{12}\cdot\hat{n}_{34})k_{56}{}^{-2}\langle k_{12}|A^{(2)}|k_{56}\rangle\\&\quad\times\langle k_{34}|A^{(L)}|k_{56}\rangle\cos^2\delta_L(k_{56})+(\text{antisymmetric expression})\quad\text{for continuum states},\\&=(2\pi^2)^{-1}(\hbar^2/M)^2\delta_{m_1m_3}\delta_{m_2m_4}\delta_{q_1q_3}\delta_{q_2q_4}\sum_{L=0}^{\infty}(2L+1)P_L(\hat{n}_{12}\cdot\hat{n}_{34})k_{12}{}^{-1}k_{34}{}^{-1}(k_{12}{}^2+\gamma^2)(k_{34}{}^2+\gamma^2)\\&\quad\times\phi_{\gamma L}(k_{12})\phi_{\gamma L}(k_{34})+(\text{antisymmetric expression})\quad\text{for bound states.}\end{aligned}\quad(11)$$

In this last equation the bound-state energies are related to γ^2 by $\omega(\gamma)=-\hbar^2\gamma^2/M$. The functions $\phi_{\gamma L}(k)$ are the Fourier transforms of the radial bound-state wave functions as defined by Eq. (I.74).

The expression given as Eq. (9) is called a *pair function*, and before the Λ transformation the interaction terms in the grand potential are expressed in terms of pair functions. Similarly, after the Λ transformation the interaction terms in the grand potential are expressed in terms of *transformed pair functions* defined as follows:

$$\begin{aligned}{}^{t_2}\begin{bmatrix}k_1k_2\\ k_3k_4\end{bmatrix}'_{t_1}&\equiv[\exp t_1(\omega_1'+\omega_2'-\omega_3'-\omega_4')]\mathfrak{g}_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}[\exp t_2(\omega_1'+\omega_2'-\omega_5-\omega_6)][\exp t_1(\omega_5+\omega_6-\omega_3'-\omega_4')]\\&\quad+f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right),\end{aligned}\quad(12)$$

where

$$\begin{aligned}&=\mathfrak{g}_1(k_1k_2|k_3k_4)=\epsilon\,\mathfrak{g}_1(k_2k_1|k_3k_4)=\mathfrak{g}_1(k_2k_1|k_4k_3)\\&=f_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)\left[P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\right].\end{aligned}\quad(13)$$

The quantity $\nu(k)$ of Eq. (4) is also changed by the Λ transformation to a quantity $\nu'(k)$ which is given by the expression

$$\nu'(k)=\frac{\exp\beta[\mathfrak{g}-\omega'(k)]}{1-\epsilon\exp\beta[\mathfrak{g}-\omega'(k)]}, \qquad (14)$$

Moreover, the quantity $N'(k)$ of Eq. (6) can be generated from $\nu'(k)$ with the aid of the integral equation

$$N'(k)=\nu'(k)\left[1+N'(k)\int_0^\beta dt\, L'(\beta,t,k,N')\right]. \qquad (15)$$

Upon combining Eqs. (6) and (15), one then obtains for the momentum distribution the general expression

$$\langle n(k)\rangle=\nu'(k)+\epsilon\nu'(k)[1+\epsilon\nu'(k)]\times\int_0^\beta dt\, L'(\beta,t,k,N'). \qquad (16)$$

The general prescriptions for calculating $L'(t_2,t_1,k)$, $\Delta(k,\beta)$, and the grand potential f are given in II. These may be explicitly calculated as power series in $\nu'(k)$ and the transformed pair-function as soon as one determines these two functions for a particular system.

In order to proceed, it is necessary to make two assumptions concerning the behavior of $\Delta(k,\beta)$ and $\nu'(k)$; namely,

(1) $\Delta(k,\beta)\rightarrow\Delta(k)=$ finite as $\beta\rightarrow\infty$ $(T\rightarrow 0)$

and

(2) $$\rho\equiv\Omega^{-1}\sum_k\langle n(k)\rangle=\Omega^{-1}\sum_k\nu'(k). \qquad (17)$$

These assumptions must always be checked during any calculation. Thus, although assumption (2) appears in the present theory to have a general validity, it certainly has not yet been proved to all orders of the pair-expansion mentioned above. Indeed, it may only be approximately true.

The implication of the above assumptions can be seen immediately. Thus, with the aid of Eq. (14) we see that assumption (1) implies that in the zero-temperature limit $\nu'(k)$ can only take on the values zero or one. We next define the Fermi energy E_F by the expressions

$$\rho\equiv(6\pi^2)^{-1}(2S+1)(2I+1)k_F^3,\quad E_F\equiv\hbar^2k_F^2/2M, \qquad (18)$$

where S is the intrinsic spin of a Fermion in the system and I is its isotopic spin. Using this definition we investigate the significance of assumption (2) above. The possible functional forms of $\nu'(k)$ at $T=0$ are indicated in Fig. 1. The definition (18) fixes the scale of the parameters of Fig. 1 in terms of E_F. If cases (b) and/or (c) are permitted in the calculation of $\Delta(k,\beta)$, then the extra parameters must be determined by minimizing the system energy with respect to their variation. Of the three possible cases, the one which gives the lowest energy is then the one which must be chosen. The function $\nu'(k)$ can, in general, also depend upon the direction of $\mathbf{k}$ as well as upon its magnitude,[8] but in this paper we shall for simplicity consider only "spherical" distributions and case (a) of Fig. 1. For this situation it is a simple consequence of assumption (2) that the thermodynamic potential at $T=0$ is given by

$$\mathfrak{g}=\omega'(k_F)=E_F+\Delta(k_F). \qquad (19)$$

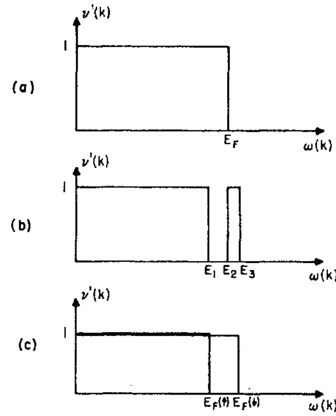

FIG. 1. Possible functional forms of $\nu'(k)$ at $T=0$. (a) A free Fermion momentum distribution. (b) A momentum distribution with a gap. (c) A momentum distribution which depends upon the spin projection along some preferred direction of the system. (If no direction is preferred, then this case may occur for an arbitrary direction).

With the above discussion in mind we outline a general procedure which can be used to calculate the physical quantities which characterize a low-temperature Fermi system. By "calculate" we shall always mean an expansion in powers of the transformed pair function (12) and the quantity $\nu'(k)$. That this is a *natural* expansion is clear; that it is a *convergent* one can only be checked by explicit calculation for a given system.

Outline of Procedure

(1) Make the assumptions (1) and (2) of (17). (2) Determine $L'(t_2,t_1,k)$ using the prescription in II. (3) Calculate the momentum distribution from Eq. (16). (4) Check assumption (2) of (17). (5) Calculate $\Delta(k,\beta)$ using the prescription in II, and then check assumption (1) of (17). The calculation involves solving a rather complex integral equation, because $\Delta(k,\beta)$ is functionally contained in the general expression (12) for the transformed pair function. This step is therefore the "self-consistency step" of the procedure. (6) Calculate the energy and other thermodynamic quantities of interest.

We note that this procedure can be applied to a large finite system, i.e., one for which the number of particles N is large $(N\gg 1)$, as well as to an infinite system (one for which the density ρ remains finite as N, $\Omega\rightarrow\infty$).

The remainder of this paper will be devoted first to applying the above procedure to the nuclear many-body problem and then to interpreting the results obtained.

[8] J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960).

II. MOMENTUM DISTRIBUTION AND EFFECTIVE SINGLE-PARTICLE ENERGIES

According to the procedure outlined at the end of Section I, we are to make assumptions (1) and (2) of (17) before proceeding with any calculations. We have seen that these assumptions enable us to specify the functional behavior of $\nu'(k)$, Eq. (14), to one of the three cases of Fig. 1. For simplicity we select the free-Fermion distribution of Fig. 1(a) for the present analysis. We now show that a further consequence of the assumptions (17) is a physical identification of the single-particle energies when $k=k_F$.

Consider the second of the expressions (2) for the thermodynamic potential. Its meaning is that the separation energy of a single particle from a system, defined to be the difference between the equilibrium energies of two systems with $(N+1)$ and N particles at constant volume and temperature, is equal to $\mathfrak{g}$ when $N\gg 1$. But if according to assumption (2) of (17) the *average* momentum distribution of a zero-temperature system is $\nu'(k)$ (of Fig. 1a), then the separated particle must have a momentum $k=k_F$. Moreover, according to Eq. (19) its energy can also be written as $\omega'(k_F)=\mathfrak{g}$. Therefore, the energy-momentum relation for particles at the Fermi surface $(k=k_F)$ is given by Eq. (7), and it is *plausible* to call $\omega'(k)$ the effective single-particle energy of the system for all k. We shall return to this point again in Sec. VI.

We have shown in Sec. I that for a zero-temperature bound system at equilibrium, the thermodynamic potential is equal to the binding energy and is therefore negative.

$$\mathfrak{g}(T=\mathcal{P}=0)=\omega'(k_F)=\frac{\langle E\rangle}{\langle N\rangle}<0 \qquad \text{(for a bound Fermi system).} \tag{20}$$

If now we assume that $\omega'(k)$ is a monotonically increasing function of k, we have the important result that the effective single-particle energies below the Fermi surface are all negative whenever $\mathfrak{g}$ is negative. This is the case for nuclear matter at its observed density.

We next examine the temperature dependence of the transformed pair function (12). According to Eq. (II.68) this function only occurs when $t_2>t_1$. Moreover, each contribution to the sum of the second term of Eq. (12) is $\sim\exp(t_2-t_1)(\omega_1'+\omega_2'-\omega_5-\omega_6)$ times the first term, where

$$\exp t(\omega_1'+\omega_2'-\omega_5-\omega_6)\to 0 \quad \text{for } t\sim\beta\to\infty \quad \text{when } (\omega_1'+\omega_2')<(\omega_5+\omega_6). \tag{21}$$

But the sum over k_5 and k_6 is over all states of the two-nucleon system, and therefore

$$(\omega_5+\omega_6)_{\text{minimum}}=(\text{binding energy of deuteron}).$$

We now *restrict our considerations to those densities of nuclear matter at $T=0$ for which the thermodynamic potential is less than one-half the binding-energy of the deuteron*, i.e., for which

$$\mathfrak{g}<\tfrac{1}{2}\,(\text{binding energy of deuteron}). \tag{22}$$

Then, for example,

$$\nu'(k_1)\nu'(k_2)\;{}^{t_2}\!\begin{bmatrix}k_1k_2\\ k_1k_2\end{bmatrix}'_{t_1}\underset{T\to 0}{\longrightarrow}\mathfrak{g}_1(k_1k_2|k_1k_2).$$

We see that this restriction immensely simplifies the calculation of $L'(t_2,t_1,k)$, $\Delta(k,\beta)$, and the grand potential, because many terms with exponential factors of the type (21) become negligibly small in the low-temperature limit. And, of course, the condition (22) is indeed satisfied for nuclear matter at its equilibrium density.

We turn our attention to the calculation of $L'(t_2,t_1,k)$. We write

$$\begin{aligned}L'(t_2,t_1,k)&=L_>'^{(1)}(t_2,t_1,k)+L_>'^{(2)}(t_2,t_1,k)\\&\quad+(\text{3-pair terms})+\cdots\text{ if } t_2>t_1\\&=L_<'^{(2)}(t_2,t_1,k)+(\text{3-pair terms})+\cdots\\&\qquad\qquad\text{if } t_2<t_1,\end{aligned} \tag{23}$$

where the N-pair terms are those in which N pair functions appear. Moreover, with the restriction (22) the one-pair term $L_>'^{(1)}(t_2,t_1,k)$ does not give an important low-temperature contribution.

It is found in the lowest-order calculations of this paper that when $t_2>t_1$, only the combination $\nu'(k)\times L'(t_2,t_1,k)$ ever occurs. Therefore, we only write down the expression $L_>'^{(2)}(t_2,t_1,k_1)$ for $k_1<k_F$. In this case, one obtains [with the restriction (22)]

$$\begin{aligned}L'_>{}^{(2)}(t_2,t_1,k_1)=-\tfrac{1}{2}\sum_{k_2k_3k_4}\nu'(k_2)[1-\nu'(k_3)][1-\nu'(k_4)][\exp(t_2-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')]\mathfrak{g}_1(k_3k_4|k_1k_2)\\ \times\Bigg[P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\mathfrak{g}_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)\\ \times P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)f_2(k_1k_2|k_5k_6|k_3k_4)\Bigg]\end{aligned}$$

$$+\tfrac{1}{2}\sum_{k_2k_3k_4}[1-\nu'(k_2)]\nu'(k_3)\nu'(k_4)[\exp t_1(\omega_3'+\omega_4'-\omega_1'-\omega_2')$$

$$\times P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\mathfrak{g}_1(k_1k_2|k_3k_4)\mathfrak{g}_1(k_3k_4|k_1k_2)$$

$$-\tfrac{1}{2}\sum_{k_2k_3k_4k_5k_6}[1-\nu'(k_2)]\nu'(k_3)\nu'(k_4)[\exp\beta(\omega_3'+\omega_4'-\omega_5-\omega_6)][\exp t_1(\omega_5+\omega_6-\omega_1'-\omega_2')]$$

$$\times P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\mathfrak{g}_1(k_1k_2|k_3k_4)f_2(k_3k_4|k_5k_6|k_1k_2)$$

$$+(\text{negligibly small terms}). \quad (24)$$

In deriving Eq. (24), we have used Eq. (15) to write

$$N'(k)=\nu'(k)[1+(\text{2-pair terms})+\cdots]\cong\nu'(k).$$

For $L_<'^{(2)}(t_2,t_1,k_1)$ we obtain:

$$L_<'^{(2)}(t_2,t_1,k_1)=\tfrac{1}{2}\sum_{k_2k_3k_4}[1-\nu'(k_2)]\nu'(k_3)\nu'(k_4)\mathfrak{g}_1(k_1k_2|k_3k_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)$$

$$\times[\exp(t_1-t_2)(\omega_3'+\omega_4'-\omega_1'-\omega_2')-\exp t_1(\omega_3'+\omega_4'-\omega_1'-\omega_2')]\Big\{\mathfrak{g}_1(k_3k_4|k_1k_2)$$

$$+\sum_{k_5k_6}P\left(\frac{1}{\omega_3'+\omega_4'-\omega_5-\omega_6}\right)[\exp(\beta-t_1)(\omega_3'+\omega_4'-\omega_5-\omega_6)]f_2(k_3k_4|k_5k_6|k_1k_2)\Big\}$$

$$-\tfrac{1}{2}\sum_{k_2k_3k_4k_5k_6}\nu'(k_2)\nu'(k_3)\nu'(k_4)f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)$$

$$\times[\exp(t_1-t_2)(\omega_3'+\omega_4'-\omega_1'-\omega_2')][1-\exp t_2(\omega_3'+\omega_4'-\omega_5-\omega_6)]$$

$$\times\Big\{\mathfrak{g}_1(k_3k_4|k_1k_2)+\sum_{k_7k_8}f_2(k_3k_4|k_7k_8|k_1k_2)[\exp(\beta-t_1)(\omega_3'+\omega_4'-\omega_7-\omega_8)]$$

$$\times P\left(\frac{1}{\omega_3'+\omega_4'-\omega_7-\omega_8}\right)\Big\}+(\text{negligibly small terms}). \quad (25)$$

The derivation of Eq. (24) is the most difficult step in obtaining the "two-pair approximation" to the momentum distribution. We substitute Eq. (24) into Eq. (16) and use the identity

$$\nu'(k)=[1-\nu'(k)]\{\exp\beta[\mathfrak{g}-\omega'(k)]\} \quad (26)$$

to obtain for $\langle n(k)\rangle$ at $T=0$

$$\langle n(k_1)\rangle=\nu'(k_1)+\tfrac{1}{2}[1-\nu'(k_1)]\sum_{k_2k_3k_4}[1-\nu'(k_2)]\nu'(k_3)\nu'(k_4)\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\mathfrak{g}_1(k_3k_4|k_1k_2)$$

$$\times\Big[\mathfrak{g}_1(k_1k_2|k_3k_4)\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)+\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)$$

$$\times P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)P\left(\frac{1}{\omega_1'+\omega_2'+\omega_5-\omega_6}\right)\Big]$$

$$-\tfrac{1}{2}\nu'(k_1)\sum_{k_2k_3k_4}\nu'(k_2)[1-\nu'(k_3)][1-\nu'(k_4)]\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\mathfrak{g}_1(k_1k_2|k_3k_4)$$

$$\times\Big\{\mathfrak{g}_1(k_3k_4|k_1k_2)\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)+\sum_{k_5k_6}f_2(k_3k_4|k_5k_6|k_1k_2)$$

$$\times P\left(\frac{1}{\omega_5+\omega_6-\omega_1'-\omega_2'}\right)P\left(\frac{1}{\omega_3'+\omega_4'-\omega_5-\omega_6}\right)\Big\}+(\text{3-pair terms})+\cdots. \quad (27)$$

Note that $\int_0^\beta dt\, L_>'^{(2)}(\beta,t,k)$ is actually infinite at $T=0$ as it must be according to Eq. (16) in order to give a finite contribution to $\langle n(k)\rangle$.

We next use the identity (I.77):

$$P\left(\frac{1}{x}\right)P\left(\frac{1}{y}\right)=(y-x)^{-1}\left[P\left(\frac{1}{x}\right)-P\left(\frac{1}{y}\right)\right]+\pi^2\delta(x)\delta(y). \tag{28}$$

When this identity is applied to the product of the two energy denominators in the "f_2-terms" of Eq. (27), then one sees that the δ-function products vanish. The final expression which we shall write down for the momentum distribution in nuclear matter is then

$$\begin{aligned}\langle n(k_1)\rangle=\nu'(k_1)&+\tfrac{1}{2}[1-\nu'(k_1)]\sum_{k_2k_3k_4}[1-\nu'(k_2)]\nu'(k_3)\nu'(k_4)\mathfrak{g}_1(k_3k_4|k_1k_2)h_1(k_1k_2|k_3k_4)\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)^2\\&-\tfrac{1}{2}\nu'(k_1)\sum_{k_2k_3k_4}\nu'(k_2)[1-\nu'(k_3)][1-\nu'(k_4)]\mathfrak{g}_1(k_1k_2|k_3k_4)h_1(k_3k_4|k_1k_2)\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)^2\\&+(\text{3-pair terms})+\cdots,\end{aligned} \tag{29}$$

where [see also Eq. (66)]

$$h_1(k_1k_2|k_3k_4)=f_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)\left[P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)-P\left(\frac{1}{\omega_3'+\omega_4'-\omega_5-\omega_6}\right)\right]. \tag{30}$$

Equation (29) reduces to Eq.'s (II.110) and (II.111) for the case of a low density Fermi gas. In addition it is easily verified with the aid of Eq. (29) that assumption (2) of (17) is correct up to and including two-pair terms.

We next turn our attention to the calculation of $\Delta(k,\beta)$ [which is equal to the quantity $A(k,\beta)$ defined in II when there is not an infinite repulsive core]. According to the prescription Eq.'s (II.89)–(II.91), an expression for $\Delta(k,\beta)$ occurs as a by-product of the calculation of $L'(t_2,t_1,k)$. We here give only the final result.

$$\begin{aligned}\Delta(k_1,\beta)=&-\sum_{k_2}\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_1k_2)\\&+\tfrac{1}{2}\sum_{k_2k_3k_4}\Big\{\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_3k_4)h_1(k_3k_4|k_1k_2)\left[P\left(\frac{1}{\omega_3+\omega_4-\omega_1'-\omega_2'}\right)-P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\right]\\&+\nu'(k_2)[\nu'(k_3)+\nu'(k_4)]\mathfrak{g}_1(k_1k_2|k_3k_4)h_1(k_3k_4|k_1k_2)P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\\&+\nu'(k_3)\nu'(k_4)\mathfrak{g}_1(k_3k_4|k_1k_2)h_1(k_1k_2|k_3k_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\Big\}\\&-\tfrac{1}{2}\pi^2\sum_{k_2k_3k_4k_5k_6}\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_3k_4)f_2(k_3k_4|k_5k_6|k_1k_2)\delta(\omega_5+\omega_6-\omega_1'-\omega_2')\\&\times\{[\nu'(k_3)+\nu'(k_4)-1]\delta(\omega_5+\omega_6-\omega_3'-\omega_4')+\delta(\omega_5+\omega_6-\omega_3-\omega_4)\}+(\text{3-pair terms})+\cdots.\end{aligned} \tag{31}$$

The third set of terms in this expression, involving the δ-function products, vanishes for k_1 values such that

$$[\omega'(k_1)+\mathfrak{g}]<(\text{binding energy of deuteron}).$$

Thus, these terms only contribute for k_1 values in the region $k_1\gtrsim 2k_F$, and they can be neglected for all practical purposes.

We observe from Eq. (31) that assumption (1) of (17) is satisfied up to and including two-pair terms. With this expression one can calculate the effective single-particle energies (7) and, in particular, the thermodynamic potential by Eq. (19). This is not a simple task, however, since Eq. (31) is a rather complex integral equation.

III. ENERGY PER PARTICLE

The pair-function expansion for the energy per particle in a low temperature Fermi system is somewhat tedious to derive and, we include here only a summary of the important steps. According to Eq. (3), we must first calculate the grand potential. If one again makes the restriction (22) and if for $N'(k)$ one uses Eq. (15) with $N'(k)$ replaced by $\nu'(k)$ on the right-hand side, then one can verify the following expression for f from the general prescription of Eq. (II.95) (see also the beginning of Sec. VI in II):

$$\Omega f(\mathfrak{g},\beta,\Omega)=-\sum_k \ln[1-\nu'(k)]+\beta\sum_k \nu'(k)\Delta(k,\beta)+\Omega[F^{(1)}(\mathfrak{g},\beta,\Omega)+F^{(2)}(\mathfrak{g},\beta,\Omega)]+(\text{3-pair terms})+\cdots, \quad (32)$$

where

$$\Omega F^{(1)}(\mathfrak{g},\beta,\Omega)=\tfrac{1}{2}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\int_0^\beta dt\ {}^{\beta}\!\begin{bmatrix}k_1k_2\\k_1k_2\end{bmatrix}'_t,$$

$$\begin{aligned}\Omega F^{(2)}(\mathfrak{g},\beta,\Omega)&=\tfrac{1}{4}\int_0^\beta dt_2\int_0^{t_2} dt_1\sum_{k_1k_2k_3k_4}\nu'(k_3)\nu'(k_4)\ {}^{\beta}\!\begin{bmatrix}k_3k_4\\k_1k_2\end{bmatrix}'_{t_2}\\&\times\Big\{[1-\nu'(k_1)-\nu'(k_2)]\ {}^{t_2}\!\begin{bmatrix}k_1k_2\\k_3k_4\end{bmatrix}'_{t_1}\\&+\nu'(k_1)\nu'(k_2)\ {}^{\beta}\!\begin{bmatrix}k_1k_2\\k_3k_4\end{bmatrix}'_{t_1}\\&-\{\exp t_2[\Delta(k_1)+\Delta(k_2)]\}\ {}^{t_2}\!\begin{bmatrix}k_1k_2\\k_3k_4\end{bmatrix}_{t_1}\\&\times\{\exp t_1[-\Delta(k_3)-\Delta(k_4)]\}\Big\}.\end{aligned}\quad (33)$$

[Note that the last term in Eq. (33) involves an ordinary pair-function, Eq. (9), and not a transformed pair-function. This special case arises in the derivation of Eq.'s (II.83)–(II.85)].

We are interested in writing down an expansion of the ground-state energy of nuclear matter. For this purpose we substitute Eq. (32) into Eq. (3) and use assumption (1) of (17) which was demonstrated to be true in Sec. II. In analogy with Eq.'s (II.118)–(II.122) we then obtain for the ground-state energy

$$\frac{\langle E\rangle_0}{\langle N\rangle}\equiv\lim_{T\to 0}\frac{\langle E\rangle}{\langle N\rangle}=(\rho\Omega)^{-1}\sum_k \nu'(k)\omega(k)-\lim_{T\to 0}(\rho\beta)^{-1}[F^{(1)}(\mathfrak{g},\beta,\Omega)+F^{(2)}(\mathfrak{g},\beta,\Omega)]+(\text{3-pair terms})+\cdots. \quad (34)$$

Upon using the expression below restriction (22), one can readily verify that the zero-temperature limit of the one-pair expression $\beta^{-1}F^{(1)}(\beta)$ is

$$\lim_{T\to 0}\beta^{-1}F^{(1)}(\mathfrak{g},\beta,\Omega)=(2\Omega)^{-1}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_1k_2). \quad (35)$$

The zero-temperature limit of the two-pair term $\beta^{-1}F^{(2)}(\beta)$ is more difficult to derive. After using the identity (26), one finds the result

$$\begin{aligned}\lim_{T\to 0}\beta^{-1}F^{(2)}(\mathfrak{g},\beta,\Omega)=(4\Omega)^{-1}&\sum_{k_1k_2k_3k_4}\nu'(k_3)\nu'(k_4)\mathfrak{g}_1(k_3k_4|k_1k_2)\Big\{[1-\nu'(k_1)][1-\nu'(k_2)]\mathfrak{g}_1(k_1k_2\tfrac{2}{3}k_3k_4)\\&\times P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)-f_1(k_1k_2|k_3k_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_3'-\omega_4'}\right)\\&+\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)\left[P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\right.\\&\left.-P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)\right]-[\nu'(k_1)+\nu'(k_2)]\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)\\&\times P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\Big\}.\end{aligned}\quad (36)$$

Equation (36) can be simplified by using the identity (28) and by then observing that none of the δ-function products contributes for nuclear matter. The result obtained, together with Eq. (35), can then be substituted

into Eq. (34) to give

$$\frac{\langle E\rangle_0}{\langle N\rangle}=(\rho\Omega)^{-1}\sum_k \nu'(k)\omega(k)-(2\rho\Omega)^{-1}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_1k_2)-(4\rho\Omega)^{-1}\sum_{k_1k_2k_3k_4}\nu'(k_3)\nu'(k_4)$$
$$\times\mathfrak{g}_1(k_3k_4|k_1k_2)h_1(k_1k_2|k_3k_4)\left[P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)-P\left(\frac{1}{\omega_1+\omega_2-\omega_3'-\omega_4'}\right)\right.$$
$$\left.-[\nu'(k_1)+\nu'(k_2)]P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\right]+(3\text{-pair terms})+\cdots, \quad (37)$$

where $h_1(k_1k_2|k_3k_4)$ is defined by Eq. (30) and where

$$\lim_{\Omega\to\infty}(\rho\Omega)^{-1}\textstyle\sum_k \nu'(k)\omega(k)=\frac{3}{5}E_F. \quad (38)$$

It is to be emphasized that Eq. (37) is valid for a large finite system ($N\gg1$) as well as for an infinite system (with finite ρ).

We conclude this section by defining a quantity $u_2(k_1,k_2)$.

$$u_2(k_1,k_2)\equiv-\mathfrak{g}_1(k_1k_2|k_1k_2)-\tfrac{1}{2}\sum_{k_3k_4}\mathfrak{g}_1(k_1k_2|k_3k_4)\left\{[1-\nu'(k_3)-\nu'(k_4)]\mathfrak{g}_1(k_3k_4|k_1k_2)P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\right.$$
$$-f_1(k_3k_4|k_1k_2)P\left(\frac{1}{\omega_3+\omega_4-\omega_1'-\omega_2'}\right)+\sum_{k_5k_6}f_2(k_3k_4|k_5k_6|k_1k_2)P\left(\frac{1}{\omega_5+\omega_6-\omega_1'-\omega_2'}\right)$$
$$\times\left[P\left(\frac{1}{\omega_3'+\omega_4'-\omega_5-\omega_6}\right)-P\left(\frac{1}{\omega_3+\omega_4-\omega_5-\omega_6}\right)\right]+[\nu'(k_3)+\nu'(k_4)]\sum_{k_5k_6}f_2(k_3k_4|k_5k_6|k_1k_2)$$
$$\left.\times P\left(\frac{1}{\omega_5+\omega_6-\omega_1'-\omega_2'}\right)P\left(\frac{1}{\omega_5+\omega_6-\omega_3'-\omega_4'}\right)\right\}. \quad (39)$$

In terms of this quantity, the ground-state energy (34) can be written

$$\frac{\langle E\rangle_0}{\Omega}=\Omega^{-1}\sum_k \nu'(k)\omega(k)+(2\Omega)^{-1}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)u_2(k_1,k_2)$$
$$+(3\text{-pair tersm})+\cdots. \quad (40)$$

If we next use the identity (28) for products of energy denominators in Eq. (39), then by referring to Eq. (31), we obtain the following alternate expression for $\omega'(k)$ at $T=0$

$$\omega'(k_0)=\omega(k_0)+\Delta(k_0)$$
$$=\omega(k_0)+\sum_{k_2}\nu'(k_2)u_2(k_0,k_2)$$
$$+\tfrac{1}{2}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\frac{\delta u_2(k_1,k_2)}{\delta\nu'(k_0)}$$
$$+(3\text{-pair terms})+\cdots, \quad (41)$$

where $\delta/\delta\nu'(k)$ indicates a functional differentiation. We finally observe from these last two equations that the expression for $\omega'(k)$ can be derived from $\langle E\rangle_0/\Omega$ by using the relation

$$\omega'(k)=\delta(\langle E\rangle_0/\Omega)/\delta\nu'(k). \quad (42)$$

Care must be exercised in using this relation, however, since when the energy is expressed as in Eq. (37) the δ-function products in Eq. (31) are not obtained. We shall return to a discussion of these last three expressions in Sec. V.

IV. PAIR-CORRELATION FUNCTION

In Sec. VI we shall write down the wave function for nuclear matter to leading order in the pair-function expansion. In order to conclusively demonstrate that this wave function is correct we shall derive here an expression for the closely-related pair-correlation function in nuclear matter.

The pair-correlation function $P(k_0,k_0')$ is defined to be the probability per unit volume that one particle in a system has momentum (and spin) k_0 and another particle in the system has momentum k_0'. The general expression for $P(k_0,k_0')$ is

$$P(k_0,k_0')$$
$$\equiv[\langle n(k_0)n(k_0')\rangle-\tfrac{1}{2}(1+\epsilon)\langle n(k_0)\rangle\delta_{k_0,k_0'}]$$
$$=\sum_{N=1}^{\infty}(N!)^{-1}\sum_{k_1\cdots k_N}\langle k_1k_2\cdots k_N|\rho_N^{(s)}|k_1\cdots k_N\rangle$$
$$\times\left(\sum_{i=1}^{N}\delta_{k_i,k_0}\right)\left(\sum_{j\neq i}^{N}\delta_{k_j,k_0'}\right), \quad (43)$$

where $\langle k_1\cdots k_N|\rho_N^{(s)}|k_1\cdots k_N\rangle$ is a symmetrized (or antisymmetrized) matrix element of the N-particle density operator. When $k_0'=k_0$ as can only occur for Bose statistics, then the right hand side of this equation is the average value per unit volume of $N(k_0)[N(k_0)-1]$, where $N(k_0)$ is the number of particles in the state k_0. The right-hand side is clearly equal to zero for Fermi statistics when $k_0'=k_0$.

In analogy with the derivation of Eq. (II.5) one can readily show that the pair-correlation function is related to the grand partition function by functional differentiation with respect to $\omega(k_0)$ and $\omega(k_0')$ (while holding the pair functions R constant).

$$\begin{aligned}P(k_0,k_0')&=\beta^{-2}e^{-\Omega f}\left[e^{-\beta\omega(k_0)}\frac{\delta}{\delta\omega(k_0')}e^{\beta\omega(k_0)}\frac{\delta}{\delta\omega(k_0)}e^{\Omega f}\right]_R\\&=\langle n(k_0)\rangle\langle n(k_0')\rangle+\beta^{-2}e^{-\beta\omega(k_0)}\left[\frac{\delta}{\delta\omega(k_0')}e^{\beta\omega(k_0)}\frac{\delta}{\delta\omega(k_0)}\Omega f\right]_R\\&=\langle n(k_0)\rangle\langle n(k_0')\rangle+\epsilon[\nu(k_0)]^2\delta_{k_0,k_0'}+2\epsilon\delta_{k_0,k_0'}[\nu(k_0)]^2[1+\epsilon\nu(k_0)]\left[\frac{\delta}{\delta\nu(k_0)}\Omega f_{\text{int}}\right]_R\\&\qquad+\nu(k_0)[1+\epsilon\nu(k_0)]\nu(k_0')[1+\epsilon\nu(k_0')]\left[\frac{\delta}{\delta\nu(k_0)}\frac{\delta}{\delta\nu(k_0')}\Omega f_{\text{int}}\right]_R,\end{aligned}\tag{44}$$

where

$$f=-\epsilon\sum\nolimits_k\ln\{1-\epsilon\exp[\beta(g-\omega_k)]\}+f_{\text{int}}.\tag{45}$$

Now, in LY IV and in Sec. I of II it is shown that functional derivatives with respect to $\nu(k)$ such as appear in the third line of Eq. (44) can be expressed in terms of contracted ζ-graphs ($\zeta=0,1,2,\cdots$). For example, one finds that

$$\left[\frac{\delta}{\delta\nu(k)}f_{\text{int}}\right]_R=\epsilon\sum(\text{all contracted 1 graphs})_k,\tag{46}$$

where the subscript k indicates that the external lines of the 1 graphs in (46) are all associated with the momentum k. It is also not difficult to show that

$$\begin{aligned}&\left[\frac{\delta}{\delta\nu(k')}\frac{\delta}{\delta\nu(k)}f_{\text{int}}\right]_R\\&=\epsilon\left[\frac{\delta}{\delta\nu(k')}\sum(\text{all contracted 1 graphs})_k\right]_R\\&=\sum(\text{all contracted 2 graphs})_{k,k'}\\&\qquad+\epsilon\delta_{k,k'}[\sum(\text{all contracted 1 graphs})_k]^2.\end{aligned}\tag{47}$$

If Eqs. (46) and (47) are now substituted into Eq. (44), then with the aid of Eq. (II.9) one can derive the following expression for the pair-correlation function

$$\begin{aligned}P(k,k')=&\langle n(k)\rangle\langle n(k')\rangle(1+\epsilon\delta_{k,k'})\\&+\nu(k)[1+\epsilon\nu(k)]\nu(k')[1+\epsilon\nu(k')]\\&\qquad\sum(\text{all contracted 2 graphs})_{k,k'}.\end{aligned}\tag{48}$$

This result is equivalent to theorems 1 and 2 in Appendix D of LY IV. It shows that the pair-correlation function is given by the product of two single-particle momentum distributions plus an interaction term which depends upon the details of the particle interactions. This second term varies as Ω^{-1} for a large Fermi system, and, therefore, it is only measurable for finite Fermi systems.

Before proceeding we observe that the momentum subscripts k and k' attached to the 2 graphs in Eq. (48) are associated with the incoming as well as with the outgoing external lines of these graphs. Moreover, two 2 graphs with the same external momenta k and k' and the same topological structure may or not differ by the assignments of their external momenta. This depends upon their internal symmetry. We finally note from Eq. (43), that

$$P(k,k')=P(k',k)\underset{k'=k}{\longrightarrow}0\quad[\text{for Fermi systems}],\tag{49}$$

which implies that this property must also be true for the second term in (48).

Equation (48) for the pair-correlation function is expressed in terms of the unphysical function $\nu(k)$, which we have shown to be zero for the ground state of a bound Fermi system. In analogy with the similar treatment of the momentum distribution in II, we now attempt to re-express the pair-correlation function in terms of $N(k)$ defined by Eq. (5). But this goal is easily achieved after introducing the concept of reducible and irreducible 2-graphs (see Sec. I of II).

Definition

A contracted 2 graph is called *reducible* if by cutting two of its (solid) internal lines open the entire graph can be separated into two (or more) contracted ζ graphs one of which is a contracted 1 graph. An *irreducible*

2 graph is a contracted 2 graph which is not reducible, with its (solid) internal lines representing factors $\epsilon N(k)$ instead of $\epsilon\nu(k)$.

It is a simple matter to generate the sum over all contracted 2 graphs from the sum over all irreducible 2 graphs. Thus, one may readily verify that the following identity is true.

$$\begin{aligned}[N(k)]^2[N(k')]^2 \sum(\text{all irreducible 2 graphs})_{k,k'} \\ =[\nu(k)]^2[\nu(k')]^2 \sum(\text{all contracted 2 graphs})_{k,k'}.\end{aligned} \tag{50}$$

If this identity is substituted into Eq. (48) and if then Eqs. (II.9) and (II.10) are used, then one obtains the following expression for the pair-correlation function

$$\begin{aligned}P(k,k')=\langle n(k)\rangle\langle n(k')\rangle(1+\epsilon\delta_{k,k'}) \\ +(1+\epsilon\langle n(k)\rangle(1+\epsilon\langle n(k')\rangle)N(k)N(k') \\ \sum(\text{all irreducible 2 graphs})_{k,k'}.\end{aligned} \tag{51}$$

Equation (51) is still not an acceptable expression for the pair-correlation function, even though the right-hand side is now a functional of $N(k)$ instead of $\nu(k)$. The reason is that according to Eq. (5) $N(k)$ is still not a truly physical quantity. In fact, $N(k)$ is zero for a bound Fermi system which means that the sum over all irreducible 2-graphs must be infinite. The essence of II, however, is how to deal with this particular infinity. Thus, we first write

$$\begin{aligned}\sum(\text{all irreducible 2 graphs}) \\ =\sum(\text{all irreducible linked-pair 2 graphs}),\end{aligned} \tag{52}$$

as we have done for 1 graphs at the beginning of Sec. II in II. We then write down the easily verified expression

$$\begin{aligned}\sum(\text{all irreducible linked-pair 2 graphs})_{k,k'} \\ =\int_0^\beta dt_4dt_3dt_2dt_1\, L_2(\beta\beta|t_4t_3|kk') \\ \times G(t_4,t_2,k)G(t_3,t_1,k'),\end{aligned} \tag{53}$$

where $G(t_2,t_1,k)$ is defined by Eq. (II.22), and where

$$\int_0^\beta dt_4dt_3\, L_2(\beta\beta|t_4t_3|kk') =\sum(\text{all master 2 graphs})_{k,k'}. \tag{54}$$

Master 2 graphs are defined at the beginning of Sec. III in II, and the quantity $L_2(\beta\beta|t_4t_3|kk')$ is defined in terms of master 2 graphs by Eq. (54). This last definition can be stated in words as follows:

$$L_2(\beta\beta|t_4t_3|kk')\equiv\sum(\text{all master }L2\text{ graphs})_{k,k'}, \tag{55}$$

where a (transformed or untransformed) *master L2 graph* is defined to be a (transformed or untransformed) master 2-graph in which (1) the integrations over the temperature variables (t_3 and t_4) at the vertices to which the *incoming* external lines attach are not performed, and with which (2) a factor $\delta(t_4-t_3)$ is included if both incoming lines attach at the same vertex.

We now have the ingredients assembled for casting Eq. (51) into a more useful form. The most important step in II is the use of the Λ transformation, in which quantities are re-expressed in terms of $N'(k)$, Eq. (6), and transformed pair functions, instead of $N(k)$ and ordinary pair-functions. In order to apply the Λ transformation to the present problem we first define a *transformed master 2 graph* to be a master 2 graph expressed in terms of $N'(k)$ and transformed pair functions. We then define

$$L_2'(\beta\beta|t_4t_3|kk')\equiv\sum(\text{all transformed master }L2\text{-graphs})_{k,k'} \tag{56}$$

in analogy with (55), where transformed master $L2$-graphs are defined below (55). It is then simply a matter of using Eqs. (II.83), (II.75), (II.62), (II.63), and (II.64) to derive the following relation connecting L_2 and L_2'.

$$\begin{aligned}N(k)N(k')\int_0^\beta dt_4dt_3dt_2dt_1\, L_2(\beta\beta|t_4t_3|kk') \\ \times G(t_4,t_2,k)G(t_3,t_1,k') \\ =N'(k)N'(k')\int_0^\beta dt_4dt_3dt_2dt_1\, L_2'(\beta\beta|t_4t_3|kk') \\ \times G'(t_4,t_2,k)G'(t_3,t_1,k'),\end{aligned} \tag{57}$$

where

$$G'(t_2,t_1,k)=\delta(t_2-t_1)+\epsilon L'(t_2,t_1,k), \tag{58}$$

and $L'(t_2,t_1,k)$ is the function calculated in Sec. II of the present paper.

We substitute Eq. (57) into Eq. (51) to obtain our final general expression for the pair-correlation function $P(k,k')$.

$$\begin{aligned}P(k,k')=\langle n(k)\rangle\langle n(k')\rangle(1+\epsilon\delta_{k,k'}) \\ +[1+\epsilon\langle n(k)\rangle][1+\epsilon\langle n(k')\rangle]N'(k)N'(k') \\ \times\int_0^\beta dt_4dt_3dt_2dt_1 L_2'(\beta\beta|t_4t_3|kk') \\ \times G'(t_4,t_2,k)G'(t_3,t_1,k').\end{aligned} \tag{59}$$

It would seem at first sight that Eq. (59) represents no improvement over Eq. (51), since we know from Eq. (6) that $N'(k)$ is not much more physical than $N(k)$. The difference, of course, is that by means of Eq. (15) we have a useful iteration for $N'(k)$ in terms of the *physical* quantity $\nu'(k)$. Thus, we can obtain the leading term of the pair-correlation function by making the

replacements $\langle n(k)\rangle \cong \nu'(k)$, $N'(k)\cong\nu'(k)$, and

$$G'(t_2,t_1,k)\cong\delta(t_2-t_1)$$

in the second term on the right-hand side of Eq. (59).

$$\begin{aligned}P(k,k')=&\langle n(k)\rangle\langle n(k')\rangle(1-\delta_{k,k'})\\&+\nu'(k)\nu'(k')[1-\nu'(k)][1-\nu'(k')]\\&\times\int_0^\beta dt_2dt_1\,L_2'(\beta\beta|t_2t_1'|kk')\\&+(\text{3-pair terms})+\cdots.\end{aligned}\quad(60)$$

We wish to apply Eq. (60) to the determination of the pair-correlation function in nuclear matter. For this purpose we use the method of Sec. II to calculate the one- and two-pair contributions to $L_2'(\beta\beta|t_2t_1|kk')$. When this is done, and when restriction (22) is again adopted, then one quickly finds that the 1-pair contribution to $P(k,k')$ is zero at $T=0$. The reason is that $\nu_k'(1-\nu_k')=0$ at $T=0$, so that $\int_0^\beta dt_2dt_1\,L_2'(t_2,t_1)$ must be infinite (which is not the case for the one-pair term) in order to give a nonzero contribution to $P(k,k')$. This zero-temperature infinity is of a very simple nature, such as that which occurs in Eq. (24) for $L_>'^{(2)}$, and it must be contrasted with the zero temperature infinity which occurs in Eq. (51) in the sum over all irreducible 2 graphs. (The latter is formally treated by the Λ transformation, but its complexity can be fully appreciated by studying Sec. II of II.)

We have just indicated that the 2-pair terms give the leading contribution to the pair-correlation function (60). A considerable amount of algebra followed by the application of identities (26) and (28) eventually yields the dominant low-temperature terms in the two pair approximation. The final result is

$$\begin{aligned}P(k_1,k_2)=&\langle n(k_1)\rangle\langle n(k_2)\rangle(1-\delta_{k_1,k_2})\\&+\tfrac{1}{2}[1-\nu'(k_1)][1-\nu'(k_2)]\sum_{k_3k_4}\nu'(k_3)\nu'(k_4)\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)^2h_1(k_1k_2|k_3k_4)\mathfrak{g}_1(k_3k_4|k_1k_2)\\&+\tfrac{1}{2}\nu'(k_1)\nu'(k_2)\sum_{k_3k_4}[1-\nu'(k_3)][1-\nu'(k_4)]\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)^2\mathfrak{g}_1(k_1k_2|k_3k_4)h_1(k_3k_4|k_1k_2)\\&-\nu'(k_1)[1-\nu'(k_2)]\sum_{k_3k_4}\nu'(k_3)[1-\nu'(k_4)]\left(\frac{1}{\omega_2'+\omega_4'-\omega_1'-\omega_3'}\right)^2\mathfrak{g}_1(k_1k_3|k_2k_4)h_1(k_2k_4|k_1k_3)\\&-[1-\nu'(k_1)]\nu'(k_2)\sum_{k_3k_4}[1-\nu'(k_3)]\nu'(k_4)\left(\frac{1}{\omega_2'+\omega_4'-\omega_1'-\omega_3'}\right)^2h_1(k_1k_3|k_2k_4)\mathfrak{g}_1(k_2k_4|k_1k_3)\\&+(\text{3-pair terms})+\cdots.\end{aligned}\quad(61)$$

We shall interpret this result in Sec. VI.

V. COMPARISON WITH THE BRUECKNER METHOD

We wish to compare our work with the Brueckner method[5] for the calculation of the properties of nuclear matter. It is clear that the expansion functions for the two methods are different. In the Brueckner method, the expansion function is the "nuclear reaction matrix" G_B, which is the solution to the Bethe-Goldstone equation,[9] whereas in the present method the expansion function is the transformed pair-function (12). Nevertheless, the methods may be compared by expressing both expansion functions in terms of the two-particle interaction. This we shall now do.

It is important to emphasize in connection with the quantum statistical procedure that the pair-functions of Eqs. (9) and (12) are related to the matrix elements of $R(t_2,t_1)$, Eq. (8), and therefore are expressible in terms of the eigenfunctions of the two-particle Hamiltonian $H^{(2)}$. Such eigenfunctions are, in general, difficult to determine so that in Sec. I we have written down their general form in terms of reaction matrices. Thus, the radial wave function corresponding to the Lth partial wave, can be written for spin-independent forces, as

$$\langle r|k_0L\rangle=\cos\delta_L(k_0)\int_0^\infty dk\,F_L(kr)\left\{\delta(k_0-k)+2\pi^{-1}kP\left[\frac{\langle k|A^{(L)}|k_0\rangle}{k^2-k_0^2}\right]\right\}\underset{r\to\infty}{\longrightarrow}\sin[k_0r-L\pi/2+\delta_L(k_0)]\quad(62)$$

[9] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1957).

for continuum states, and as

$$\langle r|\gamma L\rangle = 2\pi^{-1}\int_0^\infty dk\, F_L(kr)\phi_{\gamma L}(k), \tag{63}$$

for bound states. Thus, we see how the reaction matrix enters into our procedure.

We next prove an important identity. Except for the special case in which the two-particle interaction includes an infinite repulsive core, the wave functions (62) and (63) form a complete set of eigenfunctions over all space. When this fact is used together with identity (28), then the following completeness condition can be verified[10]

$$(k^2-l^2)^{-1}[C_L(k,l)-C_L(l,k)]\equiv 0, \tag{64}$$

where

$$\begin{aligned}C_L(k,l)\equiv{}& k\langle k|A^{(L)}|l\rangle\cos^2\delta_L(l)\\ &+(2/R)(\hbar^2/M)kl\sum_{k_0}\langle k|A^{(L)}|k_0\rangle\\ &\times\langle l|A^{(L)}|k_0\rangle\cos^2\delta_L(k_0)P\left[\frac{1}{\omega(l)-\omega(k_0)}\right]\\ &+\textstyle\sum_\gamma(k^2+\gamma^2)\phi_{\gamma L}(k)\phi_{\gamma L}(l).\end{aligned} \tag{65}$$

The first sum in Eq. (65) is over continuum states k_0, and the second sum is over bound states γ (R is the radius of a large spherical box containing the system). When this completeness condition is substituted into Eq. (30) for $h_1(k_1k_2|k_3k_4)$, then one can readily verify that

$$h_1(k_1k_2|k_3k_4)=\mathfrak{g}_1(k_3k_4|k_1k_2), \tag{66}$$

where $\mathfrak{g}_1(k_1k_2|k_3k_4)$ is given by Eq. (13) and both of these functions are real in the representation which we have chosen.

The identity (66) can be substituted into expression (37) for the energy per particle.

$$\begin{aligned}\frac{\langle E\rangle_0}{\langle N\rangle}={}&(\rho\Omega)^{-1}\sum_k\nu'(k)\omega(k)-(2\rho\Omega)^{-1}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_1k_2)-(4\rho\Omega)^{-1}\sum_{k_1k_2k_3k_4}\nu'(k_1)\nu'(k_2)[\mathfrak{g}_1(k_1k_2|k_3k_4)]^2\\ &\times\left[P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)-P\left(\frac{1}{\omega_3+\omega_4-\omega_1'-\omega_2'}\right)-[\nu'(k_3)+\nu'(k_4)]P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\right]\\ &+(\text{3-pair terms})+\cdots.\end{aligned} \tag{67}$$

We see that the proper expansion function for the ground state of nuclear matter is the matrix $\mathfrak{g}_1(k_1k_2|k_3k_4)$ *of Eq.* (13).

We next derive an expression for the matrix $\mathfrak{g}_1(k_1k_2|k_3k_4)$ in terms of the two-body interaction matrix $\langle k_1k_2|V|k_3k_4\rangle$. For this purpose it is useful to first express $\mathfrak{g}_1(k_1k_2|k_3k_4)$ in terms of the "outgoing" reaction matrix elements $\langle k_1k_2|G^{(+)}|k_3k_4\rangle$ which are defined by the integral equation

$$\langle k_1k_2|G^{(+)}|k_3k_4\rangle\equiv\langle k_1k_2|V|k_3k_4\rangle+\sum_{k_5k_6}\langle k_1k_2|V|k_5k_6\rangle\left(\frac{1}{\omega_3+\omega_4-\omega_5-\omega_6+i\epsilon}\right)\langle k_5k_6|G^{(+)}|k_3k_4\rangle. \tag{68}$$

We consider the complete set of "outgoing" wave functions in the momentum representation.

$$\langle k_1k_2|k_3k_4\rangle^{(+)}=(2\pi)^6\delta^{(3)}(\mathbf{k}_1-\mathbf{k}_3)\delta^{(3)}(\mathbf{k}_2-\mathbf{k}_4)\delta_{m_1m_3}\delta_{m_2m_4}\delta_{q_1q_3}\delta_{q_2q_4}+\langle k_1k_2|G^{(+)}|k_3k_4\rangle\left(\frac{1}{\omega_3+\omega_4-\omega_1-\omega_2+i\epsilon}\right), \tag{69}$$

and

$$\langle k_1k_2|K_{34}\gamma\rangle=(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{K}_{34})<k_{12}|\gamma\rangle$$

where the normalization of the bound state wave functions $\langle k_{12}|\gamma\rangle$ is given by

$$\sum_{m_1m_2}\sum_{q_1q_2}\int d^3k_{12}\langle\gamma|k_{12}\rangle\langle k_{12}|\gamma\rangle=(2\pi)^3. \tag{70}$$

Then one can show by taking the matrix elements of $R(t_2,t_1)$, Eq. (8), with this complete set and by then using Eqs. (9) and (13) that $\mathfrak{g}_1(k_1k_2|k_3k_4)$ is given by

[10] Equation (64) is equivalent to the condition $\langle k|U_L(0,0)|l\rangle\equiv 0$, where the matrix elements of U_L are given by Eq. (I.75).

$$\mathfrak{g}_1(k_1k_2|k_3k_4)=-\langle k_3k_4|G^{(+)}|k_1k_2\rangle^*+\sum_{k_5k_6}\langle k_1k_2|G^{(+)}|k_5k_6\rangle\langle k_3k_4|G^{(+)}|k_5k_6\rangle^*$$
$$\times\left[\left(\frac{1}{\omega_1+\omega_2-W_{56}-\omega_{56}-i\epsilon}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-W_{56}-\omega_{56}}\right)\right]$$
$$+\sum_{K_{56}\gamma}(\omega_1+\omega_2-W_{56}-\omega_\gamma)(\omega_3+\omega_4-W_{56}-\omega_\gamma)\langle k_1k_2|K_{56}\gamma\rangle$$
$$\times\langle K_{56}\gamma|k_3k_4\rangle\left[\left(\frac{1}{\omega_1+\omega_2-W_{56}-\omega_\gamma}\right)-\left(\frac{1}{\omega_1'+\omega_2'-W_{56}-\omega_\gamma}\right)\right]$$
$$+(\text{antisymmetric expression}),\qquad(71)$$

where W_{56} is the center-of-mass energy corresponding to the momentum $\mathbf{K}_{56}=\mathbf{k}_5+\mathbf{k}_6$.

Equation (71) for $\mathfrak{g}_1(k_1k_2|k_3k_4)$ can be simplified by using the expression which results from substituting Eq. (69) into Eq. (68)

$$\langle k_1k_2|G^{(+)}|k_3k_4\rangle=\sum_{k_5k_6}\langle k_1k_2|V|k_5k_6\rangle\langle k_5k_6|k_3k_4\rangle^{(+)}$$

together with the Schrödinger equation for the bound-state wave functions

$$(\omega_1+\omega_2-W_{34}-\omega_\gamma))k_1k_2|K_{34}\gamma\rangle=-\sum_{k_5k_6}\langle k_1k_2|V|k_5k_6\rangle\langle k_5k_6|K_{34}\gamma\rangle.$$

With the aid of these two expressions one can show that $\mathfrak{g}_1(k_1k_2|k_3k_4)$ can be written as

$$\mathfrak{g}_1(k_1k_2|k_3k_4)=-\langle k_3k_4|G^{(+)}|k_1k_2\rangle^*+\left\langle k_1k_2\middle|V\left[\left(\frac{1}{\omega_1+\omega_2-H^{(2)}-i\epsilon}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\right)\right]V\middle|k_3k_4\right\rangle$$
$$+(\text{antisymmetric expression}).\qquad(72)$$

Equation (72) can be written as

$$\mathfrak{g}_1(k_1k_2|k_3k_4)=-\left\langle k_3k_4\middle|V\left[1+\left(\frac{1}{\omega_1+\omega_2-H^{(2)}+i\epsilon}\right)V\right]\middle|k_1k_2\right\rangle^*$$
$$+\left\langle k_3k_4\middle|V\left[\left(\frac{1}{\omega_1+\omega_2-H^{(2)}+i\epsilon}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\right)\right]V\middle|k_1k_2\right\rangle^*+(\text{antisymmetric expression})$$
$$=-\left\langle k_3k_4\middle|V\left[1+P\left(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\right)\right]V\middle|k_1k_2\right\rangle^*+(\text{antisymmetric expression})$$
$$=-\langle k_3k_4|G_1|k_1k_2\rangle^*+\langle k_4k_3|G_1|k_1k_2\rangle^*,\qquad(73)$$

where

$$G_1=V+VP\left(\frac{1}{E'-H^{(2)}}\right)V.\qquad(74)$$

Equation (73) exhibits the relationship between the function $\mathfrak{g}_1(k_1k_2|k_3k_4)$ and the matrix elements of the operator G_1 defined by Eq. (74).[11]

In the Brueckner method the ground-state energy is assumed to be expandable in terms of the reaction matrix G_B, which is defined to be the solution to the Bethe-Goldstone equation.

$$\langle k_1k_2|G_B|k_3k_4\rangle=\langle k_1k_2|V|k_3k_4\rangle+\sum_{k_5k_6>k_F}\langle k_1k_2|V|k_5k_6\rangle\left(\frac{1}{\omega_3'+\omega_4'-\omega_5'-\omega_6'}\right)\langle k_5k_6|G_B|k_3k_4\rangle.\qquad(75)$$

It is seen from the preceding three equations that the difference between the present theory of nuclear matter and the Brueckner method resides in the use of different reaction matrices as expansion functions for physical quantities. The fact that we have arrived at the matrix G_1 is a direct consequence of the Λ transformation introduced in II.

[11] The author is greatly indebted to Dr. J. S. Bell of CERN Laboratories for indicating to him and proving the result given as Eq. (73).

From Eqs. (71) and (72) one can derive an alternate form for the function $\mathfrak{g}_1(k_1k_2|k_3k_4)$. Thus, one can show that

$$\mathfrak{g}_1(k_1k_2|k_3k_4)=f_1(k_1k_2|k_3k_4)$$
$$+\left\langle k_3k_4\left|V\left[P\left(\frac{1}{\omega_1+\omega_2-H^{(2)}}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\right)\right]V\right|k_1k_2\right\rangle^*$$
$$-\left\langle k_4k_3\left|V\left[P\left(\frac{1}{\omega_1+\omega_2-H^{(2)}}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\right)\right]V\right|k_1k_2\right\rangle^*$$
$$f_1(k_1k_2|k_3k_4)=-\langle k_3k_4|G^{(+)}|k_1k_2\rangle^*$$
$$+i\pi\sum_{k_5k_6}\delta(\omega_1+\omega_2-\omega_5-\omega_6)\langle k_1k_2|G^{(+)}|k_5k_6\rangle\langle k_3k_4|G^{(+)}|k_5k_6\rangle^*+(\text{antisymmetric function}). \qquad (76)$$

According to Eq. (67) the leading contribution to the ground-state interaction energy involves only the diagonal matrix elements $\mathfrak{g}_1(k_1k_2|k_1k_2)$. Moreover, from Eq. (76) one observes that quantity $f_1(k_1k_2|k_1k_2)$ is the real part of the forward scattering amplitude [see also Eqs. (10) and (11) for spin-independent forces], which is an experimentally measurable quantity. The difference between $\mathfrak{g}_1$ and f_1 is a function which when written in the form of Eq. (13) exhibits a strong cutoff for high-momentum intermediate states. One might therefore hope that a perturbation expansion of this term would converge sufficiently rapidly so that it could be simply calculated. Whether or not Eq. (76) exhibits the function $\mathfrak{g}_1(k_1k_2|k_3k_4)$ in a form which is most suitable for calculation is a question which remains to be answered. It is clear, however, that in the calculation of the diagonal elements the isolation of an experimentally determinable part is a useful step.

VI. QUASI-PARTICLE MODEL AND THE LANDAU THEORY

We suppose that to first approximation the wave function of nuclear matter can be written as an antisymmetrized combination of products of single-particle wave functions, and that for a very large system these single particle wave functions are plane waves. Let b_k and $b_k^\dagger$ be, respectively, the annihilation and creation operator for a plane wave state in nuclear matter. We shall refer to such a state as a *quasi-particle state.* We shall also assume that the energy-momentum relation for quasi-particle states is given by Eq. (7),[12] i.e.,

$$\omega'(k)=\omega(k)+\Delta(k,\beta). \qquad (77)$$

The true ground-state wave function in nuclear matter $|G\rangle$ is, of course, not a simple antisymmetrized product $|0\rangle$ of quasi-particle states, because the quasi-particles interact. Let us now suppose that the interaction Hamiltonian for the quasi-particles is given by

$$H_{\text{q.p.}}'=-\tfrac{1}{4}\sum_{k_1k_2k_3k_4}b_3^\dagger b_4^\dagger b_1b_2\mathfrak{g}_1(k_1k_2|k_3k_4), \qquad (78)$$

where $\mathfrak{g}_1(k_1k_2|k_3k_4)$ is given by Eq. (13), and where $k=(\mathbf{k},m,q)$ denotes both momentum and spin state coordinates. The interaction Hamiltonian (78) is not Hermitian, because $\mathfrak{g}_1(k_3k_4|k_1,k_2)\neq\mathfrak{g}_1(k_1k_2|k_3k_4)$.

The first-order correction to the quasi-particle wave function $|0\rangle$ due to the interaction (78) gives the wave function $|1\rangle$ which can be written as follows

$$|1\rangle=N\left\{1+\tfrac{1}{4}\sum_{k_3,k_4>k_F}\sum_{k_1,k_2<k_F}\left[\frac{\mathfrak{g}_1(k_1k_2|k_3k_4)}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right]\times b_3^\dagger b_4^\dagger b_1b_2\right\}|0\rangle,$$

$$N^{-2}=1+\tfrac{1}{4}\sum_{k_3,k_4>k_F}\sum_{k_1,k_2<k_F}\left[\frac{\mathfrak{g}_1(k_1k_2|k_3k_4)}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right]^2. \qquad (79)$$

We can use Eq. (79) to calculate both the momentum distribution and the pair-correlation function that one woud obtain as a consequence of the hypothesis (78). For this purpose we use the following definitions of these distribution functions:

$$\langle n(k)\rangle\equiv\langle G|b_k^\dagger b_k|G\rangle,$$
$$\langle n(k)n(k')\rangle\equiv\langle G|b_k^\dagger b_{k'}^\dagger b_{k'}b_k|G\rangle, \qquad (80)$$

where according to Eq. (43) $P(k,k')=\langle n(k)n(k')\rangle$.

If the approximation $|G\rangle\cong|1\rangle$ is made in Eqs. (80) and if Eq. (66) is used, then one readily obtains Eq. (29) for $\langle n(k)\rangle$ and Eq. (61) for $\langle n(k)n(k')\rangle$. Therefore (79) is a correct approximation to the wave function of nuclear matter. We conclude that (provided higher order corrections are small) nuclear matter is correctly described by the model of quasi-particles interacting with the interaction (78).

[12] The quasi-particle energy $\omega'(k)$ is *not* the energy-momentum relation for a collective excitation in nuclear matter. However, for an excitation of momentum $(\mathbf{k}_2-\mathbf{k}_1)$, where $k_1<k_F<k_2$, the real part of the excitation energy is given by $[\omega'(k_2)-\omega'(k_1)]$, as can readily be verified to leading orders in $\mathfrak{g}_1$. The imaginary part of the excitation energy, which corresponds to the inverse of the excitation lifetime, is $O(\mathfrak{g}_1^2)$ to leading order. In this connection, see N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

We would next like to interpret Eq. (67) for the energy of nuclear matter in terms of the quasi-particle model. The most general expression for the energy can be written as

$$\langle E\rangle=\sum_k \nu'(k)\omega(k)+\tfrac{1}{2}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)u(k_1,k_2). \quad (81)$$

The first order perturbation theory calculation of $u(k_1,k_2)$, using the interaction (78), is

$$u_1(k_1,k_2)=-\mathfrak{g}_1(k_1k_2|k_1k_2), \quad (82)$$

which agrees with Eq. (67). Moreover, each quasi-particle experiences a potential energy due to interaction with the other quasi-particles which in the notation of (77) can be written

$$\Delta_1(k_1)=-\sum_{k_2}\nu'(k_2)\mathfrak{g}_1(k_1k_2|k_1k_2)=\sum_{k_2}\nu'(k_2)u_1(k_1,k_2). \quad (83)$$

This result is in agreement with Eq. (31).

If the only important correlations in nuclear matter were those between two quasi-particles, then it would in general be true that $\Delta(k_1)=\sum_{k_2}\nu'(k_2)u(k_1,k_2)$. This is not the case, however, and therefore the potential energy which a quasiparticle experiences in nuclear matter includes a second term due to higher order correlations. The most general expression for $\Delta(k)$ is

$$\Delta(k_0)=\sum_{k_2}\nu'(k_2)u(k_0,k_2)+\tfrac{1}{2}\sum_{k_1k_2}\nu'(k_1)\nu'(k_2)\frac{\delta u(k_1,k_2)}{\delta\nu'(k_0)}, \quad (84)$$

where the second term has been called the *rearrangement energy*.[13]

We have already demonstrated Eq. (84) explicitly to second order in $\mathfrak{g}_1(k_1k_2|k_3k_4)$ in the derivation of Eq. (41). From Eq. (67) we have for $u(k_1,k_2)$ to second order in $\mathfrak{g}_1$:

$$\begin{aligned}u_2(k_1,k_2)&=-\mathfrak{g}_1(k_1k_2|k_1k_2)+\tfrac{1}{2}\sum_{k_3k_4}[1-\nu'(k_3)-\nu'(k_4)]\\&\quad\times[\mathfrak{g}_1(k_1k_2|k_3k_4)]^2P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\\&\quad-\tfrac{1}{2}\sum_{k_3k_4}[\mathfrak{g}_1(k_1k_2|k_3k_4)]^2P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3-\omega_4}\right). \quad (85)\end{aligned}$$

This expression agrees with Eq. (39) in the momentum region $[\omega'(k_1)+\omega'(k_2)]<$(binding energy of deuteron), but for higher momenta it is incorrect and we must use Eq. (39) instead.

[13] The concept of a rearrangement energy was first introduced by D. J. Thouless, Phys. Rev. **112**, 906 (1958).

We now discuss the second and third terms of Eq. (85) for $u_2(k_1,k_2)$. The sums in these two terms are over the intermediate plane wave states k_3 and k_4. In fact, if $H_{\text{q.p.}}'$ were Hermitian, then the second term would be exactly the second order perturbation theory contribution of $H_{\text{q.p.}}'$. (Note that $(1-\nu_3')(1-\nu_4')=1-\nu_3'-\nu_4'+\nu_3'\nu_4'$ and that a $\nu_3'\nu_4'$ term in Eq. (81) vanishes when H' is Hermitian). Thus, we can understand the origin of the second term in (85). The third term is what we would get from second order perturbation theory by summing, without regard to the Pauli principle, over *free* nucleon intermediate states instead of quasi-particle intermediate states. The second order contribution to $u(k_1,k_2)$ is the difference between these two effects, and this difference vanishes for large k_3 and k_4 values where $\Delta(k_3)$ and $\Delta(k_4)$ are (expected to be) negligibly small. We conclude that the total second order contribution to (85) is a sum over intermediate states involving momenta which are *not* $\gg k_F$. High momentum intermediate states are unimportant!

The quasi-particle model of nuclear matter which we have just presented follows rigorously from the theory of the grand canonical ensemble in quantum statistics. This model also seems to be equivalent to the Landau Theory of the Fermi liquid,[6] although we have only explicitly exhibited this fact to second order in $\mathfrak{g}_1(k_1k_2|k_3k_4)$. Thus, Eqs. (81) and (84) are the general expressions for the system energy and the quasi-particle potential energy in the Landau Theory.[14] It should be emphasized, however, that there are important differences between our work and the Landau theory. Thus we have shown that the quasi-particle momentum distribution and pair-correlation function in nuclear matter are given, respectively, by Eqs. (29) and (61), whereas in the Landau theory the distribution functions take their free-particle forms. The free particle distribution functions are, of course, correct averages, and we have shown that the "average" momentum distribution occurs as the weighting function in expressions for thermodynamic quantities. We conclude that one cannot invert such expressions to obtain the true distribution functions by functional differentiation, as is done in the Landau theory, unless one is extremely careful. For example, if one uses Eq. (29) in the expression for $N=\sum_k\langle n(k)\rangle$, then by functional differentiation with respect to $\nu'(k)$ one obtains $\langle n(k)\rangle$ for the quasi-particle momentum distribution. Alternatively, the second term in this summation is actually zero so that one also has $N=\sum_k\nu'(k)$. Functional differentiation of this last expression yields the result that the quasi-particle momentum distribution is $\nu'(k)$, which is incorrect.

We suggest that perhaps the present quasi-particle model is applicable not only to the ground state of

[14] A general proof of the microscopic form of the Landau theory has been investigated by R. Balian and C. DeDominicis, Nuclear Phys. **16**, 502 (1950); Compt. rend. **3285**, 4111 (1960); and by A. Klein, Phys. Rev. **112**, 957 (1961).

nuclear matter, but also to the determination of the very low-temperature properties of liquid He^3.

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APPENDIX—TREATMENT OF INFINITE REPULSIVE CORE

The phenomenological analysis of nucleon-nucleon scattering leads to the current picture of a nucleon as a particle with a hard core of diameter a. Such a core can be pictured as a repulsive barrier which is higher than any of the c.m. scattering energies now available. Suppose that the barrier height is 1 Bev. It then becomes sensible to use the mathematical idealization of an infinite repulsive core in nuclear problems. In particular, for nuclear matter $E_F \sim 50$ Mev$\ll$1 Bev and therefore there are no momentum components in nuclear matter which are sensitive to the details of a core of this height.

The use of nuclear wave functions corresponding to an infinite repulsive core interaction leads to a $\delta(t_2-t_1)$ term in the pair function (9), as has been shown in Sec. V of I. To include this case we have therefore written the pair function in Eq. (II.70) as

$$
{}^{t_2}\begin{bmatrix} k_1k_2 \\ \\ k_3k_4 \end{bmatrix}_{t_1} = [\exp t_1(\omega_1+\omega_2-\omega_3-\omega_4)]f_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}[\exp t_2(\omega_1+\omega_2-\omega_5-\omega_6)][\exp t_1(\omega_5+\omega_6-\omega_3-\omega_4)]
$$
$$
\times f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)+\delta(t_2-t_1)[\exp t_2(\omega_1+\omega_2-\omega_3-\omega_4)]f_3(k_1k_2|k_3k_4), \quad \text{(A.1)}
$$

where the summation refers to *all* states of the two-body Hamiltonian $H^{(2)}$. The functions f_1 and f_2 are still given by Eqs. (10) and (11). According to Eqs. (I.45), (I.78), (I.75), (64), and (65), f_3 is given by

$$
f_3(k_1k_2|k_3k_4)=[(2\pi)^3/\Omega]^2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\tilde{f}_3(k_{12}|k_{34}), \quad \text{(A.2)}
$$

where for a spin-independent interaction

$$
\tilde{f}_3(k_{12}|k_{34})=(2\pi^2)^{-1}\delta_{m_1m_3}\delta_{m_2m_4}\delta_{q_1q_3}\delta_{q_2q_4}\sum_{L=0}^{\infty}(2L+1)P_L(\hat{n}_{12}\cdot\hat{n}_{34})(2\pi)^{-1}\langle k_{12}|U_L(0,0)|k_{34}\rangle
$$
$$
+(\text{antisymmetric expression}), \quad \text{(A.3)}
$$
$$
(2\pi)^{-1}\langle k|U_L(0,0)|l\rangle=(kl)^{-1}(k^2-l^2)^{-1}[C_L(k,l)-C_L(l,k)]\to 0 \quad \text{for a finite repulsive core.}
$$

We have also shown with Eq. (II.71) that the transformed pair function for this more general case is

$$
{}^{t_2}\begin{bmatrix} k_1k_2 \\ \\ k_3k_4 \end{bmatrix}'_{t_1} = [1+\epsilon B(k_1)][1+\epsilon B(k_2)]\{[\exp t_1(\omega_1'+\omega_2'-\omega_3'-\omega_4')]\mathfrak{g}_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}[\exp t_2(\omega_1'+\omega_2'-\omega_5-\omega_6)]
$$
$$
\times[\exp t_1(\omega_5+\omega_6-\omega_3'-\omega_4')]f_2(k_1k_2|k_5k_6|k_3k_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)
$$
$$
+\delta(t_2-t_1)[\exp t_1(\omega_1'+\omega_2'-\omega_3'-\omega_4')]f_3(k_1k_2|k_3k_4)\}, \quad \text{(A.4)}
$$

where instead of Eq. (13) we have for $\mathfrak{g}_1(k_1k_2|k_3k_4)$

$$
\mathfrak{g}_1(k_1k_2|k_3k_4)=f_1(k_1k_2|k_3k_4)+\sum_{k_5k_6}f_2(k_1k_2|k_5k_6|k_3k_4)\left[P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)-P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\right]
$$
$$
+\epsilon[\Delta(k_1)+\Delta(k_2)]f_3(k_1k_2|k_3k_4). \quad \text{(A.5)}
$$

The function $B(k)$ is defined by Eqs. (II.90) and (II.91), and we now obtain its leading contribution. This comes from the one-pair-function term and is simply

$$
B^{(1)}(k_1)=2\epsilon\Omega^{-1}[1+\epsilon B(k_1)]\sum_{k_2}\nu'(k_2)[1+\epsilon B(k_2)]\sum_{L=0}^{\infty}(2L+1)[1+\epsilon\delta_{m_1m_2}\delta_{q_1q_2}(-1)^L]\langle k_{12}|U_L(0,0)|k_{12}\rangle. \quad \text{(A.6)}
$$

If one approximates $B(k)=B^{(1)}(k)$ on the right-hand side of this last expression, then it can readily be shown that the function

$$
D^{(1)}(k)\equiv 1+\epsilon B^{(1)}(k) \quad \text{(A.7)}
$$

satisfies the integral equation (setting $\epsilon=-1$)

$$D^{(1)}(k)=[1+\Phi(k,D^{(1)})]^{-1}, \tag{A.8}$$

where

$$\Phi(k_1,D^{(1)})\equiv-2\Omega^{-1}\sum_{k_2}\nu'(k_2)D^{(1)}(k_2)\sum_{\mathbf{K}=0}^{\infty}(2L+1)[1-\delta_{m_1m_2}\delta_{q_1q_2}(-1)^L]\langle k_{12}|U_L(0,0)|k_{12}\rangle. \tag{A.9}$$

We next study the function $\langle k|U_L(0,0)|k\rangle$. One can show quite generally that this is simply the "excluded volume integral"

$$\begin{aligned}\langle k|U_L(0,0)|l\rangle&=-2\pi(kl)^{-1}\int_0^a dr\,F_L(kr)F_L(lr),\\ F_L(\rho)&=\rho j_L(\rho)\underset{\rho\to\infty}{\longrightarrow}\sin(\rho-L\pi/2).\end{aligned} \tag{A.10}$$

This integral can be evaluated to give

$$\begin{aligned}\langle k|U_L(0,0)|l\rangle&=-2\pi(kl)^{-1}(k^2-l^2)^{-1}[lF_{L-1}(la)F_L(ka)-kF_{L-1}(ka)F_L(la)],\quad \text{if } l\neq k,\\ &=-\pi ak^{-2}[F_{L-1}(ka)F_{L+1}(ka)-F_L{}^2(ka)]<0\quad \text{if } l=k,\end{aligned} \tag{A.11}$$

where $F_{L-1}(x)$ is defined to be $\cos x$ when $L=0$. One can also verify the following sum rule for the matrix elements $\langle k|U_L(0,0)|l\rangle$.

$$\begin{aligned}\sum_{L=0}^{\infty}(2L+1)P_L(\hat{n}_k\cdot\hat{n}_l)\langle k|U_L(0,0)|l\rangle&=-2\pi K^{-3}[\sin Ka-Ka\cos Ka]\quad \text{if } \mathbf{k}\neq\mathbf{l},\\ &=-\tfrac{2}{3}\pi a^3\quad \text{if } \mathbf{k}=\mathbf{l},\end{aligned} \tag{A.12}$$

where $\mathbf{K}=\mathbf{k}-\mathbf{l}$. Finally, we write down the diagonal matrix element $\langle k|U_L(0,0)|k\rangle$ for the special case $L=0$

$$\langle k|U_L(0,0)|k\rangle=\pi ak^{-2}[(2ka)^{-1}\sin 2ka-1]\rightarrow-\tfrac{2}{3}\pi a^3[1+O(ka)^2]\quad \text{if } (ka)\ll 1 \tag{A.13}$$

We see that when $k_Fa\ll 1$, Eq. (A.9) is essentially an excluded volume correction, i.e., it is $\sim\rho a^3$ (for nuclear matter $k_Fa\cong 0.6$).

We observe from Eq. (A.11) that $\langle k|U_L(0,0)|k\rangle$ is negative definite, and therefore that $\Phi(k_1,D^{(1)})$, Eq. (A.9), is positive definite. This implies that $D^{(1)}(k)$ is less than unity. This factor therefore operates in Eq. (A.4) so as to *reduce* the magnitude of the transformed pair-function when the latter is calculated using infinite hard-core wave functions.