Shape-Independent Theory of High-Energy Nucleon-Nucleon Scattering*

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A shape-independent formula for nucleon-nucleon phase shifts, approximately valid over wide energy ranges is derived, assuming only general properties of the wave function and potential. It is found to approximately reproduce the S, P, and D proton-proton phase shifts in the ranges 10–150 Mev and 10–310 Mev with two, or at most three, free parameters per state. A generalization, which includes part or all of the outer potential exactly, is derived at the same time.

INTRODUCTION

THERE have been in general three approaches to nucleon-nucleon scattering at moderately high energies, i.e., the range 10–300 Mev. On the one hand there have been attempts at correlating the data by means of more or less phenomenological potentials containing central, tensor and spin-orbit components.[1–5] On the other hand there have been phase shift analyses at various energies,[6] the latest giving phase shifts varying continuously with energy,[7,8] although a completely unique solution does not yet appear to have been found.

A third approach[1,2,6] has been to impose boundary conditions with and without an additional potential. This approach was initiated by Feshbach and Lomon,[9] who assumed that the logarithmic derivative at an appropriate radius varies slowly with energy. Raphael[10] studied the restrictions such an assumption places on certain standard potentials, and proposed an expansion in powers of the energy for this logarithmic derivative. It has been shown by Noyes[11] that the S-state phase shifts given by the Gammel-Thaler potential are approximately reproduced by the Raphael formula, although it has recently been shown by Perring and Phillips[12] and by Signell and Yoder[13] that the Raphael formula disagrees with certain potentials if low-energy parameters are used. In general, this approach assumes that most of the contribution which is not included in some assumed outer potential is due to an inner potential which is large compared with the kinetic energy. However, most successful potentials have large regions which are comparable with the energy variations considered, even at small distances, and there is no obvious reason why their contribution should be small.

Of the above approaches, the second is purely phenomenological, while the first and third are based on quite specific models or assumptions. In the present work an attempt is made to derive approximate formulas based only on some well-known qualitative or semiquantitative features of the potential and wave function. The resulting formulas are equivalent to an energy-dependent boundary condition approach, but are independent of any specific model.

VARIATIONAL PRINCIPLE FOR THE PHASE SHIFTS

We shall now obtain a generalization of the usual variational principle for phase shifts.[14] This will then be used as a starting point for the derivation of shape-independent formulas, which may or may not include exactly part or all of the outer potential.

Consider the radial Schrödinger equation that applies in some angular momentum state outside a hard repulsive core of radius r_c (which may be zero). In the appropriate units,

$$u''+[k^2-(W+U)]u=0, \tag{1}$$

where k is the momentum of one nucleon in the center of mass system. W contains the centrifugal term, the Coulomb potential (if there is any), and, perhaps, a part of the outer nuclear potential. U contains the rest of the nuclear potential V, which, in general, we may take to be a symmetric integral operator. Thus

$$Vu=\int_{r_c}^{\infty} K(r,r')u(r')dr', \tag{2}$$

where $K(r,r')$ is symmetric in r and r'.

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[1] R. J. N. Phillips, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22, p. 562.
[2] J. L. Gammel and R. M. Thaler, *Progress in Elementary Particle and Cosmic-Ray Physics* (North Holland Publishing Company, Amsterdam, 1960), Vol. 5, p. 97.
[3] T. Hamada *et al.*, Progr. Theoret. Phys. (Kyoto) **22**, 566 (1959); **23**, 366 (1960).
[4] R. A. Bryan, Nuovo cimento **16**, 895 (1960).
[5] T. Hamada, Progr. Theoret. Phys. **24**, 220, 1033 (1960); **25**, 246 (1961).
[6] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nuclear Sci. **10**, 291 (1960).
[7] G. Breit *et al.*, Phys. Rev. **120**, 2227 (1961).
[8] H. P. Stapp, M. J. Moravcsik, and H. P. Noyes (private communication) to whom the author is deeply indebted for supplying the phase shifts used in this paper.
[9] H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956).
[10] R. B. Raphael, Phys. Rev. **102**, 905 (1956); **107**, 1135 (1957).
[11] H. P. Noyes, University of California Radiation Laboratory Report UCRL-5521-T, 1959 (unpublished); *Proceedings of the International Conference on Nuclear Forces and the Few Nucleon Problem* (Pergamon Press, New York, 1960), p. 39.
[12] J. K. Perring and R. J. N. Phillips (to be published).
[13] P. Signell and R. Yoder, Phys. Rev. **122**, 1897 (1961). (The author is indebted to Dr. H. P. Noyes for calling his attention to the work of Perring and Phillips and Signell and Yoder.)
[14] J. Schwinger, Phys. Rev. **78**, 135 (1950).

Now let $v(r)$ be that particular solution of the equation

$$v''+[k^2-W]v=0, \tag{3}$$

which has the same asymptotic form as $u(r)$.

Finally, let us consider the integral

$$J(a)=\int^{\infty} dr[(u'^2-v'^2)+(W-k^2)(u^2-v^2)+u^2U], \tag{4}$$

where the lower limit, whenever it is omitted, will always be $r=r_c$ for terms containing u or u', and $r=a$ for terms containing v or v'. For the present, a can have any value. If U is not a local operator the expression u^2U will always be understood to stand for uUu, and similarly for v^2W and u^2W.

For exact u and v, we have, using Eqs. (1) and (3),

$$J(a)=\int^{\infty} d/dr(u'u-v'v)dr=v'(a)v(a),$$

since $u(r_c)=0$.

Writing $[v'(a)/v(a)]=\Gamma(a)$, we have

$$v^2(a)\Gamma(a)=J(a). \tag{5}$$

Now it can be easily shown by the usual methods[14] that $\Gamma(a)$, as given by Eq. (5), is stationary with respect to variations in u and v as long as $u\rightarrow v$ as $r\rightarrow\infty$, $u(r_c)=0$, and V has the form given by Eq. (2). Thus if we insert approximate values of u and v into Eq. (5), we obtain $\Gamma(a)$ with errors of a higher order than the errors in u and v.

Let us henceforth normalize our wave functions to be such that $v(a)=1$ and let $f(k,r)$ be that solution of Eq. (3) which $\rightarrow F_l(k,r)$ as $r\rightarrow\infty$ and $g(k,r)$ that which $\rightarrow G_l(k,r)$ as $r\rightarrow\infty$, where F_l and G_l are the usual regular and irregular Coulomb wave functions. In the absence of the Coulomb potential these become $rj_l(kr)$ and $-rn_l(kr)$ where j_l and n_l are the spherical Bessel and Neumann functions, respectively. Equation (5) now becomes

$$\Gamma(a)=\{[(d/dr)f(k,r)+\tan\delta(d/dr)g(k,r)]_{r=a}/[f(k,a)+\tan\delta g(k,a)]\}=J(a), \tag{6}$$

where δ is the nuclear phase shift.

SHAPE-INDEPENDENT APPROXIMATIONS

Our aim will now be to find a value of a for which u is a slowly varying function of energy in those regions where its contribution to $J(a)$ is not small.

Let a be some point such that u and v are already approximately equal to each other at $r=a$, so that (u^2-v^2) and $(u'^2-v'^2)$ are small for $r>a$. This means that a is a measure of the range of U with most of the contribution of U within $r=a$. If, in addition, a is within, or at about the first maximum of u, then u is approximately independent of energy within $r=a$, indeed, for some distance outside that radius, even over variations of energy comparable with the magnitude of the potential itself (see Figs. 1–3). Beyond that, the integrand of $J(a)$ is negligible, since U is small and v practically equal to u.

As a first approximation we may thus replace u and v as they come into Eq. (4) by u_0 and v_0, which we will normally take to be functions resembling u and v in an average way over a wide range of energies for those radii where the contribution to $J(a)$ is not small. This procedure is usually (but not necessarily always) more appropriate for our purpose than just approximating u and v by their values at some particular energy, as is done in the usual low-energy effective range theory approach. This latter approximation is appropriate if we try to reproduce very accurately measured phase shifts over a small energy range,

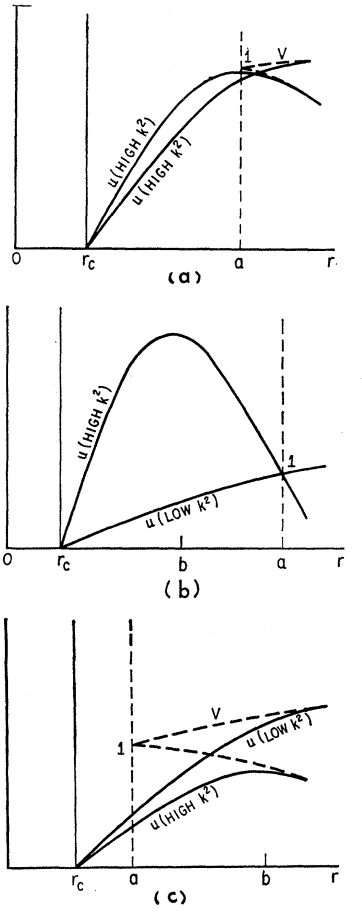

FIG. 1. (a). Variation of the wave function with energy in the S state as long as a is near the first maximum. The wave function does not vary much at small distances, mainly because of the insensitivity of its shape, although the stronger the potential the less u varies. It is obvious that a is not a sensitive parameter. (More generally the wave function within $r=a$ varies less when a is near a maximum than when it is anywhere else, except within the first maximum.) (b). If a is too large we have large variations of u within the range b of the force. (c). If a is too small we have large variations of u and v within the range b of the force.

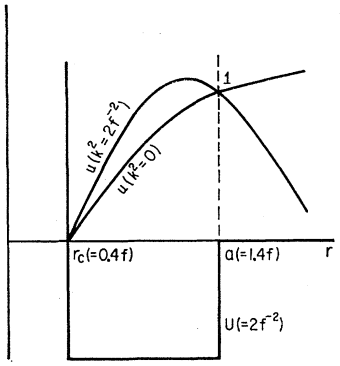

FIG. 2. The wave function in the $^1S_0$ state for a square well of depth 83 Mev and an energy variation of 166 Mev with $W=0$. This potential does give the approximate values of the phase shift at high energies.

whereas what we are usually trying to do here is to approximate not-too-accurate phase shifts over a wide range.

Thus Eq. (6) becomes

$$\Gamma(a)\cong A-k^2B, \tag{7}$$

where

$$A=\int^{\infty} dr[u_0'^2-v_0'^2)+(u_0^2-v_0^2)W+u_0^2U], \tag{8a}$$

$$B=\int^{\infty} dr[u_0^2-v_0^2], \tag{8b}$$

and, since $\Gamma(a)$ is stationary with respect to variations of u and v, the error in $\Gamma(a)$, as given by Eq. (7), is of a higher order than the errors in the magnitudes of the wave function and its first derivative resulting from our approximation, which was why we started with a variational principle in the first place. Equation (7) is equivalent to solving Eq. (3) with the boundary condition that the logarithmic derivative at $r=a$ is linear in energy.

Thus, if we have $\Gamma(a)$ as given by some theory or model or phase-shift analysis at two (or more) widely spaced energies in some energy range, Eq. (7) will give the approximate values of $\Gamma(a)$ (and hence of the phase shifts) at all other energies in that range.

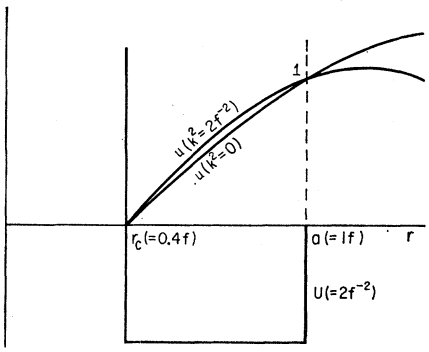

FIG. 3. The wave function for a square well of depth 83 Mev and an energy variation of 166 Mev with the potential for $r>a$ $=1$ f included in W. The variation of the wave function $\lesssim 10\%$, which should make the error associated with $\Gamma(a)$ of the order of 1% or less. This variation should be smaller for deeper potentials.

The above situation actually holds for high-energy nucleon–nucleon scattering, even if W does not include any of the nuclear force. In this latter case an appropriate choice of a would be the pion Compton wavelength ($\cong$1.4 fermi), which would include most of the nuclear force and for which the first maximum would be close to or outside of $r=a$ (see Fig. 2). Other possible choices of a will be brought up in the next section. In obtaining A and B it would be dangerous to rely too heavily on the low-energy parameters, since it is precisely at low energies that the outer parts of the wave function are most important, and these can be expected to vary rapidly with energy. Thus, if one wishes to use the low-energy data at all, one should use at most one parameter, say the phase shift at 5 Mev. In the higher angular momentum states, this outer region plays an important role up to even higher energies, but it is only when the inner region becomes important, and hence Eq. (7) is valid, that this phase shift would begin to contribute to the cross sections to any extent. Besides, the one-pion exchange (OPEC) approach[15] is valid until this happens.

To remove the above difficulty and permit a simultaneous description of both low- and high-energy scattering, one would have to include in W the outer forces, at least approximately. With $a=1.4$ f, it should be sufficient just to include the one-pion exchange potential (OPEP). However, to obtain the most accurate results one should include as much of the known potential as possible, since then $(v-u)$ and U become small at smaller radii. This will enable us to choose a smaller value of a, and, naturally, the smaller a is, the less will be the variation of the wave function with energy in the inner region, both because the region is smaller and because the potential there is larger (compare Figs. 2 and 3). A good choice would be the OPEP together with a potential having a form, if not actually given by meson theory, at least suggested by it. Such a form, which we will call the MSOP (meson-suggested outer potential), may be expected to include most of the potential for $r\gtrsim 1$ f,[1,16] making it appropriate to take $a\cong 1$ f. Moreover, with this approach, the inner region is unimportant in the higher angular momentum states, making it possible to take A and B equal to zero, or else making rough estimates of them by obtaining them from some simple potential consistent with the appropriate low angular momentum parameters. The parameter B, in particular, can be estimated quite well in this way.

To obtain a formula to any desired degree of accuracy,

[15] M. J. Moravcsik, University of California Radiation Laboratory Report UCRL-5317-T, 1958 (unpublished).

[16] M. Taketani *et al.*, Suppl. Progr. Theoret. Phys. (Kyoto) 1, No. 3 (1956).

we could always take an approximation

$$u^2 \cong u_k^2 = \sum_{i=0}^{N} u_i^2 k^{2i},$$

such that the error resulting from the use of the approxmation

$$u_k'^2 \cong \sum_{i=0}^{N+1} (u')_i^2 k^{2i}$$

is of a higher order than the error $(u-u_k)$ in u. Thus Eq. (6) becomes

$$\Gamma(a) \cong A - k^2 \sum_{i=0}^{N} B_i k^{2i}. \tag{9}$$

APPLICATION TO HIGH-ENERGY PROTON-PROTON SCATTERING

We shall only consider the case where W contains just the Coulomb and centrifugal terms. In addition, at high energies and short distances, we can take, except perhaps for normalization,

$$F_l(k,r) \cong r j_l(kr), \quad G_l(k,r) \cong -r n_l(kr),$$

and so, since a is a small distance, Eq. (7) becomes, on using Eq. (6) and the usual properties of Bessel functions,

$$A_{jl} - k^2 B_{jl} \cong -\frac{l}{a} + k \frac{j_{l-1}(ka) - \tan\delta_{jl} n_{l-1}(ka)}{j_l(ka) - \tan\delta_{jl} n_l(ka)},$$

$$\text{if } l>0 \tag{10a}$$

$$\cong k \cot(ka+\delta_{jl}), \quad \text{if } l=0, \tag{10b}$$

where δ_{jl} is the nuclear phase shift in the state with total angular momentum j and orbital angular momentum l.

Expanding the Bessel functions, and dropping the j subscript for the singlet states, we have for the $^1S_0$, ^3P_j, and $^1D_2$ waves

$$k \cot(ka+\delta_0) \cong C_0 - k^2 B_0, \tag{11a}$$

$$ka \cot(ka+\delta_{jl}) \cong 1 - k^2 a/(C_{jl} - k^2 B_{jl}), \tag{11b}$$

$$ka \cot(ka+\delta_2) \cong 1 - \frac{k^2 a^2}{3 - k^2 a/(C_2 - k^2 B_2)}, \tag{11c}$$

where $C_{jl} = (l/a) + A_{jl}$.

We shall not consider the tensor force in the $^3P_2$ state explicitly, i.e., we assume it can be approximated by an equivalent potential, as might be given, say, by the WKB approximation.[17]

At this point, we could take a to be such that $B=0$ to obtain the Raphael formula.[10,11] This would tend to make the wave function stable in the immediate neighborhood of the repulsive core, and would thus be appropriate if the potential is strongest at very small distances. Another choice would be to take a such that $\Gamma(a)=0$ at some suitable energy, say near the upper end of the energy range being considered. This would correspond to taking a near the first maximum and

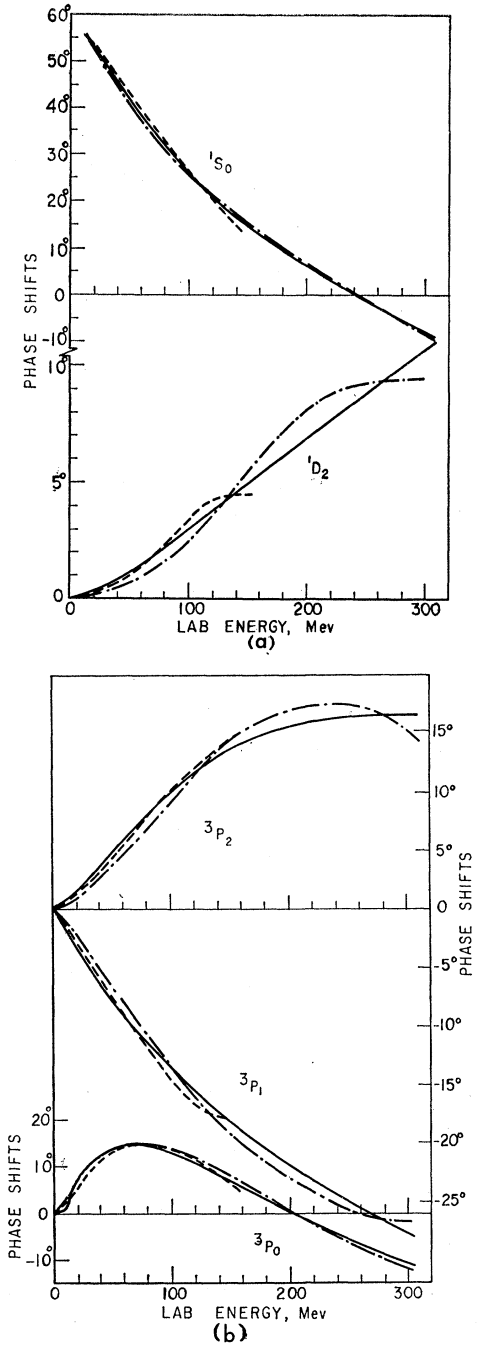

FIG. 4. Plots of the calculated phase shifts as compared with those obtained from the Stapp, Moravcsik, and Noyes phase shift analysis (denoted by full lines). The dashed curve is calculated from the parameters of Table II while the dot-dash-dot curve is calculated from those of Table III in each case. The experimental errors are of the order of a degree.

[17] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).

TABLE I. Phase shifts calculated from Eqs. (11) with C and B obtained from the results of the Signell-Marshak potential in the range 18–150 Mev.

State	a (in f)	C (in f^{-1})	B (in f)	Phase shifts from Eqs. (11) 18 Mev	40 Mev	100 Mev	150 Mev	Signell-Marshak phase shifts[18] 18 Mev	40 Mev	100 Mev	150 Mev
$^1S_0$	1.4	$0.22a^{-1}$	$0.84a^{-1}$	49.3°	45.1°	29.3°	16.7°	51.4°	42.9°	28.1°	19.2°
	1.2	$0.17a^{-1}$	$0.42a^{-1}$	50.1°	44.3°	28.9°	17.5°				
	1.0	$0.14a^{-1}$	$0.15a^{-1}$	50.3°	44.6°	29.3°	18.6°				
	0.8	$0.12a^{-1}$	$-0.04a^{-1}$	49.6°	44.1°	28.9°	18.4°				
$^1D_0$	1.4	1.75	−0.40	0.2°	0.8°	4.0°	5.1°	0.2°	1.0°	3.8°	5.7°
$^3P_0$	1.4	1.38	0.14	2.7°	7.7°	16.3°	15.7°	3.1°	8.5°	15.8°	15.5°
$^3P_1$	1.4	4.80	1.30	−2.4°	−6.5°	−15.3°	−17.1°	−2.5°	−6.3°	−14.0°	−17.3°
$^3P_2$	1.4	1.64	0.22	1.5°	4.4°	10.5°	11.0°	1.5°	4.6°	9.8°	11.4°

would tend to make the wave function stable in those regions where it is most likely to be important. However, the easiest and perhaps the most appropriate procedure would be to take $a=1.4$ f. This may be a little beyond the first maximum in the $^1S_0$ state but appears to be adequate.

Thus if we plot $ka\cot(ka+\delta_0)$, $k^2a/[1-ka\cot(ka+\delta_{jl})]$, and

$$\frac{k^2a}{3-k^2a^2/[1-ka\cot(ka+\delta_2)]},$$

with phase shifts from some phase-shift analysis or theory, and draw the best straight lines through them, we obtain the constants C_0, B_0; C_{jl}, B_{jl}; C_2, B_2. The magnitudes of the errors should be taken into account in drawing the lines.

This procedure was first applied to the Signell-Marshak potential[18] (for which the choice $a=1.4$ f is meaningful) in the range 18–150 Mev. The results are shown in Table I, which also shows the effect of varying a in the $^1S_0$ state. There is some improvement for smaller values of a, probably because the first maximum is within 1.4 f and the potential is weak except at very small distances, but the errors are of the same order of magnitude. The phase shifts obtained from Eqs. (11) should be compared with those obtained by Signell and Marshak.[18]

Table II shows the result of applying the above procedure in the range 10–150 Mev to a phase-shift analysis recently made by Stapp *et al.* at the University of California,[8] the results of which are reproduced at certain energies in Table IV. The same procedure was also applied in the range 10–310 Mev, although it is necessary to use Eq. (9) with $N=1$ instead of (7) in the $^1S_0$ and $^3P_0$ states (see Table III). In the former case this would probably be due again to the first maximum being within $a=1.4$ f at very high energies and, in the latter, to the presence of strong repulsion at small distances followed by a long range force, as would be the case with the usual description by a strong short-range spin orbit, together with a long-range tensor potential. It should be noted that the differences between the parameters in the 10–150 Mev range and the corresponding ones in the 10–310 Mev range are not necessarily very small when we are using different average forms for u and v in the respective ranges.

USE OF EQUIVALENT INNER POTENTIALS IN OTHER PROBLEMS OF CLASSICAL NUCLEAR PHYSICS

We have seen that if one incorporates the outer potential, even approximately, into W, one can obtain accurate phase shifts from Eq. (7) at both low and high energies. A particularly appropriate choice for the outer potential, as we have seen, is the MSOP. This potential which we shall assume from now on, would be a good approximation to the correct potential for $r>2$ f and at least a fair approximation for 1 f$\lesssim r\lesssim$2 f, with the error small at $r=2$ f but becoming steadily larger as the distance becomes smaller.

Since Eq. (7) is shape-independent and depends on only two parameters, any reasonable two-parameter inner potential will give the phase shifts to the same accuracy if it gives values of $\Gamma(a)$ equal to $(A-k^2B)$ at two widely spaced energies in our range, i.e., if it gives the same A and B as the correct potential. We

TABLE II. Phase shifts calculated from Eqs. (11) with C and B obtained from the phase shifts of Table IV in the range 10–150 Mev.

State	a (in f)	C (in f^{-1})	B (in f)	Phase shifts at various energies 9.68 Mev	25.63 Mev	39.4 Mev	68.3 Mev	95 Mev	143 Mev
$^1S_0$	1.4	$1.40a^{-1}$	$0.69a^{-1}$	55.4°	50.8°	45.8°	35.9°	27.5°	14.4°
$^1D_2$	1.4	2.05	−0.25	0.0°	0.2°	0.7°	1.8°	3.0°	4.6°
$^3P_0$	1.4	1.08	−0.16	1.9°	7.0°	11.0°	15.0°	14.3°	7.7°
$^3P_1$	1.4	4.00	0.80	−0.9°	−3.2°	−5.5°	−10.0°	−13.7°	−17.8°
$^3P_2$	1.4	1.80	0.35	0.4°	1.8°	3.2°	6.6°	9.6°	13.7°

[18] P. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958).

TABLE III. Phase shifts calculated from Eqs. (11) with C and B obtained from the phase shifts of Table IV in the range 10–310 Mev. In the $^1S_0$ and $^3P_0$ states we have to add $-Dk^4$ to $(C-k^2B)$ in the Eqs. (11).

State	a (in f)	C (in f^{-1})	B (in f)	D (in f^3)	Phase shifts at various energies: 9.68 Mev	25.63 Mev	39.4 Mev	68.3 Mev	95 Mev	143 Mev	210 Mev	310 Mev
$^1S_0$	1.4	$0.105a^{-1}$	$0.445a^{-1}$	$0.165a^{-1}$	56.4°	49.0°	43.3°	33.7°	26.6°	16.8°	5.7°	−9.7°
$^1D_2$	1.4	2.62	0.095		0.0°	0.2°	0.4°	1.2°	2.2°	5.0°	8.5°	9.5°
$^3P_0$	1.4	0.98	−0.34	0.077	2.3°	8.0°	11.8°	15.3°	14.1°	8.5°	−1.1°	−12.7°
$^3P_1$	1.4	3.55	0.50		−0.8°	−2.8°	−4.8°	−9.2°	−13.1°	−18.5°	−23.3°	−26.6°
$^3P_2$	1.4	1.92	0.42		0.2°	1.2°	2.4°	5.3°	8.4°	13.7°	17.2°	14.3°

TABLE IV. Some phase shifts taken from solution 6 of the Stapp, Moravcsik, and Noyes phase shift analysis.[8] The error is of the order of a degree.

State	Phase shifts at various energies: 9.68 Mev	25.63 Mev	39.4 Mev	68.3 Mev	95 Mev	143 Mev	210 Mev	310 Mev
$^1S_0$	56.2°	49.7°	44.5°	34.5°	26.7°	15.9°	5°	−8.8°
$^1D_2$	0.1°	0.4°	0.8°	1.8°	2.8°	4.7°	7.3°	10.9°
$^3P_0$	1.0°	8.3°	12.0°	15.0°	13.5°	8.0°	−0.8°	−12.2°
$^3P_1$	−1.3°	−4.0°	−6.2°	−10.0°	−13.1°	−17.6°	−22.6°	−28.2°
$^3P_2$	0.5°	2.0°	3.5°	6.8°	9.6°	13.3°	15.6°	16.4°

shall call such a two-parameter potential an equivalent inner potential. The generalization using Eq. (9) is obvious.

Suppose we now simultaneously integrate in from infinity using the equivalent and the correct inner potentials respectively. Since they give the same phase shifts (within an error of order 1% or less), the two wave functions will be essentially equal until we come to $r\approx 1$ f. At about this point they begin to deviate somewhat from each other, because the potentials begin to differ by large amounts. This error ϵ in the wave function is small, however, since it is mainly within the first maximum of u, where the shape of the wave function is not sensitive even to large changes of the potential, and also since both potentials have the same parameter B, which gives an approximate measure of the integral of the square of the inner wave function [see Eq. (8b) and Fig. 5].

Now it is well known that all the usual problems of classical nuclear physics require either a correct phase for the asymptotic wave function or a good over-all wave function. The first class includes such problems as nuclear reactions, binding energies, etc., while the second arises from the coupling of nucleons to other fields. Thus any potential which will give these properties of the wave function accurately will give accurate results for such problems. We shall now give two arguments to show that the MSOP together with any equivalent inner potential will give these with negligible error (i.e., with an error of an order of magnitude smaller than ϵ). These two arguments can always be used together, with one applying to some forces and the other to the rest. The first argument only considers two-body forces, while the second takes multibody forces into account, but is probably somewhat less satisfactory in other respects.

(a) Consider a collection of nucleons (and, perhaps, other particles which interact in a known way with nucleons) under the usual conditions of classical nuclear physics, and suppose that the correct inner potential between just two of the nucleons is distorted into an equivalent inner potential. If these were the only nucleons present, then, as we saw, the only non-negligible error in the wave function is of order ϵ at small distances. If there are additional particles in the neighborhood, there is an additional distortion (not error). Although this becomes large at larger distances, it is small at smaller distances (within 1 f, say, where it is of order $\Delta\lesssim 10\%$), again mainly because of the insensitivity of the wave function within its first maximum with our normalization. The resultant addi-

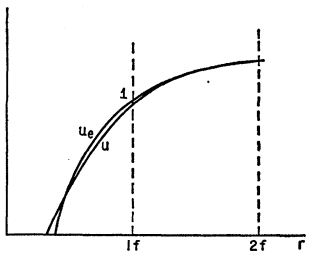

FIG. 5. Error ϵ in the wave function for $r\lesssim 1$ f due to the use of the equivalent instead of the correct inner potential in the S state. We can expect ϵ to be of the order of at most 10%, which is the error associated with a distortion of the inner potential comparable with the magnitude of that potential itself (see Fig. 3). It is also smaller in the higher angular momentum states and at higher energies because of the increased importance of the centrifugal and energy terms in Eq. (1).

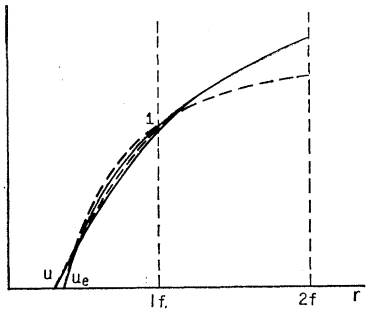

FIG. 6. Additional distortion (not error) due to neighboring particles. It is small for small r, where it is comparable with ϵ, but becomes large for larger r. The dashed curves are those of Fig. 4. It is obvious that the over-all wave function has an error of at most only a few percent within 2 f, even at low energies.

tional error in the asymptotic wave function is, however, an order of magnitude smaller than ϵ or Δ. (This is shown variationally in Appendix I, although it may also be seen qualitatively.) Thus the error in the over-all wave function is also negligible, since if we now integrate in from infinity as before, we see that the only non-negligible error in the wave function is still of order ϵ, which, however, resides mainly within a very small region. Since the typical internucleon distance is 2 f, whereas ϵ is confined largely to within a distance of 1 f, such regions altogether only occupy about $R\approx 10\%$ of the total volume. Thus the effect on the over-all wave function is negligible over the average volume occupied by the two nucleons (see also Fig. 6).

This fact in turn means that the neighboring particles are affected even less by our distortion from the correct to the equivalent inner potential, which, incidentally, was why the effect of these particles in producing the additional distortion in the wave function was essentially the same in both cases. Thus, if we similarly distort the potentials, one by one, between all the nucleons, we still have the correct asymptotic and over-all wave function to first order (i.e., with negligible error) over the average volume occupied by these nucleons. The reason why the percentage error remains within the same order of magnitude every time we do this is because only a few nucleons interact through the inner potential simultaneously. Indeed we shall show in part (b) that essentially only two nucleons interact through the inner potential.

(b) In our second argument we may take multibody forces into account, assuming, however, that such forces are not excessively large attractively at small distances. Suppose we know the approximate three-body potential for distances $\gtrsim 1$ f, i.e., for all distances for which the nucleons do not simultaneously come within 1 f of each other. We would again take for such a potential a form, if not actually given by meson theory, at least suggested by it. Such a form would thus have a negligible absolute error at 2 f and a steadily increasing error for smaller distances. We will now show that the multibody correlation errors (not the multibody correlations themselves) resulting from the use of this approximate three-body potential, together with our two-body MSOP plus equivalent inner potential, are negligible.

To see this, we note that under the usual conditions of classical nuclear physics, the probability P^2 that three or more nucleons simultaneously come within 1 f of each other is negligible, since the typical internucleon distance is of the order of 2 f which makes $P^2\approx 1\%$. (Even if this were large, however, the distortion in the wave function would not be too great because of its insensitivity to large perturbations at small r.) P^2 is further reduced by the repulsive core, which pushes out the wave function, and, for like nucleons, by the existence of the so-called Pauli repulsion.[19] This argument shows that three-body correlation errors are negligible for $r\lesssim 1$ f for any reasonable assumed potential, which is also why we could neglect multibody forces for $r\lesssim 1$ f. Since only three- and two-body forces have ranges $\gtrsim 1$ f, this also shows why we only needed two- and three-body potentials.

Although most of the error in our assumed approximate potentials lies within about 1 f, there is also some error for $r\gtrsim 1$ f. This error, which is small at $r=2$ f, becomes increasingly larger for smaller r, but at the same time the probability for three nucleons to simultaneously come within that radius decreases also, keeping the three-body correlation errors comparable with those in the preceding paragraph.

We are now left with only two-body correlation errors to consider. This means that if the effect of distorting the two-body inner potential between just two nucleons gives the asymptotic and over-all wave functions with negligible error for the average volume occupied by them, we could do the same for all the other two-body potentials and obtain only negligible errors. But this former is shown to be true in the first paragraph of part (a), which thus completes the argument.

CONCLUSION

We thus see that it is possible to describe nucleon-nucleon scattering in terms of only a few parameters per angular momentum state, and assuming only certain qualitative or semiquantitative properties of the potential. If the outer forces are not taken into account explicitly, one has to describe the low and high energies separately. If, however, an approximate outer force (specifically the MSOP) is assumed, one can describe both low- and high-energy phase shifts in terms of a few parameters.

Moreover, any equivalent inner potential obtained from these parameters, when used with the MSOP, should give negligible errors when it is applied to problems which can be described by potentials acting

[19] S. D. Drell and K. Huang, Phys. Rev. **91**, 1527 (1953).

between nucleons.[20] Different types of inner potentials may be appropriate for different problems. It is probably more convenient to use a local potential plus hard repulsive core for two-body problems, while it may be more convenient to use, say, a nonlocal separable inner potential for multibody problems. Such a potential can be made to have the same effect as a very strong core for two-body problems, while causing none of the usual difficulties associated with such a core in multibody problems.

Thus we have a definite approximate method of relating the nucleon-nucleon S matrix in a limited range of energies to all such problems. This is probably the most one can extract from the nucleon-nucleon S matrix for application to such problems, since the interaction is almost certainly nonlocal for small distances, which means that there is not much point in, say, deriving a more accurate local potential from the S matrix, because the errors due to the nonlocality and lack of uniqueness are then of the same order as our other errors. At the same time a general nonlocal potential cannot be derived uniquely from the phase shifts. However, the errors arising from the use of an equivalent inner potential are of the order of 1% or less. Such potentials should therefore be adequate for most purposes.

ACKNOWLEDGMENTS

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APPENDIX 1

We shall show that the asymptotic wave function in any particular angular momentum state, and hence in every angular momentum state, is correct to first order with any equivalent inner potential in the presence of neighboring particles.

Let us first consider the case where the potential is correct. Let ΔU be the additional effective potential (which may be explicitly energy-dependent, and, indeed, depend on anything else we please) due to the presence of other particles.

Then, replacing W by $(W+\Delta U)$ in Eqs. (3) and (4), and with the MSOP included in W, Eqs. (4) and (6) become

$$\Gamma(a)=\int^{\infty} dr[(u'^2-v'^2)+(W-k^2)(u^2-v^2)+u^2U] + \int^{\infty}(u^2-v^2)\Delta U dr. \quad \text{(A1)}$$

Note that any uniform part of ΔU can be incorporated into k^2. Indeed in what follows, we shall assume as much of it has been so incorporated as to make the distortion Δ from the free wave function at small distances as small as possible for a given k^2, even though the meaning of k^2 is different from that for free scattering now.

Now the wave functions u and v in (A1) do not differ much from those for free scattering at small r, i.e., in those regions where their contribution to the integral in (A1) is not small, the difference being of order Δ. Thus it follows from the variational principle that, within an error of order Δ^2, the first integral in (A1) is just $\Gamma_f(a)$, the $\Gamma(a)$ for free scattering. Now since the insertion of approximate u and v into both $\Gamma_f(a)$ and into $\Gamma(a)$ gives results with an error of order higher than the error in u and v, the insertion of an approximate u^2 and v^2 into $\int^{\infty}(u^2-v^2)\Delta U dr$ alone will give a value of $\Gamma(a)$ to the same order of accuracy. In other words, we are justified in treating ΔU as a perturbation as far as $\Gamma(a)$ is concerned, even though ΔU may be large. In particular we may replace u^2 by u_e^2 and v^2 by v_e^2, where u_e is the wave function for the equivalent inner potential U_e. The resulting error is thus only of order ϵ^2.

Thus, to this order of accuracy,

$$\Gamma(a)\cong\int^{\infty} dr[(u'^2-v'^2)+(W-k^2)(u^2-v^2)+u^2U] + \int^{\infty}(u_e^2-v_e^2)\Delta U dr \quad \text{(A2)}$$

$$\cong\Gamma_f(a)+\int^{\infty}(u_e^2-v_e^2)\Delta U dr. \quad \text{(A3)}$$

But $\Gamma_f(a)$ is given accurately by the equivalent inner potential also, and, as before, one may use the wave functions u_e and v_e for scattering in the presence of other particles instead of the corresponding free scattering wave functions. The error, as before, is only of order Δ^2.

Thus, to that order, (A3) becomes

$$\Gamma(a)\cong\int^{\infty} dr[(u_e'^2-v_e'^2)+(W-k^2)(u_e^2-v_e^2)+u_e^2U_e] + \int^{\infty}(u_e^2-v_e^2)\Delta U dr, \quad \text{(A4)}$$

which is just the $\Gamma(a)$ for the equivalent potential. Note that because of the variational principle, only

[20] A trivial modification of the argument (a) in the preceding section can be given even if we have internucleon distances somewhat less than 2 f, since such distances would be accompanied by larger energies, giving correspondingly smaller values of the error ϵ.

errors of order Δ^2 and ϵ^2 were involved in going from (A1) to (A4) [in addition to the negligible error involved in going from the $\Gamma_f(a)$ for the correct potential to the corresponding $\Gamma_f(a)$ for the equivalent potential]. This is not a trivial result. Had we used a boundary condition at $r=a$ instead of an equivalent inner potential, for instance, the error in the asymptotic wave function would have been of order Δ, which is not negligible by our definition.

APPENDIX 2. TENSOR FORCES

If we have tensor forces, we will have coupled equations of the form

$$\begin{aligned} u_l''+[k^2-(W_l+U_l)]u_l-(W_{ll'}+U_{ll'})u_{l'}=0,\\ u_{l'}''+[k^2-(W_{l'}+U_{l'})]u_{l'}-(W_{ll'}+U_{ll'})u_l=0. \end{aligned} \tag{A5}$$

Now let v_l, $v_{l'}$ be that particular pair of solutions of the equations

$$\begin{aligned} v_l''+[k^2-W_l]v_l-W_{ll}v_{l'}=0,\\ v_{l'}''+[k^2-W_{l'}]v_{l'}-W_{ll'}v_l=0, \end{aligned} \tag{A6}$$

having the same asymptotic form as that pair of solutions u_l, $u_{l'}$ of Eq. (A5) which we are considering.

Consider the integrals

$$\begin{aligned} J_l(a)&=\int^{\infty} dr\{(u_l'^2-v_l'^2)-(k^2-W_l)(u_l^2-v_l^2)+u_l^2U_l\},\\ J_{l'}(a)&=\int^{\infty} dr\{(u_{l'}'^2-v_{l'}'^2)-(k^2-W_{l'})(u_{l'}^2-v_{l'}^2)\\ &\qquad\qquad +u_{l'}^2U_{l'}\},\\ J_{ll'}(a)&=\int^{\infty} dr\{W_{ll'}(u_lu_{l'}-v_lv_{l'})+U_{ll'}u_lu_{l'}\}, \end{aligned} \tag{A7}$$

where, as for central forces, the lower limit, whenever it is omitted, will be $r=r_c$ for terms containing u or u' and $r=a$ for terms containing v or v'.

For exact u_l, $u_{l'}$; v_l, $v_{l'}$ it can be easily shown that, if $[v_l'(a)/v_l(a)]=\Gamma_l(a)$, $[v_{l'}'(a)/v_{l'}(a)]=\Gamma_{l'}(a)$, then

$$v_l^2(a)\Gamma_l(a)+v_{l'}^2(a)\Gamma_{l'}(a)=J_l(a)+J_{l'}(a)+J_{ll'}(a). \tag{A8}$$

Now it can be shown in a straightforward manner that if, for a particular solution, $\Gamma_l(a)$ and $\Gamma_{l'}(a)$ depend on a parameter η (one of our three real generalized phase shifts for instance), then η [as it occurs in $\Gamma_l(a)$ and $\Gamma_{l'}(a)$] is stationary with respect to variations in the wave function as long as $u\to v$ as $r\to\infty$ and $u(r_c)=0$.

Using exactly the same arguments as for central forces, we have, in the shape independent approximation, an expression of the form

$$\begin{aligned} v_l^2(a)\Gamma_l(a)+v_{l'}^2(a)\Gamma_{l'}(a)&=v_l^2(a)[A_l-k^2B_l]\\ &\quad+v_{l'}^2(a)[A_{l'}-k^2B_{l'}]+v_l(a)v_{l'}(a)F_{ll'}. \end{aligned} \tag{A9}$$

In addition, we can make the approximation $v_{l'}(a)/v_l(a)\cong E=$ constant. This is certainly true for small energy variations, and can also be seen from the WKB approximation,[17] for which $[u_{l'}(r)/u_l(r)]$ is energy-independent. At $r=a$, this ratio is $\cong[v_{l'}(a)/v_l(a)]$ which is thus approximately energy-independent. This approximation is really similar to the other approximations made and should be of about the same accuracy as the others within or at the first maximum. Equation (A5) now takes on the form

$$\Gamma_l(a)+E^2\Gamma_{l'}(a)=G-k^2H. \tag{A10}$$

There are three equations of the form (A10), corresponding to three different solutions of (A5). Of these, two will be linearly independent, and the third will be some linear combination of the independent solutions. Thus we can solve for our three generalized phase shifts at each energy using Eqs. (A10).

Thus if we are given the phase shifts at two (or more) widely spaced energies in some energy range, the Eqs. (A10) will give the phase shifts at all other energies in that range. E would be determined by taking it equal to $[v_{l'}(a)/v_l(a)]$ at those given energies and taking an appropriate average. After that, of course, it has to be always treated as a constant. This constant has to be found first, before we find G and H from our phase shifts at those two (or more) energies.

The extension to higher approximations is obvious, just as for central forces. Thus (A10) would become

$$\Gamma_l(a)+\Big(\sum_{i=0}^{N}E_ik^{2i}\Big)\Gamma_{l'}(a)=G-k^2\sum_{i=0}^{N}H_ik^{2i}. \tag{A11}$$

APPENDIX 3

It was pointed out that Eq. (7) is equivalent to solving Eq. (3) with the boundary condition that the logarithmic derivative at $r=a$ is just $J(a)=A-k^2B$. We will now give a variational method of solving this problem. Consider the integral

$$\begin{aligned} J'(a)=\int_a^{\infty} dr\Big[&(v'^2-h'^2)\\ &+\Big(\frac{e^2}{r}+\frac{l(l+1)}{r^2}-k^2\Big)(v^2-h^2)+v^2W\Big], \end{aligned} \tag{A12}$$

where h is the asymptotic form for v satisfying the equation

$$h''+\Big(k^2-\frac{l(l+1)}{r^2}-\frac{e^2}{r}\Big)h=0. \tag{A13}$$

For exact v and h, we have

$$h^2(a)[h'(a)/h(a)]=J'(a)+v^2(a)J(a). \tag{A14}$$

It can easily be shown that $[h'(a)/h(a)]$ is stationary with respect to variations in v and h as long as $v\to h$ as $r\to\infty$. Thus, any approximate form for h and v

will give $[h'(a)/h(a)]$ (and hence the phase shift) correct to a higher order than the v and h. If v and h depend on free parameters, these may be found in the usual way by using the stationary character of $[h'(a)/h(a)]$.

If the Born approximation conditions are valid for $r>a$, we may take $v\cong h$. In that case Eq. (A14) becomes

$$[h'(a)/h(a)]\cong[h(a)]^{-2}\int_a^\infty h^2 W dr+(A-k^2B). \quad \text{(A15)}$$

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Σ-Λ Relative Parity and the $\Sigma^0\to\Lambda^0+\gamma+\gamma$ Decay

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It is shown that the π^0-pole term predicts a large difference by nearly two to three orders of magnitude for the branching ratio of the $\Sigma^0\to\Lambda^0+2\gamma$ decay mode, depending upon the value of the $\Sigma-\Lambda$ relative parity. It is further argued that this difference is not masked, even if we include other diagrams. It is thus suggested that a study of the branching ratio of the $\Sigma^0\to\Lambda^0+2\gamma$ decay may serve to determine the $\Sigma-\Lambda$ relative parity.

THE determination of $\Sigma-\Lambda$ relative parity is at present an urgent problem for the development of the theory of elementary particles. Suggestions have been made[1] to determine this parity unambiguously from correlation effects in the Dalitz decay, $\Sigma^0\to\Lambda^0+e^+ +e^-$. In this letter we wish to point out that a study of the branching ratio of the $\Sigma^0\to\Lambda^0+\gamma+\gamma$ decay mode could also serve this purpose.

We will denote even and odd $\Sigma-\Lambda$ parities by $P=\pm1$, respectively. The corresponding parameters will often be denoted by the superscripts $\pm$. First of all, we notice that under the assumption of charge-independent strong interactions and minimal electromagnetic interactions the decay $\Sigma^0\to\Lambda^0+2\gamma$ is strictly forbidden for either parity, as long as we switch on only those interactions which involve the particles with integral isotopic spin (Σ, Λ, and π) and the photon. This is because the above-mentioned class of interactions is invariant under the isotopic rotation $e^{i\pi T_2}$ (under which $\Sigma^0\to-\Sigma^0$, $\Lambda^0\to\Lambda^0$) together with $\gamma\to-\gamma$ (γ from the isovector current). The decay takes place only when[2] the strong and electromagnetic (minimal) interactions involving the particles with half-integral isotopic spin (N, Ξ, and K) are switched on. Thus if we neglect the K-meson cloud, the decay has to occur only through baryon (half-integral isotopic spin) loops, the main contribution of which may be expected to be given by the π^0-pole term, shown in Fig. 1.

Let us, therefore, first study the contribution of Fig. 1, hoping that it dominates. The possible importance of other intermediate states will be discussed at the end. The $\pi^0\to2\gamma$ vertex in Fig. 1, involving the baryon (half-integral isotopic spin) loops, can be estimated from the observed rate of π^0 decay. We denote the $\pi^0\to2\gamma$ matrix element by

$$(2\pi)^4\delta^4(p_\pi-k_1-k_2)F_\pi\epsilon_{\alpha\beta\gamma\delta}e_{1\alpha}e_{2\beta}k_{1\gamma}k_{2\delta},$$

where four-vector $e_{1,2}$ denotes the polarizations of the photons with four-momenta $k_{1,2}$, respectively. We may safely assume that the form factor F_π is nearly a constant. Then in the rest frame of Σ^0, the rate of

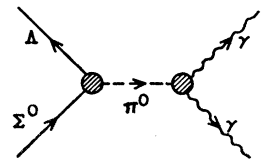

FIG. 1. The π^0 pole diagram for $\Sigma^0\to\Lambda^0+2\gamma$ decay.

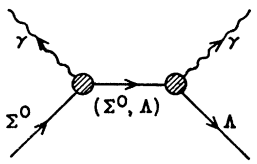

FIG. 2. The (Σ^0,Λ)-pole diagram for $\Sigma^0\to\Lambda^0+2\gamma$ decay.

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[1] J. Sucher and G. A. Snow, Nuovo cimento **18**, 195 (1960); N. Byers and H. Burkhardt, Phys. Rev. **121**, 281 (1961); L. Michel and H. Rouhaninejad, Phys. Rev. **122**, 242 (1961); and S. Chiba (to be published).

[2] In this case one photon comes from the isoscalar and the other from the isovector part of the current.