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Classical Theory of Radiation Solutions for the Coulomb Field

P. A. Clavier
Aeronutronic Division, Ford Motor Company, Newport Beach, California
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Eliezer has contended that the Dirac equations for the motion of a particle in a Coulomb field, when the radiative reaction is included, do not have physically realizable solutions. The problem is reinvestigated and Eliezer's conclusion proved false. Two-dimensional solutions (spirals) are found, which can be described with the exclusive use of integrable functions. One-dimensional solutions (radial trajectories) are found for both a repulsive and an attractive pole. The radial solutions involve distributions (in the sense of Schwartz). A consequence of the solutions is the emission of a strong pulse of radiation when a particle hits an attractive pole or when it leaves a respulsive one.

I. INTRODUCTION

The emission of electromagnetic energy by an accelerated charged particle is well proved experimentally. However it has remained, to date, a theoretical controversy. While no one seriously doubts the fundamental nature of Dirac's[1] radiative reaction (nor Wheeler and Feynman's[2] interpretation of it), well-behaved solutions of the resulting set of equations have been proved nonexisting in cases of physical significance. Our intuition, however, demands proper answers in such cases.

The chief destroyer of proper solutions has been Eliezer[3] who showed in a series of papers that physical solutions do not exist for three particular fields of force: the field of a thin infinite charged plate, the attractive Coulomb field, and the repulsive Coulomb field. At this point, the opinion was even held that the three-dimensional relativistic set of equations proposed by Dirac does not possess any solution when the particle is influenced by an external force. However, Plass[4] showed, that this is not the case and that a proper solution exists as long as the external force does not become infinite at any point on the trajectory. Plass even showed that certain infinities were permissible but that others were not. The pole of a Coulomb field remained forbidden. While Plass' work limits Eliezer's conclusion for the Coulomb field to the case when the pole is on the trajectory, it nevertheless leaves our intuition baffled in one very important physical case. It should be noted at this point that Eliezer's reasoning is complete only for a particle moving radially from or toward the pole of the field. Eliezer himself does not claim that he has a proof for the nonexistence of a solution when the particle spirals around the pole either before it falls into it or after it is released from it; he only advances some heuristic arguments to substantiate his contention.

We shall prove that there is indeed a well-behaved solution which is tangent to a logarithmic spiral at the pole. This solution, however, is not very satisfactory because, while some such spirals will pass through any point in space, the velocity of the particle at that point is fixed in magnitude and must remain tangent to a fixed cone apexed at the point in question. One has the feeling that many more solutions must exist which spiral into or out of the pole of the field and pass through a given point in space. One would expect that at a given point in space one can choose the magnitude and the direction of the velocity of the particle within some dense and rather large domain and obtain a trajectory which reaches or has previously reached the pole. In particular, proper solutions are expected for particles moving on the straight line defined by the point in question and the pole.

This apparent paradox can be resolved if one remembers that the integrable functions do not form a differentiable group. In order to form such a group one must add to the integrable functions the measures and the distributions defined by Schwartz.[5] Eliezer's proof of the nonexistence of a radial solution which includes the pole is based on the use of integrable functions exclusively. It happens, as will be shown later, that the proper solution in this case is not describable with integrable functions exclusively. The solution exists but its description requires the introduction of one of Hadamard's pseudo-functions. The spiral solutions can also, in general, only be described with the introduction of distributions. We shall, however, restrict in this paper the search for solutions in the case of a Coulomb field to the spirals describable with integrable functions exclusively and to the properties of the radial trajectories.

II. COULOMB FIELD, SPIRAL SOLUTION DESCRIBABLE EXCLUSIVELY BY INTEGRABLE FUNCTIONS

In Cartesian coordinates the relativistic classical equations of motion of a charged particle including radiative reactions are

$$\dot{u}_i = (e/mc)F_{ik}u^k + 1/b(\ddot{u}_i - (1/c^2)u_i\dot{u}^k\dot{u}_k). \qquad (1)$$

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[2] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).
[3] C. J. Eliezer, Revs. Modern Phys. **19**, 147 (1947).
[4] G. N. Plass, Revs. Modern Phys. **31**, 37 (1961).
[5] L. Schwartz, *Theorie des distributions* (Hermann et Cie, Paris, 1950).

The tensor F_{ik} is the electromagnetic field tensor:

$$F_{ik}=\begin{vmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{vmatrix}. \tag{2}$$

The parameter b is the reciprocal of the time taken by a light signal to travel a distance equal to two-thirds of the classical electron radius:

$$b=\tfrac{3}{2}(mc^3/e^2). \tag{3}$$

The mass m is the rest mass of the particle with charge e, c is the velocity of light, and E and H are the retarded electric and magnetic fields. The velocity u is the rate of change of location x_i per change of proper time τ. The dots are for differentiation in function of proper time, thus:

$$u_i=\dot{x}_i. \tag{4}$$

We shall use intrinsic coordinates. Let $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ be the unit vectors in the direction of the tangent, principal normal, and binormal, respectively. Let $\mathbf{u}$ be the three-dimensional vector representing the proper velocity and u be its magnitude. We obtain:

$$\begin{aligned} \mathbf{u}&=u\mathbf{t}, \\ d\mathbf{u}/d\tau&=\dot{u}\mathbf{t}+k_1u^2\mathbf{n}, \end{aligned} \tag{5}$$

$$d^2\mathbf{u}/d\tau^2=(\ddot{u}-k_1^2u^3)\mathbf{t}+(3k_1u\dot{u}+(dk_1/ds)u^3)\mathbf{n}-k_1k_2u^3\mathbf{b}.$$

The following Frenet formulas have been used:

$$\begin{aligned} d\mathbf{t}/ds&=k_1\mathbf{n}, \\ d\mathbf{b}/ds&=k_2\mathbf{n}, \\ d\mathbf{n}/ds&=-k_1\mathbf{t}-k_2\mathbf{b}. \end{aligned} \tag{6}$$

In the Frenet formulas, k_1 is the curvature, k_2 is the torsion, and s is the arc length along the trajectory. After the introduction of intrinsic coordinates, the Dirac equations become

$$\ddot{u}-b\dot{u}-\frac{u\dot{u}^2}{c^2+u^2}=k_1^2u^3\left(1+\frac{u^2}{c^2}\right)-\frac{eb}{mc}(c^2+u^2)^{\frac{1}{2}}E_t, \tag{7}$$

$$k_1^2u^2+k_1(3u\dot{u}-bu^2)=\frac{eb}{mc}[uH_b-(c^2+u^2)^{\frac{1}{2}}E_n], \tag{8}$$

$$k_2=\frac{eb}{mc}\,\frac{uH_n+(c^2+u^2)^{\frac{1}{2}}E_b}{k_1u^3}. \tag{9}$$

To find, in the case of the Coulomb field, a spiral solution exclusively describable by functions, one replaces E_t and E_n in Eqs. (7) and (8) by

$$E_t=(A/\rho^2)\cos\varphi, \tag{10}$$

$$E_n=(A/\rho^2)\sin\varphi. \tag{11}$$

The angle φ is the angle between the tangent to the trajectory and the radius of that trajectory. The polar coordinates (ρ,θ) originate at the pole of the field. If φ has a limit φ_0 when ρ decreases to zero, the trajectory will be tangent at the pole to a logarithmic spiral of the form

$$\rho=\rho_0\exp(\theta/\tan\varphi_0). \tag{12}$$

The arc length is given by

$$s=\rho/\cos\varphi_0. \tag{13}$$

The curvature is given by

$$k_1=\sin\varphi_0/\rho=\tan\varphi_0/s. \tag{14}$$

Expanding the velocity in powers of the arc length one finds that the velocity reaches a finite limit u_0 at the pole, given by

$$u_0^3=(eb/m)A/\sin\varphi_0. \tag{15}$$

Equation (15) obtains, provided

$$\tfrac{1}{2}|(eb/mc^3)A|=(\cos2\varphi_0)^{\frac{3}{2}}/(1-\cos2\varphi_0). \tag{16}$$

The bars are for absolute value.

There is always a solution for a value of φ_0 between 0 and $\pi/4$. This solution is such that

$$u_0A>0. \tag{17}$$

It satisfies Eq. (17) which is the condition for a proper solution. Indeed, Eq. (17) states that when the field is attractive the velocity near the pole is directed toward the pole and when the field is repulsive, the velocity near the pole is directed away from the pole.

If one is not contented with the logarithmic spiral, one can improve the accuracy of the result by expanding ρ and u in series of s.

Let us write

$$\rho=s\cos(\varphi_0+\varphi_1s). \tag{18}$$

One also defines

$$u=u_0+u_1s. \tag{19}$$

The parameters φ_1 and u_1 are finite constants. It is of course not necessary for φ to be simply

$$\varphi=\varphi_0+\varphi_1s. \tag{20}$$

The definition of polar coordinates yields

$$(ds/d\theta)^2=\rho^2+(d\rho/d\theta)^2. \tag{21}$$

Using Eq. (18) and Eq. (21), one obtains

$$ds/d\theta=(s/\tan\varphi_0)[1-\varphi_1s(\tan\varphi_0+2/\tan\varphi_0)], \tag{22}$$

$$d\rho/d\theta=(s\cos\varphi_0/\tan\varphi_0)[1-\varphi_1s(3\tan\varphi_0+2/\tan\varphi_0)]. \tag{23}$$

For any curve, however, one has

$$\tan\varphi=\rho/(d\rho/d\theta). \tag{24}$$

The introduction of Eq. (24) leads to

$$\varphi=\varphi_0+2\varphi_1s. \tag{25}$$

For any curve, one also has

$$k_1=d(\varphi+\theta)/ds. \tag{26}$$

Thus, the curvature becomes

$$k_1=(\tan\varphi_0/s)[1+\varphi_1 s(\tan\varphi_0+4/\tan\varphi_0)]. \quad (27)$$

Replacing the radius by the arc length in Eqs. (10) and (11) one finds

$$E_t=A/s^2\cos\varphi_0, \quad (28)$$

$$E_n=(A\tan\varphi_0/s^2\cos\varphi_0)[1+2\varphi_1 s(\tan\varphi_0+1/\tan\varphi_0)]. \quad (29)$$

Replacing in Dirac's equations, one obtains

$$2\varphi_1\left(\tan\varphi_0+\frac{4}{\tan\varphi_0}\right)=\frac{u_1}{u_0}\left(\frac{c^4A\cos\varphi_0}{eu_0(c^2+u_0^2)^{\frac{3}{2}}\sin^2\varphi_0}-3-\frac{2u_0^2}{c^2+u_0^2}\right), \quad (30)$$

$$u_0^2=\frac{eA(c^2+u_0^2)^{\frac{1}{2}}}{mc\cos\varphi_0}\left[\frac{u_1u_0}{c^2+u_0^2}+2\varphi_1\left(\tan\varphi_0+\frac{1}{\tan\varphi_0}\right)\right]. \quad (31)$$

The values of u_1 and φ_1 are extracted easily from Eqs. (30) and (31).

We could improve step by step our knowledge of the solution. Barring the occurrence of indeterminacies at a later step, the only degree of freedom is the trivial rotation of the trajectory around the straight line defined by a point on the trajectory and the pole of the field.

When the pole is due to a single positive charge and the moving particle is an electron, one obtains

$$A=-e, \quad (32)$$

$$\varphi_0\sim\pi/6, \quad (33)$$

$$u_0=-1.44c, \quad (34)$$

(the corresponding real-time velocity is $0.82c$)

$$\varphi_1=-5.25\times10^{10} \quad (\mathrm{cm}^{-1}), \quad (35)$$

$$u_1=-2.88\times10^{11}c \quad (\mathrm{cm}^{-1}). \quad (36)$$

One notes that within the approximation, the proper velocity decreases toward the pole. This reversal of expected behavior will be clarified in the next section of the paper. One finds in general that the electron first increases its velocity toward the pole, that the velocity passes by a maximum and then, very near the pole, decreases rapidly. The decrease of the kinetic energy is, of course, accompanied by the emission of a considerable amount of radiation.

In the case of a repulsive single pole, one obtains

$$\begin{aligned}&\varphi_0\sim\pi/6, \quad u_0=1.44c,\\ &\varphi_1=1.83\times10^{11} \quad (\mathrm{cm}^{-1}),\\ &u_1=-1.00\times10^{12}c \quad (\mathrm{cm}^{-1}).\end{aligned} \quad (37)$$

The radiative reaction is so intense that the electron at first loses kinetic energy while spiraling away from the pole.

The behavior of both the attractive and the repulsive systems suggests an analogy with nuclear models. The penetration of an attracted particle or escape of a repulsed one is hampered by a short-range force.

III. RADIAL MOTION IN A COULOMB FIELD

The behavior of the only solution describable exclusively with integrable functions suggests the reason for the lack of other solutions of that kind. The acceleration changes sign near the pole. Suppose now that this change of sign occurs in general at the pole itself, then the solution must lead to an indeterminacy at the pole. The velocity of the particle will have two values at the pole, one reached by analytic continuation of the neighborhood value toward the pole and the other obtained at the pole itself. Thus, an attracted particle will be continuously accelerated toward the pole and, when reaching it, will suddenly be decelerated. On the contrary, a particle which is repulsed by the pole will suddenly lose a certain amount of kinetic energy before it leaves the pole. The sudden loss of kinetic energy in both cases is compensated by the emission of a burst of electromagnetic radiations.

We shall restrict our treatment of this problem to radial motion and indeed find that the *a priori* conclusions hold. The Dirac equation of motion becomes

$$\ddot{u}-b\dot{u}-\frac{u\dot{u}^2}{c^2+u^2}=-\frac{ebA}{mc}(c^2+u^2)^{\frac{1}{2}}\frac{1}{s^2}. \quad (38)$$

We analyze first the case of an attractive pole. In this case, one must reach the pole at a time

$$\tau_1>0. \quad (39)$$

We may assume

$$s>0. \quad (40)$$

A proper solution will be such that

$$u<0, \quad A<0. \quad (41)$$

One introduces

$$G=\frac{\dot{u}}{(c^2+u^2)^{\frac{1}{2}}}\exp(-b\tau). \quad (42)$$

The formal integration of Eq. (42) leads to

$$u+(c^2+u^2)^{\frac{1}{2}}=[u_0+(c^2+u_0^2)^{\frac{1}{2}}]\times\exp\left[\int_0^\tau G\exp(b\tau)d\tau\right]. \quad (43)$$

The velocity u_0 is the value of u at the origin of time.

Integrating Eq. (43), one finds

$$s=s_0+\tfrac{1}{2}[u_0+(c^2+u_0^2)^{\frac{1}{2}}]\int_0^\tau \exp\left[\int_0^\tau G\exp(b\tau)d\tau\right]d\tau$$
$$-\frac{c^2}{2[u_0+(c^2+u_0^2)^{\frac{1}{2}}]}\times\int_0^\tau \exp\left[-\int_0^\tau G\exp(b\tau)d\tau\right]d\tau. \quad (44)$$

The arc s_0 defines the position of the particle at time zero.

Equation (38) can be written

$$\dot{G}s^2\exp(b\tau)=-ebA/mc. \quad (45)$$

It leads to

$$G=-\frac{ebA}{mc}\int_0^\tau \frac{\exp(-b\tau)}{s^2}d\tau+G_0. \quad (46)$$

The constant G_0 is an integration constant.

It will, however, be more useful to write

$$G=-\frac{ebA}{mc}\int_0^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau+\frac{ebA}{mc}\int_\tau^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau+G_0. \quad (47)$$

From Eq. (44), for given $G(\tau)$ and τ_1, it is possible to find a starting location and a starting velocity

$$s_0>0, \quad (48)$$

$$u_0<0, \quad (49)$$

provided

$$\frac{\int_0^{\tau_1}\exp\left[-\int_0^\tau G\exp(b\tau)d\tau\right]d\tau}{\int_0^{\tau_1}\exp\left[\int_0^\tau G\exp(b\tau)d\tau\right]d\tau}>0. \quad (50)$$

It is the case as long as G is finite. From Eq. (46), a difficulty can only occur at the pole, when s becomes zero. As a matter of fact it has been shown by Eliezer[3] that a solution which would involve only integrable functions does not exist at the pole.

Equation (46) is, however, not the complete solution of Eq. (45). One can certainly add to G a punctual distribution P located at τ_1, such that

$$Ps^2=0. \quad (51)$$

The distribution theory of Schwartz[5] is still not widely used in the United States. While the best work on the subject is Schwartz's own book, publications in English include introductions by Halperin[6] and Güttinger.[7]

The natural choice for P leads to

$$G=-\frac{ebA}{mc}\left(\mathrm{Pf}\int_0^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau\right)+\frac{ebA}{mc}\left(\mathrm{Pf}\int_\tau^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau\right)+G_0. \quad (52)$$

The symbol Pf is for pseudo-function or Hadamard's finite part of the integral. The form of Eq. (52) is such that G tends toward infinity near the pole but never reaches it. Because G never reaches infinity, Eq. (50) is satisfied and a correct solution exists up to the pole. The proof that Eq. (52) is solution of Eq. (45) is given in the Appendix.

We define the constant

$$G_1=G_0-\frac{ebA}{mc}\left(\mathrm{Pf}\int_0^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau\right). \quad (53)$$

It permits us to replace Eq. (52) by

$$G=\frac{ebA}{mc}\left(\mathrm{Pf}\int_0^{\tau_1}\frac{\exp(-b\tau)}{s^2}d\tau\right)+G_1. \quad (54)$$

We can, in an analogous manner, replace Eq. (44) by

$$s=-\tfrac{1}{2}[u_0+(c^2+u_0^2)^{\frac{1}{2}}]\exp\left[\int_0^{\tau_1}G\exp(b\tau)d\tau\right]\times\int_\tau^{\tau_1}\exp\left[-\int_\tau^{\tau_1}G\exp(b\tau)d\tau\right]d\tau$$
$$+\frac{c^2}{2[u_0+(c^2+u_0^2)^{\frac{1}{2}}]}\exp\left[-\int_0^{\tau_1}G\exp(b\tau)d\tau\right]\times\int_\tau^{\tau_1}\exp\left[\int_\tau^{\tau_1}G\exp(b\tau)d\tau\right]d\tau. \quad (55)$$

When τ tends toward τ_1, the right-hand side in Eq. (54) is of the form

$$G(\tau)=ebA/mc[F(\tau_1)-F(\tau)-F(\tau_1)(1-Y_{\tau_1})]+G_1. \quad (56)$$

In Eq. (56), $F(\tau)$ is positive, $F(\tau_1)$ is infinite and Y_{τ_1} is Heaviside's step function starting at τ_1. Thus, near τ_1, $G(\tau)$ is given by

$$G(\tau)=-(ebA/mc)F(\tau)+G_1. \quad (57)$$

At τ_1, $G(\tau)$ becomes

$$G(\tau_1)=G_1. \quad (58)$$

[6] I. Halperin, *Introduction to the Theory of Distributions* (University of Toronto Press, Toronto, 1952).

[7] W. Güttinger, Phys. Rev. 89, 1004 (1953).

Because of Eq. (58), the velocity at the pole is given by

$$u_1=\tfrac{1}{2}[u_0+(c^2+u_0^2)^{\frac{1}{2}}]\exp\left[\int_0^{\tau_1}G\exp(b\tau)d\tau\right]-\frac{c^2}{2[u_0+(c^2+u_0^2)^{\frac{1}{2}}]}\exp\left[-\int_0^{\tau_1}G\exp(b\tau)d\tau\right]. \quad (59)$$

The velocity at the pole must be negative.

Because of Eq. (57) the velocity is discontinuous near the pole and tends to infinity before it reaches the value given by Eq. (59). The important fact, however, is that, because G is integrable from 0 to τ_1, Eq. (55) is valid. The infinite term in the velocity is the one which involves

$$\exp\left[\int_\tau^{\tau_1}G\exp(b\tau)d\tau\right], \quad (60)$$

it is not the one which involves

$$\exp\left[-\int_\tau^{\tau_1}G\exp(b\tau)d\tau\right]. \quad (61)$$

As expected *a priori*, the velocity remains negative, tends toward infinity and finally changes abruptly at the pole to the finite value given by Eq. (59). The kinetic energy lost in the abrupt change of velocity at the pole is accompanied by the emission of a burst of electromagnetic energy. One should note also that the solution is a function of the conditions at the origin of time. It thus includes all possible radial motion toward an attractive pole. A simple mathematical form for $s(t)$ has not been found. While we know the behavior of the solution, exact numerical computations require programming on a computer. It was done by Plass[4] for cases where the pole is not on the trajectory.

To get a possibly better feeling for the significance of the discontinuity at the pole, assume that even near the pole

$$s=u_1(\tau_1-\tau). \quad (62)$$

Of course Eq. (62) can only be valid at the pole and not much can be expected from such an approximation. Replacing in Eq. (54), one obtains

$$G=-\frac{ebA}{mc}\frac{\exp(-b\tau_1)}{u_1^2}\left(\frac{1}{\tau_1-\tau}\right)+G_1. \quad (63)$$

Replacing in Eq. (55) and differentiating, one obtains

$$u=\tfrac{1}{2}[u_0+(c^2+u_0^2)^{\frac{1}{2}}]\exp\left[\int_0^{\tau_1}G\exp(b\tau)d\tau\right]\times\exp\left[\frac{ebA}{mcu_1^2}\ln\left(\frac{\tau_1-\tau}{\tau_1-\tau_1}\right)\right]-\frac{c^2}{2[u_0+(c^2+u_0^2)^{\frac{1}{2}}]}\times\exp\left[-\int_0^{\tau_1}G\exp(b\tau)d\tau\right]\times\exp\left[-\frac{ebA}{mcu_1^2}\ln\left(\frac{\tau_1-\tau}{\tau_1-\tau_1}\right)\right]. \quad (64)$$

The value obtained is infinite for τ different from τ_1. The important fact is that, because A is negative, the infinite value is negative.

In the case of a repulsive pole, it has been shown by Plass[4] that a proper solution exists provided the particle does not start at the pole. Suppose a particle starts at

$$\tau=\tau_1+\epsilon.$$

The time delay is made arbitrarily small but finite. The time τ_1 is the time at which the particle started from the pole. One can write

$$s=s_1+\tfrac{1}{2}[u_1+(c^2+u_1^2)^{\frac{1}{2}}]\times\int_{\tau_1+\epsilon}^{\tau}\exp\left[\int_{\tau_1+\epsilon}^{\tau}G\exp(b\tau)d\tau\right]d\tau-\frac{c^2}{2[u_1+(c^2+u_1^2)^{\frac{1}{2}}]}\times\int_{\tau_1+\epsilon}^{\tau}\exp\left[-\int_{\tau_1+\epsilon}^{\tau}G\exp(b\tau)d\tau\right]d\tau. \quad (65)$$

One can also write

$$G=-\frac{ebA}{mc}\int_{\tau_1+\epsilon}^{\tau}\frac{\exp(-b\tau)}{s^2}d\tau+G_1. \quad (66)$$

In the present case, we have the following inequalities:

$$A>0,\quad u>0,\quad s>0,$$
$$\tau>\tau_1,\quad u_1>0,\quad s_1>0.$$

Because G is still given by Eq. (42), the value of G at infinity is

$$G(\infty)=0.$$

One also has

$$G_1=\frac{ebA}{mc}\int_{\tau_1+\epsilon}^{\infty}\frac{\exp(-b\tau)}{s^2}d\tau; \quad (67)$$

it yields

$$G=\frac{ebA}{mc}\int_{\tau}^{\infty}\frac{\exp(-b\tau)}{s^2}d\tau. \quad (68)$$

Because G is positive, a proper solution is obtained. When ϵ is made infinitely small, there is little reason for Eq. (68) not to hold.

The following limit does not hold:

$$\lim_{\epsilon\to 0}\exp\left[\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]=1. \quad (69)$$

The velocity u_1 in Eq. (65) does not become the velocity at the pole. If u_1 is to be the velocity at the pole, one

must write

$$s=\tfrac{1}{2}[u_1+(c^2+u_1^2)^{\frac{1}{2}}]\exp\left[-\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]$$
$$\times\int_{\tau_1}^{\tau}\exp\left[\int_{\tau_1}^{\tau}G\exp(b\tau)d\tau\right]d\tau-\frac{c^2}{2[u_1+(c^2+u_1^2)^{\frac{1}{2}}]}$$
$$\times\exp\left[\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]$$
$$\times\int_{\tau_1}^{\tau}\exp\left[-\int_{\tau_1}^{\tau}G\exp(b\tau)d\tau\right]d\tau. \quad (70)$$

Replace G by its finite part. It is of the form

$$\mathrm{Pf}G=ebA/mc[F(\tau)-F(\tau_1)(1-Y_{\tau_1})]. \quad (71)$$

The function $F(\tau)$ is positive; the function $F(\tau_1)$ is infinite.

One obtains for Eq. (70)

$$s=\tfrac{1}{2}[u_1+(c^2+u_1^2)^{\frac{1}{2}}]\exp\left[-\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]$$
$$\times\int_{\tau_1}^{\tau}\exp\left[\int_{\tau_1}^{\tau}\mathrm{Pf}G\exp(b\tau)d\tau\right]d\tau-\frac{c^2}{2[u_1+(c^2+u_1^2)^{\frac{1}{2}}]}$$
$$\times\exp\left[\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]$$
$$\times\int_{\tau_1}^{\tau}\exp\left[-\int_{\tau_1}^{\tau}\mathrm{Pf}G\exp(b\tau)d\tau\right]d\tau. \quad (72)$$

Equation (72) yields the same result as Eq. (70) outside of the pole, but, at the pole itself. Eq. (72) yields

$$u=\tfrac{1}{2}[u_1+(c^2+u_1^2)^{\frac{1}{2}}]\exp\left[-\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]$$
$$-\frac{c^2}{2[u_1+(c^2+u_1^2)^{\frac{1}{2}}]}$$
$$\times\exp\left[\lim_{\epsilon\to 0}\int_{\tau_1}^{\tau_1+\epsilon}G\exp(b\tau)d\tau\right]. \quad (73)$$

It is infinite.

Thus, before the particle leaves the pole, it loses a considerable amount of kinetic energy and emits a burst of electromagnetic radiation.

Because u_1 can be chosen at will, we have obtained the complete set of solutions for a particle leaving radially a repulsive Coulomb pole.

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APPENDIX: ORDER OF INFINITUDE OF A PSEUDOFUNCTION

We wish to show that

$$Ps^2=0. \quad (74)$$

The distribution P is defined by

$$P=\frac{d}{d\tau}\mathrm{Pf}\int_{\tau}^{\tau_1}\frac{d\tau}{s^2}. \quad (75)$$

The following condition holds:

$$s(\tau_1)=0. \quad (76)$$

Let one write

$$U(\tau)=\int d\tau/s^2; \quad (77)$$

the pseudofunction can be made explicit in the following manner:

$$\mathrm{Pf}\int_{\tau}^{\tau_1}\frac{d\tau}{s^2}=U(\tau_1)-U(\tau)-U(\tau_1)(1-Y_{\tau_1}). \quad (78)$$

The function Y is Heaviside's step function. The definition of P yields

$$P=U(\tau_1)\delta_{\tau_1}. \quad (79)$$

The symbol δ is for Dirac's delta function. From Eq. (77), one obtains

$$s^2(\tau_1)U(\tau_1)=0. \quad (80)$$

The form of Eq. (74) is similar to

$$f(\tau)\delta_0=0. \quad (81)$$

The solution of Eq. (81) is

$$f(0)=0. \quad (82)$$

Thus Eq. (74) is fulfilled. One proves Eq. (81) by projection on an appropriate functional space, the space of the class D of functions. The functions of the class D have all compact domains; they are also absolutely continuous and always differentiable. The derivative of any function of the class is a function of the class. Thus, for any function φ of the class D

$$f(\tau)\delta_0\cdot\varphi=\int_{-\infty}^{+\infty}f(\tau)\delta_0\varphi(\tau)d\tau=[f(\tau)\varphi(\tau)]_{(\tau=0)}. \quad (83)$$

Because $\varphi(0)$ is finite, or zero, the projection on the right-hand side of Eq. (83) will be identically zero when Eq. (82) holds.

Schwartz's major contribution to algebra has been the replacement of the coordinates of common algebra (the Dirac delta functions) by the much more useful, because much better behaved, functions of the class D.