atom tends to form a molecular ion because it tends to drift naturally, i.e. without any specific attractive energy terms, to a position nearer the anion than is possible in the configurations considered here. A preliminary calculation in this direction is not conclusive. It will also be necessary in accurate future work to take into account the polarization models of Lundquist[9] and Woods *et al.*,[11] for ions and atoms that are close together and to consider other than exponential forms for the repulsive interactions.[12,13]

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Magnetic Field Dependence of the Superconducting Penetration Depth in Thin Specimens

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The magnetic field dependence of the superconducting penetration depth for very thin films as predicted by the Ginzburg-Landau theory is considered. The results obtained depend upon the boundary conditions on the film. For the usual case of equal magnetic fields on opposite sides of the film, the penetration depth increases smoothly toward infinity as the critical field is approached, corresponding to a second-order phase transition. For the less common case of unequal fields on opposite sides, the penetration depth increases toward a finite value as the critical field is approached, corresponding to a first-order phase transition. The results for the latter case are shown to agree remarkably well with the very precise experiments of Garwin, Erlbach, and Sarachik on the field dependence of the penetration depth of a 250 A film of Pb. The penetration depth in zero field as a function of thickness is also considered.

I. INTRODUCTION

THE magnetic field dependence of the superconducting penetration depth has provoked considerable interest. This interest has been chiefly concerned with bulk superconductors. Because most bulk superconductors satisfy the nonlocal condition (coherence length ξ greater than penetration depth λ), the interpretation of experimental results can be quite difficult. The reason for this difficulty is that there is as yet no satisfactory theory that considers nonlocal effects in the presence of a strong magnetic field. One way to skirt this problem is to limit the coherence length by making the dimensions of the superconductor small, as in the case of a thin evaporated film. By making the thickness d of the film, and hence ξ, less than λ, the superconductor will satisfy the local limit ($\xi<\lambda$). There is fortunately a satisfactory local theory that is valid for all fields up to the critical field; it is the non-linear phenomenological theory of Ginzburg-Landau[1] (GL). Since the GL equations have been derived from the microscopic theory by Gor'kov,[2] the solutions of the GL equations will have the same rigor as those of the microscopic theory. Thus, by considering thin films the nonlocal problem can be avoided, and the comparison between theory and experiment can be made with considerably less ambiguity. Therefore, in this paper we shall consider the magnetic field dependence of the penetration depth in *thin* specimens.

In the GL theory the penetration depth, λ, is inversely proportional to the order parameter, ψ, which is a function of coordinates, magnetic field, and temperature. However, if $d/\lambda \ll \kappa^{-1} \approx 10$, where d is the thickness of the specimen and κ is the nonlinear coupling constant of the theory, then ψ (and λ) is independent of coordinates. Thus, if we restrict ourselves to this condition, the dependence on coordinates is eliminated and the field dependence of the penetration depth may be expressed as

$$\lambda(T,0)/\lambda(T,H)=\psi(T,H)/\psi(T,0). \tag{1}$$

Thus the problem of finding the field dependence of λ reduces to solving the GL equations for $\psi(T,H)/\psi(T,0)$ for a specific geometry and specific boundary conditions.

There is evidence[3] that $\lambda(T,0)$ increases as the thickness decreases, and Tinkham[4] has suggested a simplified way of calculating this increase. In Sec. IIA a more fundamental calculation of this effect will be presented. The field dependence of the penetration depth for equal and unequal values of the field on opposite sides of the film will be considered in Sec. IIB. In Sec. III comparison of these results with an experiment of Garwin, Erlbach, and Sarachik[5] will be made.

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† Operated with support from the U. S. Army, Navy and Air Force.
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[3] W. B. Ittner, III, Phys. Rev. **119**, 1591 (1960).
[4] M. Tinkham, Phys. Rev. **110**, 26 (1958).
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II. SIZE EFFECTS IN THE GINZBURG-LANDAU THEORY

A. Variation of Penetration Depth with Thickness in Zero Field

The main topic of this paper is the effect of a magnetic field on the penetration depth but before the effects of a magnetic field can be considered, we must first see how the penetration depth depends on thickness.

Gor'kov[6] has also derived the GL equation for the case of impurity scattering of the electrons. He finds that the penetration depth is modified by collisions and depends on a dimensionless loss parameter ρ (ratio of the coherence length to the distance between collisions) in the following way:

$$\lambda(T,0)=\lambda_L(T,0)\chi^{-\frac{1}{2}}(\rho), \tag{1}$$

in which

$$\chi(\rho)=\frac{8}{7\zeta(3)}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2(2m+1+\rho)} \tag{2}$$

$$=\frac{8}{7\zeta(3)}\left(\frac{1}{\rho}\right)\left[\frac{\rho}{1+\rho}+\frac{\Phi'(\frac{1}{2})}{4}+\frac{1}{2\rho}\left\{\Phi(\tfrac{1}{2})-\Phi\left(\frac{\rho+1}{2}\right)\right\}\right], \tag{3}$$

$$\rho=\xi_0/v_F\tau, \tag{4}$$

where τ is the collision time, ξ_0 is the bulk coherence length, v_F is the Fermi velocity, ζ is the Riemann zeta-function, Φ is the digamma function, Φ' is the trigamma function,[7] and $\lambda_L(T,0)$ is the London penetration depth (corresponding to $\tau=\infty$). Equation (3) has the following asymptotic forms: as ρ approaches 0 (infinite collision time),

$$\chi(\rho)\rightarrow 1,$$

and as ρ approaches ∞ (very short relaxation time),

$$\chi(\rho)\rightarrow\frac{8}{7\zeta(3)}\frac{1+\frac{1}{4}\Phi'(\frac{1}{2})}{\rho}=\frac{1.173}{\rho}.$$

In order to apply these formulas to any experimental results, one needs to know what the collision rate τ^{-1} is. (The following analysis is similar to that of Tinkham.[4]) In a bulk superconductor one would expect the rate to be just v_F/ξ_0. If impurities of mean separation l are present the rate would be v_F/l and boundary scattering gives a rate v_F/d, where d is the size of the specimen. If it can be assumed that these processes are independent and equivalent, then the total collision rate is

$$1/\tau=v_F/\xi_0+v_F/d+v_F/l. \tag{5}$$

Putting (5) into (4) we obtain (for our considerations we will neglect impurity scattering)

$$\rho=1+\xi_0/d. \tag{6}$$

It is seen that the bulk state corresponds to $\rho=1$. The penetration depth, expressed in terms of the bulk penetration depth $\lambda_B(T,0)$, is

$$\lambda(T,0)=G(\rho)\lambda_B(T,0), \tag{7}$$

where

$$G(\rho)=[\chi(1)/\chi(\rho)]^{\frac{1}{2}}. \tag{8}$$

Equation (8) has been calculated and is shown in Fig. 1.

B. Variation of Penetration Depth in a Magnetic Field

In this section we shall solve the GL equations for a thin film in an external field. We shall show that the field dependence of ψ (and λ) depends quite strongly on the boundary condition. The criterion for the destruction of superconductivity also changes with boundary conditions. The case of unequal fields is especially interesting and will be discussed.

The GL equations of a superconducting film thin in the x direction are[8]

$$\partial^2a_y/\partial\eta^2=\phi^2a_y,\quad \partial^2a_z/\partial\eta^2=\phi^2a_z, \tag{9a, b}$$

$$\partial^2\phi/\partial\eta^2=\phi\kappa^2[a_y^2+a_z^2+\phi^2-1], \tag{10}$$

$$\partial\phi/\partial\eta=0\quad\text{at surface}, \tag{11}$$

where

$$\phi=\psi(T,H)/\psi(T,0),\quad \mathbf{a}=\mathbf{A}/\sqrt{2}\lambda(T,0)H_{\text{cb}},$$
$$\eta=x/\lambda(T,0),\quad \mathbf{h}=\mathbf{H}/(\sqrt{2}H_{\text{cb}}),$$

ψ is the order parameter, $\mathbf{A}$ the vector potential, λ the penetration depth, κ the nonlinear coupling constant (about equal to 0.1 for most superconductors), and H_{cb} is the bulk critical field. Equations (9) and (10) are three coupled nonlinear differential equations in ϕ, a_y, and a_z. However, when $\kappa d/\lambda(T,0)\ll 1$, ϕ becomes independent of coordinates and the equations can be readily solved. We shall solve the equations under this condition.

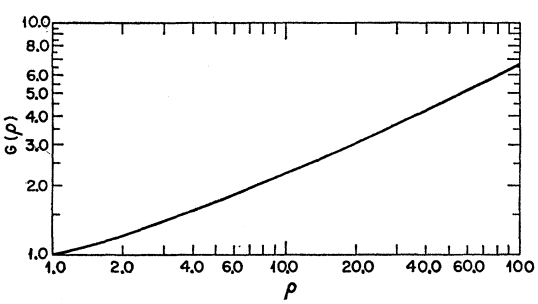

FIG. 1. $G(\rho)$ vs ρ.

[6] L. P. Gor'kov, J. Exptl. Theoret. Phys. (USSR) **37**, 1407 (1959); Soviet Physics—JEPT **10**, 998 (1959).

[7] These functions are tabulated in Harold T. Davis, *Tables of the Higher Mathematical Functions* (Principia Press, Inc., Bloomington, Indiana, 1935).

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Consider a thin plate with the normal to the plate in the x direction. The plate is bounded by the planes $x=0$, d. Now subject the side of the superconductor at $x=0$ to an external field H_1 and the side of the superconductor at $x=d$ to an external field H_2, both being in the z direction. This gives for boundary conditions at $\eta=0$,

$$h_z=h_1, \quad h_y=0;$$

and at $\eta=d/\lambda(T,0)$,

$$h_z=h_2, \quad h_y=0.$$

Using the relation curl $\mathbf{a}$ equals $\mathbf{h}$, we obtain for the solutions of Eq. (9)

$$a_z=0$$

$$a_y=\frac{h_2-h_1\cosh[\phi d/\lambda(T,0)]}{\phi\sinh[\phi d/\lambda(T,0)]}\cosh(\phi\eta)+\frac{h_1}{\phi}\sinh(\phi\eta). \tag{12}$$

Using (11) the integration of (10) becomes

$$(1-\phi^2)=\frac{\lambda(T,0)}{d}\int_0^{d/\lambda(T,0)} a_y^2 d\eta. \tag{13}$$

Putting (12) into (13) we obtain

$$1-\phi^2=\frac{\lambda}{2d\phi^3\sinh^2[\phi d/\lambda(T,0)]}\left\{(h_1-h_2)^2\cosh^2\left(\frac{\phi d}{2\lambda(T,0)}\right)\left[\sinh\left(\frac{\phi d}{\lambda(T,0)}\right)+\frac{\phi d}{\lambda(T,0)}\right]\right.$$
$$\left.+(h_1+h_2)^2\sinh^2\left(\frac{\phi d}{2\lambda(T,0)}\right)\left[\sinh\left(\frac{\phi d}{\lambda(T,0)}\right)-\frac{\phi d}{\lambda(T,0)}\right]\right\}.$$

We are interested in the thin-film limit, $\phi d/\lambda(T,0)\ll 1$. For this limit the above equation reduces to

$$1-\phi^2=\left[\frac{\lambda(T,0)}{d}\right]^2\frac{1}{\phi^4}(h_1-h_2)^2+\frac{1}{12}\left[\frac{d}{\lambda(T,0)}\right]^2\left(\frac{h_1+h_2}{2}\right)^2. \tag{14}$$

Let us consider first the case of unequal fields when $h_1\neq h_2$. [More precisely:

$$|h_1-h_2|\gg(12)^{-\frac{1}{2}}(d/\lambda)(h_1+h_2)/2].$$

Equation (14) reduces to

$$(h_2-h_1)^2=\phi^4(1-\phi^2)[d/\lambda(T,0)]^2, \tag{15}$$

or

$$(H_2-H_1)^2=2\phi^4(1-\phi^2)[d/\lambda(T,0)]^2H_{\text{cb}}^2. \tag{16}$$

If $|H_2-H_1|$ is much less than H_{cb} then

$$\phi^2=1-\tfrac{1}{2}[\lambda(T,0)/d]^2[(H_2-H_1)/H_{\text{cb}}]^2. \tag{17}$$

Combining (17) and (1) we obtain for the field dependence of the penetration depth

$$[\lambda(T,0)/\lambda(T,H)]^2=1-\tfrac{1}{2}[\lambda(T,0)/d]^2[(H_2-H_1)/H_{\text{cb}}]^2. \tag{18}$$

The critical conditions are obtained by holding H_1 (or H_2) constant and finding in Eq. (16) the maximum value of H_2 (or H_1) with respect to ϕ; this means

$$(\partial H_2/\partial\phi)_{H_1}=0 \quad [\text{or } (\partial H_1/\partial\phi)_{H_2}=0].$$

This calculation gives

$$\phi_c^2=\tfrac{2}{3}, \tag{19}$$

$$(H_2-H_1)_c^2=(8/27)[d/\lambda(T,0)]^2H_{\text{cb}}^2. \tag{20}$$

If the field is zero on one side, say H_2, then we get

$$H_{1c}^2=(8/27)[d/\lambda(T,0)]^2H_{\text{cb}}^2, \tag{21}$$

which is the result obtained by Ginzburg[8] for this limiting case.

For the case of equal fields $h_1=h_2$ [the fields may differ slightly as long as

$$|h_1-h_2|\ll(12)^{-\frac{1}{2}}(d/\lambda)(h_1+h_1)/2],$$

Eq. (14) reduces to

$$h_1^2=12(1-\phi^2)[\lambda(T,0)/d]^2, \tag{22}$$

or

$$H_1^2=24(1-\phi^2)[\lambda(T,0)/d]^2H_{\text{cb}}^2. \tag{23}$$

Solving (23) for ϕ^2 and using (1) we obtain for the field dependence of the penetration depth

$$[\lambda(T,0)/\lambda(T,H)]^2=1-(1/24)[d/\lambda(T,0)]^2(H_1/H_{\text{cb}})^2, \tag{24}$$

or, using (27)

$$[\lambda(T,0)/\lambda(T,H)]^2=1-(H_1/H_{1c})^2. \tag{25}$$

The critical values obtained from (23) from $\partial H_1/\partial\phi=0$ are

$$\phi_c=0, \tag{26}$$

$$H_{1c}^2=24[\lambda(T,0)/d]^2H_{\text{cb}}^2, \tag{27}$$

which are again the results obtained by Ginzburg[9] for *this* case. Thus, Eq. (14) has the proper limiting cases. It should be noted that, as the critical field is approached, in the first case the order parameter decreases to a finite critical value, while in the second case it decreases to zero.

For the intermediate case, between the equal and

[9] V. L. Ginzburg, Doklady Akad. Nauk. SSSR **83**, 385 (1952).

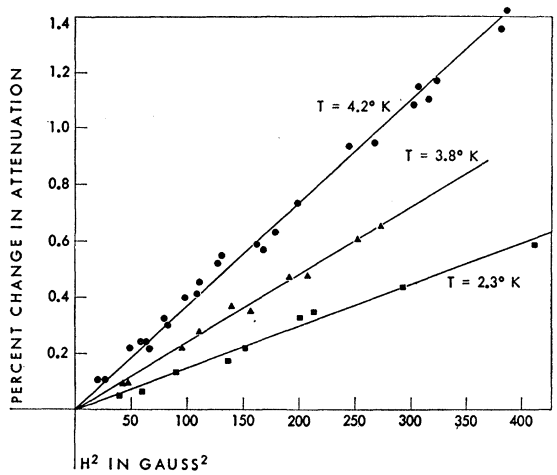

FIG. 2. Attenuation ($\propto\lambda^{-2}$) vs magnetic field for 250 A Pb film (Erlbach, Garwin, and Sarachik).

unequal limits, Eq. (14) can be easily solved when the fields are much less than their critical values (when $H\ll H_c$, then $\phi\sim 1$):

$$\phi^2=1-\left[\left(\frac{\lambda(T,0)}{d}\right)^2(h_1-h_2)^2+\frac{1}{12}\left(\frac{d}{\lambda(T,0)}\right)^2\left(\frac{h_1+h_2}{2}\right)^2\right].$$

This gives for the field dependence of the penetration depth

$$\left[\frac{\lambda(T,0)}{\lambda(T,H)}\right]^2=1-\left[\frac{1}{2}\left(\frac{\lambda(T,0)}{d}\right)^2\left(\frac{H_1-H_2}{H_{cb}}\right)^2+\frac{1}{24}\left(\frac{d}{\lambda(T,0)}\right)^2\left(\frac{H_1+H_2}{2H_{cb}}\right)^2\right].$$

It is seen that the above equation reduces to Eqs. (18) and (24) for the appropriate limits.

It is to be noted that the critical fields for these two cases have a different dependence on thickness, unequal fields giving a value less than H_{cb} and equal fields giving a value greater than H_{cb}.

It is appropriate to consider here the consequences of this last statement. Let us consider the application of Eq. (20) to a thin cylindrical multiply-connected film. Let H_1 be the field at the inner surface and H_2 the field at the outer surface. Starting with $H_1=H_2=0$, let us increase H_2 to the critical value H_{2c}. At this point, the superconducting state is destroyed, the film becomes normal, and the flux will leak in; thus $H_1=H_{2c}$. Since the field is now the same on both sides of the film and is less than the critical field for these boundary conditions, the film cannot remain in the normal state. It must return to the superconducting state! Upon further increase in the external field, this process will repeat again and again until the critical field for equal fields is reached. This explains why the critical field measurements of Ginzburg and Shal'nikov[10] on cylindrical films could apparently be explained by using the expression (27) for equal fields.

III. APPLICATION TO EXPERIMENT

Recently Erlbach, Garwin and Sarachik[5] have performed a very precise measurement of the field dependence of the penetration depth on a cylindrical film of lead of thickness 250±50 A. If an external field H_2 is impressed on the outer surface, then the relationship between H_2 and the resulting field H_1 at the inner surface has been found for both the London theory[5] and the Pippard theory[11] to be

$$(H_1/H_2)\propto\lambda^2, \tag{28}$$

when $d\ll\lambda(T,0)$, where d is the thickness of the film. [In the derivation of (28) it was assumed that the radius of curvature was very much larger than d.] In this experiment they placed transmitting and receiving coils, operated at 2.2 Mc/sec, on opposite sides of the film. The attenuation, A, which is proportional to H_2(2.2 Mc/sec)/H_1(2.2 Mc/sec), was measured as a function of H_2, the externally applied dc field. Assuming λ(2.2 Mc/sec)$=\lambda$(0 Mc/sec) we obtain

$$A(H)/A(0)=[\lambda(T,0)/\lambda(T,H)]^2. \tag{29}$$

The data, unexplained by them, are shown in Fig. 2.

It is clear that the conditions of this experiment correspond to the first case considered in Sec. II, and the field dependence of the penetration depth is described by Eq. (18). Therefore, the attenuation is

$$\frac{A(H)}{A(0)}=\left[\frac{\lambda(T,0)}{\lambda(T,H)}\right]^2=1-\frac{1}{2}\left[\frac{\lambda(T,0)}{d}\right]^2\left(\frac{H_2-H_1}{H_{cb}}\right)^2. \tag{30}$$

Since for this film $H_1\approx 10^{-4}H_2$, we may neglect H_1 in comparison to H_2. Hence,

$$\frac{A(H)}{A(0)}=\left[\frac{\lambda(T,0)}{\lambda(T,H)}\right]^2=1-\frac{1}{2}\left[\frac{\lambda(T,0)}{dH_{cb}}\right]^2H_2^2. \tag{31}$$

In order to calculate the coefficient of H_2^2 (quadratic coefficient) in Eq. (31), the following relations were used: $H_{cb}=H_{cb}(0)(1-t^2)$, $H_{cb}(0)=805$ gauss, $t=T/T_c$, $\lambda(T,0)=G(\rho)\lambda_B(0,0)(1-t^4)^{-\frac{1}{2}}$, $\lambda_B(0,0)=370$A, $\xi_0=830$A. The coefficients calculated in this way are as follows[12]:

Temperature	Experiment	Theory
4.2°K	$37.5\pm16\times10^{-6}$ gauss^{-2}	$19.3\pm11\times10^{-6}$ gauss^{-2}
3.8	23.7±9	15.4±9
2.3	15.0±6	9.3±5

[10] N. I. Ginzburg and A. I. Shal'nikov, J. Exptl. Theoret. Phys. (USSR) 37, 399 (1959); Soviet Phys.—JETP 10, 285 (1960).
[11] M. Peter, Phys. Rev. 109, 1857 (1958).
[12] Dr. Sarachik has pointed out that the absolute calibration of the magnetic field could be in error by as much as 20%.

This experiment is a very sensitive test of the Ginzburg-Landau theory, and the good agreement, considering the uncertainty of some of the parameters, should be interpreted as strong support of the theory.

The second case considered in Sec. II, where the fields are equal on opposite sides of the film, is the more common one experimentally. The field dependence of the penetration depth for this case as expressed by Eq. (25) will affect several experiments. For example, the magnetization of a thin $[d\ll\lambda(T,0)]$ superconducting plate is

$$M=-(48\pi)^{-1}(d/\lambda)^2H_1. \tag{32}$$

If Eq. (25) is substituted into (32) we obtain

$$M=-(48\pi)^{-1}\left[\frac{d}{\lambda(T,0)}\right]^2\left[1-\left(\frac{H_1}{H_{1c}}\right)^2\right]H_1. \tag{33}$$

Equation (33) predicts considerable rounding of the magnetization curve with a broad maximum at $H_1/H_{1c}=0.578$, which is to be contrasted with the triangular-shaped curves for the bulk state. Lock's[13] data on the magnetization of superconducting plates show that the magnetization curves become more rounded as the specimens become thinner. In fact, even though the inequality $d\ll\lambda(T,0)$ is not satisfied, the thinnest specimens all appear to have a broad maximum at about the predicted value.

There should also be an experimental correction to nuclear resonance measurements on superconductors. Because of the incomplete penetration of the dc magnetic field, there are contributions[14] to both the linewidth and the shift in the line going as λ^{-2}. Thus, Eq. (25) should be applicable. Usually, however, either this type of contribution is small or the operating field is much less than the critical field, so that this effect has not yet been observed.

It should also be pointed out that Prozorova[15] has considered the field dependence of the surface impedance of thin films by using Eq. (25).

IV. DISCUSSION OF RESULTS

This work shows that the field dependence of the penetration depth of superconductors of small dimensions can be very well explained by the Ginzburg-Landau theory. In addition to depending on size, the penetration depth also depends on the boundary conditions. For the case where the external field is equal on both sides of a thin film the phase transition is second order, and the penetration depth goes smoothly to infinity as the critical field is approached. In fact, Ginsburg[9] has shown that this is true up to $d/\lambda(T,0)=\sqrt{5}$. On the other hand, if the fields are unequal on opposite sides, the transition is first order and the penetration depth approaches a finite value at the critical field.

By going to thin specimens nonlocal effects can be avoided. There would appear to be no loss in generality or fundamental information in doing this. Since the basic equations of the GL theory are differential ones containing fundamental lengths of the order of 1000 A, one would expect the character of the solutions to change as the thickness is made less than this value. The same fundamental constants would appear in both limits, but, as this work shows, they are more accessible for the thin limit.

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[14] F. Reif, Phys. Rev. **106**, 208 (1957).
[15] L. A. Prozorova, J. Exptl. Theoret. Phys. (USSR) **34**, 14 (1958); Soviet Phys.—JETP **7**, 9 (1958).