Dependence of the Free-Carrier Faraday Ellipticity in Semiconductors on Scattering Mechanisms*

J. K. FURDYNA AND M. E. BRODWIN
Department of Electrical Engineering, Northwestern University, Evanston, Illinois
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The theory of the Faraday ellipticity in semiconductors is developed, via the Boltzmann transport equation, under the assumption of an isotropic energy-dependent time of relaxation τ. Equations relating ellipticity to semiconductor parameters are derived for various ranges of the collision, cyclotron, and applied frequencies. It is observed that, besides its dependence on the value of the scattering parameter, Faraday ellipticity is rather sensitive to the type of scattering mechanism as such, and to the distribution function. Some specific experiments are suggested in the ranges where ellipticity appears particularly promising as a tool for investigating these aspects of the scattering process. Numerical examples, calculated for thermal and ionized impurity scattering in nondegenerate carrier systems, are contrasted with the results of the constant-τ approximation, showing the inadequacy of the latter approach. Finally, the effect of spheroidal surfaces of constant energy on Faraday ellipticity is briefly discussed.

1. INTRODUCTION

FARADAY ellipticity arises from the difference in attenuation of the left- and right-handed circularly polarized components of an initially linearly-polarized plane electromagnetic wave, transmitted through a substance in the direction of an external magnetic field. In terms of the complex effective dielectric constant $(\epsilon_\pm)_{\text{eff}}=\epsilon_\pm'\mp i\epsilon_\pm''$, with $\epsilon_\pm'$ and $\epsilon_\pm''$ real, the attenuation constant for a nonmagnetic material is given by

$$\beta_\pm=\omega(\mu_0/2)^{\frac{1}{2}}[(\epsilon_\pm'^2+\epsilon_\pm''^2)^{\frac{1}{2}}-\epsilon_\pm']^{\frac{1}{2}}, \tag{1}$$

where ω is the angular frequency of the wave, μ_0 the permeability of free space, and the subscripts + and − refer to the left- and right-handed modes of circular polarization. The ellipticity, i.e., the ratio of the minor to the major axis of the resulting electric field pattern, is given by[1,2]

$$E=\tanh[\tfrac{1}{2}(\beta_- -\beta_+)t]\approx\tfrac{1}{2}(\beta_- -\beta_+)t, \tag{2}$$

t being the thickness of the specimen. This expression can be simplified for the range of low losses ($\epsilon_\pm'>\epsilon_\pm''$) as well as high losses ($\epsilon_\pm''>\epsilon_\pm'$).

The theory of the Faraday ellipticity in semiconductors has been previously discussed on the basis of the free-carrier model, assuming a constant relaxation time.[1,2] Because of the strong and unique dependence of ellipticity on scattering, the latter assumption is a rather serious drawback in discussing this phenomenon. In the present paper the theory is extended, via the Boltzmann transport equation, to include the dependence of the relaxation time on energy. It will be shown by the resulting equations that, apart from its dependence on the relaxation parameter predicted by the previous formulation, ellipticity is a strong function of the scattering mechanism itself, and of the carrier distribution function. Faraday ellipticity can therefore be expected to be a very powerful tool for the experimental investigation of the collision processes in semiconductors.

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[1] R. R. Rau and M. E. Caspari, Phys. Rev. **100**, 632 (1955).

[2] J. K. Furdyna and S. Broersma, Phys. Rev. **120**, 1995 (1960).

2. APPLICATION OF THE CONDUCTIVITY TENSOR

The effect of the energy dependence of the relaxation time τ on the Faraday ellipticity can be conveniently seen by developing the problem in terms of the conductivity tensor $\boldsymbol{\sigma}$. For isotropic materials in a magnetic field,

$$\boldsymbol{\sigma}=\begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ -\sigma_{12} & \sigma_{11} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}, \tag{3}$$

where axis 3 coincides with the direction of the field.

The real and imaginary parts of the effective dielectric constant for the two senses of polarization can now be written[3]

$$\epsilon_\pm'=\epsilon_{\text{st}}'+(\sigma_{11}''\pm\sigma_{12}')/\omega, \tag{4a}$$

$$\epsilon_\pm''=(\sigma_{11}'\mp\sigma_{12}'')/\omega. \tag{4b}$$

Here ϵ_{st}' is the dielectric constant of the material without free carriers, and the single and double prime denote the real and imaginary parts. The Faraday ellipticity then becomes[4] for the region of small losses

$$E=\tfrac{1}{4}(\mu_0/\epsilon_{\text{st}}')^{\frac{1}{2}}t[2\sigma_{12}''+(\sigma_{11}'\sigma_{12}'-\sigma_{11}''\sigma_{12}'')/\omega\epsilon_{\text{st}}'], \tag{5}$$

and for high values of the loss tangent

$$E=\tfrac{1}{2}(\mu_0\omega/2)^{\frac{1}{2}}t[\sigma_{12}'(1-a)+\sigma_{12}'']/[\sigma_{11}'(1-a)]^{\frac{1}{2}}, \tag{6}$$

where

$$a=(\epsilon_{\text{st}}'\omega+\sigma_{11}'')/\sigma_{11}'=(\epsilon_+'+\epsilon_-')/(\epsilon_+''+\epsilon_-''),$$

which can generally be neglected when the loss tangent is very large.

In the present discussion, the conductivity tensor is

[3] B. Lax and L. M. Roth, Phys. Rev. **98**, 549 (1955).

[4] J. K. Furdyna and S. Broersma (unpublished). In the present article we do not discuss explicitly the narrow range where $0>\epsilon_\pm''/\epsilon_\pm'>(-1)$. In this region of extreme attenuation experiments involving transmission are impractical.

found by solving the Boltzmann transport equation for components of the electric current, in the case of orthogonal high-frequency electric field and dc magnetic field. Components of the tensor are given explicitly in Appendix II. Second-order terms in the electric field are neglected, and the effects of orbital quantization at higher magnetic fields are not considered. The existence of an isotropic relaxation time is assumed in the form $\tau=\tau_0\epsilon^p$, where ϵ is the carrier energy relative to the band edge. The main part of this article is further restricted to single-band conduction, spherical surfaces of constant energy (except in Sec. 6), and nondegenerate carrier systems. It is, however, a simple matter to extend the equations to more complicated semiconductor models by using appropriate conductivity tensors in Eqs. (5) and (6).

3. SMALL LOSSES

A. $\omega_c<\tau^{-1}$, $\omega<\tau^{-1}$

When the collision frequency τ^{-1} greatly exceeds the cyclotron frequency ω_c as well as the frequency of the electromagnetic wave ω, and the dielectric loss tangent $\epsilon_\pm''/\epsilon_\pm'$ is small compared to unity,[5] the ellipticity is given by

$$E=\left(\frac{\mu_0}{\epsilon_{st}'}\right)^{\frac{1}{2}}t\sigma\mu_H B\omega\langle\tau\rangle\left(\frac{\langle\tau^3\rangle}{\langle\tau^2\rangle\langle\tau\rangle}-\frac{1}{4}\frac{\omega_p^2}{\omega^2}\right), \tag{7}$$

where second order terms $(\omega\tau)^2$ and $(\omega_c\tau)^2$ have been neglected in the conductivity tensor components. Here $\sigma=ne^2\langle\tau\rangle/m^*$ is the dc conductivity, $\mu_H=(e/m^*)(\langle\tau^2\rangle/\langle\tau\rangle)$ the Hall mobility, B the magnetic field, τ the relaxation time, $\omega_p=(ne^2/m^*\epsilon_{st}')^{\frac{1}{2}}$ the classical plasma frequency, with carrier concentration, effective mass, and electronic charge denoted by n, m^*, and e, respectively—all in mks units. For a Maxwell-Boltzmann distribution, the symbol $\langle g(x)\rangle$ represents the integral

$$\langle g(x)\rangle=\frac{4}{3\sqrt{\pi}}\int_0^\infty g(x)x^{\frac{3}{2}}e^{-x}dx;\quad x\equiv\epsilon/kT, \tag{8}$$

where k is the Boltzmann constant, and T the absolute temperature. The validity of this distribution is assumed for all numerical examples in this article.

Equation (7) is expressed in terms of σ, μ_H, and ω_p which are known (or can be calculated) from standard dc measurements, or from the Faraday rotation[1,2,6] (see also Appendix I). This form shows that ellipticity is a strong function of the value of $\langle\tau\rangle$, as well as of the scattering mechanism and the distribution function which determine the dimensionless quantity $\langle\tau^3\rangle/\langle\tau^2\rangle\langle\tau\rangle$. When the form $\tau=\tau_0\epsilon^p$ is assumed and Eq. (8) applied, this quantity is conveniently expressed as a ratio of gamma functions

$$\frac{\langle\tau^3\rangle}{\langle\tau^2\rangle\langle\tau\rangle}\equiv\eta=\frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{2}+3p)}{\Gamma(\frac{5}{2}+2p)\Gamma(\frac{5}{2}+p)}, \tag{9}$$

which becomes 1.50 for ideal lattice scattering ($p=-\frac{1}{2}$), and 3.05 for ionized impurity scattering ($p=\frac{3}{2}$), as compared to unity predicted by the energy-independent model. The quantity η is more sensitive to the type of scattering than the parameters more commonly used in investigating collision processes, e.g., the ratio of the Hall-to-drift mobilities.[7]

The range in which Eq. (7) applies is typical for room temperature microwave experiments on relatively pure materials. Note that in this range ellipticity exhibits a change of sign, i.e., a reversal in the sense of elliptical polarization, at $\omega_p^2/\omega^2=4\eta$. This is of interest to the experimentalist, since the condition for the node involves few parameters, and can be determined with high precision. The feature should be particularly useful for the investigation of scattering as a function of temperature. Since the quantity ω_p is constant through the exhaustion range of a semiconductor, it should be possible to study the relative behavior of η in this range by obtaining a temperature plot of signal frequencies at which ellipticity vanishes.

B. $\omega_c>\tau^{-1}$, $\omega_c>\omega$

In the case of small losses and high magnetic fields, i.e., when the cyclotron frequency exceeds both the experimental frequency and the average collision frequency,[8] the Faraday ellipticity is given by

$$\begin{aligned}E&=\left(\frac{\mu_0}{\epsilon_{st}'}\right)^{\frac{1}{2}}t\frac{\sigma\omega\langle\tau\rangle}{\mu_H^3B^3}\left(1-\frac{1}{4}\frac{\omega_p^2}{\omega^2}\right)\langle\tau\rangle\left\langle\frac{1}{\tau}\right\rangle\left(\frac{\langle\tau^2\rangle}{\langle\tau\rangle^2}\right)^3\\&=(\mu_0\epsilon_{st}')^{\frac{1}{2}}t\frac{\omega_p^2\omega}{\omega_c^3}\left\langle\frac{1}{\tau}\right\rangle\left(1-\frac{1}{4}\frac{\omega_p^2}{\omega^2}\right),\end{aligned} \tag{10}$$

where ω_c is the cyclotron frequency eB/m^*, with remaining symbols as previously defined. The first form is expressed in terms of standard semiconductor quantities σ and μ_H which are generally known for a given sample, but which themselves depend on the type of scattering. The quantity $\langle\tau\rangle\langle 1/\tau\rangle(\langle\tau^2\rangle/\langle\tau\rangle^2)^3$ equals 1.85 for ideal lattice scattering, and as much as 24.4 for ionized impurity scattering, while the energy-independent model predicts unity.

The second form of Eq. (10) is expressed in terms of ω_p and ω_c, which do not depend on scattering, to show

[5] In this frequency range the loss tangent is approximately equal to $\sigma/\omega\epsilon_{st}'$, where σ is the dc conductivity of the material.

[6] M. J. Stephen and A. B. Lidiard, J. Phys. Chem. Solids **9**, 43 (1959).

[7] For a general discussion of the relation between scattering mechanisms and fundamental galvanomagnetic coefficients see, e.g., A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1954), 2nd ed., pp. 231–242; also F. J. Blatt, in *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1957), Vol. 4, pp. 238–258.

[8] In the high-magnetic-field limit, the loss tangent reduces to $\sigma/(\omega\epsilon_{st}'\mu_H^2B^2)$.

that the high-field Faraday ellipticity directly measures the quantity $\langle\tau^{-1}\rangle$ rather than $\langle\tau\rangle$. Note that here, unlike the low-field case, the sign-reversal condition does not depend on scattering. Thus, if ω_p is unknown, it should be possible to determine its value from an ellipticity node at high fields, and apply this value in interpreting measurements obtained at low fields or at other frequencies.

C. $\omega>\omega_c$, $\omega>\tau^{-1}$

When losses are small and the frequency of the wave is larger than both the collision and the cyclotron frequency,[9] the Faraday ellipticity is given by

$$E=(\mu_0\epsilon_{\rm st}')^{\frac{1}{2}}t\frac{\omega_p{}^2\omega_c}{\omega^3}\left\langle\frac{1}{\tau}\right\rangle\left(1+\frac{3}{4}\frac{\omega_p{}^2}{\omega^2}\right). \qquad (11)$$

Similarly as in the high-magnetic-field range, ellipticity in this region measures $\langle\tau^{-1}\rangle$, i.e., the average collision frequency. However, it does not possess the experimentally convenient feature of a sign change as a function of ω. The second term in the brackets will be relatively unimportant at infrared frequencies, but it is likely to dominate in microwave experiments at very low temperatures.

4. HIGH LOSSES

The range of very high values of the loss tangent $\epsilon_\pm''/\epsilon_\pm'$ will be considered for the case when the collision frequency exceeds both the cyclotron frequency and the frequency of the wave. For small $\omega_c\tau$, Eq. (6) becomes, after neglecting terms in a,

$$E=-\tfrac{1}{2}\left(\frac{\mu_0\epsilon_{\rm st}'\omega}{2}\right)^{\frac{1}{2}}t\omega_p\omega_c \times\frac{\langle\tau^2(1+\omega_c{}^2\tau^2-2\omega\tau-\omega^2\tau^2)/(1+\omega^2\tau^2)^2\rangle}{\langle\tau/(1+\omega^2\tau^2)\rangle^{\frac{1}{2}}}. \qquad (12)$$

When terms in $(\omega_c\tau)^2$ and $(\omega\tau)^2$ are neglected, this relation can be expressed rather simply in terms of the semiconductor parameters σ and μ_H as

$$E=-\tfrac{1}{2}(\mu_0\omega\sigma/2)^{\frac{1}{2}}t\mu_H B(1-2\omega\langle\tau\rangle\eta), \qquad (13)$$

where η is defined by Eq. (9).

It follows from Eqs. (12) and (13) that in the limit of very small $\omega\langle\tau\rangle$, Faraday ellipticity in lossy materials depends on scattering only through σ and μ_H, and cannot provide additional information. The effect of scattering becomes increasingly significant, however, when $\omega\langle\tau\rangle$ exceeds approximately 1/10.

The condition under which Eq. (12) reverses sign as a function of frequency or temperature in the small field limit is strongly dependent on the type of scattering. It is easily seen that in the energy-independent approximation ellipticity will display a change of sign when $\omega\langle\tau\rangle=0.41$. Using tabulated integrals,[10] it is found that when scattering is dominated by thermal lattice vibrations, the zero in Eq. (12) occurs when $\omega\langle\tau\rangle=0.33$, and when it is primarily due to ionized impurities, the zero occurs at $\omega\langle\tau\rangle=0.17$. Equation (13) vanishes at values of $\omega\langle\tau\rangle$ sufficiently close to the above to be useful as a good approximation for this range. When $\epsilon_\pm'/\epsilon_\pm''$ are smaller than unity but not negligible, Faraday ellipticity can be similarly analyzed by retaining terms in a in Eq. (6).

The region of high frequencies or high fields is not discussed in this section. The value of the loss tangent is itself a function of the parameters $\omega\tau$ and $\omega_c\tau$ [as can be seen from Eq. (4) and the tensor components given in the Appendix], and in general decreases as these quantities are increased. Hence, in the limit $\omega\tau\gg1$ or $\omega_c\tau\gg1$ the situation will generally belong to the range of small losses, described by the equations in the previous section.

5. GENERAL MAGNETIC FIELD DEPENDENCE OF ELLIPTICITY

As shown above, Faraday ellipticity varies as the first power of the magnetic field in the range $\omega_c<\tau^{-1}$, and inversely as the third power of the field when $\omega_c\gg\tau^{-1}$. It has been previously observed[2] that the variation of ellipticity with magnetic field will in general display a maximum when $\omega_c\approx\tau^{-1}>\omega$. Similarly, a peak will appear at cyclotron resonance, $\omega_c\approx\omega>\tau^{-1}$. The shape of the maximum, its magnitude, and the value of the field at which it occurs will depend to some extent on the scattering mechanism. The analysis of this dependence is, however, quite tedious, and therefore much less revealing than the analysis of the ellipticity in the high and low limits of the field.

There is, however, a range in which the ellipticity vs field curve shows a rather unique behavior, which is extremely sensitive to the scattering mechanism and is particularly suited for curve fitting. A brief qualitative discussion of this behavior is therefore given. If a material is characterized by high losses at small values of B, the sign of ellipticity is essentially determined by $(2\omega\langle\tau\rangle\eta-1)$, Eq. (13). On the other hand, if the low-loss range is reached by increasing B sufficiently, the sign is determined by $(1-\frac{1}{4}\omega_p{}^2/\omega^2)$, Eq. (10). Thus, under the conditions $\frac{1}{4}\omega_p{}^2/\omega>1$, $2\omega\langle\tau\rangle\eta>1$, ellipticity will reverse sign at some value of the magnetic field. The shape of the resulting E vs B curve, particularly the relative values of the extrema and the zero position, are a sensitive function of the type of scattering. A typical example is shown in Fig. 1. These curves are calculated exactly, using Eqs. (1) and (2), with the effective dielectric constants given by Eq. (4), for a

[9] In the high-frequency limit, the loss tangent is approximately given by $\sigma/(\omega\epsilon_{\rm st}'\omega^2\tau^2)$.

[10] R. B. Dingle, D. Arndt, and S. K. Roy, Appl. Sci. Research **B6**, 144, 245 (1957).

typical semiconductor sample at liquid nitrogen temperature. Tabulated integrals[10] were used in evaluating the conductivity contribution to Eq. (4), assuming $\tau=\tau_0\epsilon^p$ and $p=-\frac{1}{2}$ for lattice scattering, $\frac{3}{2}$ for ionized impurity scattering, and 0 for the energy-independent model. To make the comparison physically meaningful, the quantity τ_0 was chosen in each case so as to give the same value of $\langle\tau\rangle$. This value, as well as the values of ϵ_{st}', n, m^*, and ω used in the calculations, are indicated in the figure.

6. EFFECTIVE MASS ANISOTROPY

Equations (4) to (6), expressed in terms of the conductivity tensor, apply to isotropic materials. By the word "isotropic" we only make the restriction that $\sigma_{11}=\sigma_{22}$, in which case the initial linearly-polarized wave can be resolved into two independent circularly polarized components.[3] When $\sigma_{11}\neq\sigma_{22}$, there arises an additional problem analogous to optical birefringence.

In the case of spheroidal constant energy surfaces (such as in n-type silicon or germanium), the relation $\sigma_{11}=\sigma_{22}$ holds for all orientations of the magnetic field in the range $\omega_c\ll\tau^{-1}$. In this case Eqs. (7) and (13) can be used if

$$\sigma=\frac{ne^2(2m_1+m_2)\langle\tau\rangle}{3m_1m_2}=\omega_p{}^2\langle\tau\rangle\epsilon_{st}',$$

and

$$\mu_H=\frac{(m_1+2m_2)}{(2m_1+m_2)}\frac{e}{m_2}\frac{\langle\tau^2\rangle}{\langle\tau\rangle},$$

where m_1 and m_2 are the longitudinal and transverse components of the effective mass tensor. Since the many-valley model permits both intravalley and intervalley scattering, this will affect the values of the scattering-dependent quantities[11] such as η. The Faraday ellipticity can therefore be used to study this aspect of scattering in germanium and silicon.

For the spheroidal surfaces of constant energy the conductivity tensor becomes anisotropic ($\sigma_{11}\neq\sigma_{22}$) in the range $\omega_c>\tau^{-1}$, $\omega_c>\omega$ for an arbitrary orientation of the magnetic field. However, in the special case when B is oriented along [111] or [100] crystallographic directions, σ_{11} becomes equal to σ_{22} and Eqs. (4) to (6) can again be applied. As an example, equations for Faraday ellipticity are given for a material with six constant energy spheroids along the [100] directions, such as n-type silicon. For the magnetic field B along a [111] axis,

$$E=(\mu_0\epsilon_{st}')^{\frac{1}{2}}t\frac{\omega_p{}^2\omega}{\Omega_a{}^3}\left\langle\frac{1}{\tau}\right\rangle\left(1-\frac{1}{4}\frac{\omega_p{}^2}{\omega^2}\right). \qquad (14)$$

For B along a [100] axis,

$$E=(\mu_0\epsilon_{st}')^{\frac{1}{2}}t\frac{\omega_p{}^2\omega}{\Omega_b{}^3}\left\langle\frac{1}{\tau}\right\rangle\left(1-\frac{1}{4}\gamma\frac{\omega_p{}^2}{\omega^2}\right), \qquad (15)$$

[11] C. Herring, Bell System Tech. J. 34, 237 (1955).

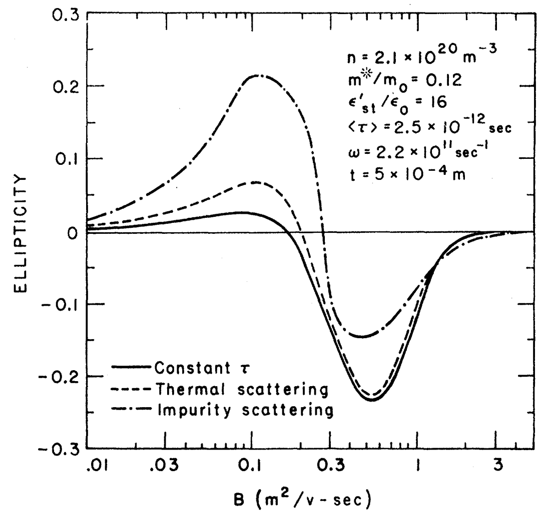

FIG. 1. Faraday ellipticity as a function of the magnetic field B, for the frequency range $\omega_p{}^2/\omega^2>4$, $2\omega\langle\tau\rangle\eta>1$. The curves are calculated exactly using Eqs. (1) and (2). The values used in the calculation are indicated in the figure. They are representative of a wide range of semiconductors near the liquid nitrogen temperature. The usual form $\tau=\tau_0\epsilon^p$ is assumed in averaging the components of the conductivity tensor, with $p=-\frac{1}{2}$ for lattice scattering, $\frac{3}{2}$ for ionized impurity scattering, and 0 for the constant-τ approximation.

where

$$\Omega_a{}^3\equiv e^3B^3(m_1+2m_2)(2m_1+m_2)/9m_1{}^2m_2{}^3,$$
$$\Omega_b{}^3\equiv e^3B^3/m_1m_2{}^2,$$

and

$$\gamma\equiv 3m_1(m_1+2m_2)/(2m_1+m_2)^2.$$

The form of the dependence of ellipticity on scattering in the high-field limit is therefore unaffected by the anisotropy of the effective mass. It is interesting to note that the conditions for the nodes do not depend on scattering, but can give some information regarding the m^* anisotropy.

7. SUMMARY AND REMARKS

The equations developed in the present article show that the Faraday ellipticity in semiconductors depends in a rather sensitive and unique manner on the mechanism of scattering, the carrier distribution function, and the value of the average relaxation time even in the range $\omega\tau\ll 1$. This information can be conveniently obtained from ellipticity experiments at various conditions, in conjunction with standard galvanomagnetic data known from dc measurements or from the related Faraday rotation. When the conductivity and mobility are only roughly known (so that an approximate value of the loss tangent can be established), ellipticity alone can provide much useful information. Here the ellipticity nodes, which can be observed, e.g., by varying the applied frequency, are especially interesting because they can be determined with precision and involve few parameters.

From the equations developed above it can be seen that the behavior of ellipticity predicted by the constant-τ approximation agrees qualitatively with the more rigorous results of the present article. However, since the constant-τ model neglects the influence of the type of scattering and of the distribution function, it cannot be used in a quantitative discussion of ellipticity. Comparison shows that, in the range $\omega\tau<1$ (typical of microwave experiments), the effect of the energy dependence of τ is equivalent to an increase of the value of $\omega\tau$ wherever this parameter appears explicitly in equations derived from the constant-τ model.[2] It is interesting that in the energy-independent analysis of the preliminary experimental data reported in reference 2, the best fit was in fact obtained with values of τ considerably larger than those calculated from the dc mobility and effective-mass information.

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APPENDIX I. THE FARADAY ROTATION

Faraday ellipticity is always accompanied by Faraday rotation. Although the explicit dependence of the rotation on scattering is in general not very interesting, measurements of this effect can be helpful in interpreting ellipticity data. An outline of the problem of rotation should therefore be useful in parallel with the development of ellipticity presented above. The equations for the Faraday rotation in terms of the conductivity tensor are,[4] for the region of small losses

$$\theta=-\tfrac{1}{2}(\mu_0/\epsilon_{st}')^{\frac{1}{2}}t\sigma_{12}', \tag{A1}$$

and for the high-loss range

$$\theta=-\tfrac{1}{2}(\mu_0\omega/2)^{\frac{1}{2}}t[\sigma_{12}'(1+a)-\sigma_{12}'']/[\sigma_{11}'(1+a)]^{\frac{1}{2}}, \tag{A2}$$

where a is defined in Sec. 2. In the range of very small ω_c and ω, these quantities can be shown to depend on scattering only through σ and μ_H. In the high-frequency[6] or high-field[2] limit (where the low-loss formulas apply), rotation is scattering-independent.

The effect of the scattering mechanism is, however, quite significant in the range where $\omega\tau\approx 1$, as can be seen by using the conductivity tensor components given below. This will be of importance in low-temperature microwave experiments. It can be readily shown that when $\omega\langle\tau\rangle\lesssim 1$, the theoretical value of rotation is considerably lower than that obtained with the same values of ϵ_{st}', n, m^*, ω_c, ω, and $\langle\tau\rangle$ using the constant-τ approximation. The values of rotation calculated with the energy-dependent model give improved agreement with Faraday rotation measurements on germanium at 77°K in the low-field region.[2]

APPENDIX II. COMPONENTS OF THE CONDUCTIVITY TENSOR

For reference, the components of the conductivity tensor are listed for the case of isotropic energy-dependent τ and isotropic m^*:

$$\sigma_{11}'=(ne^2/m^*)\langle\tau(1+\omega^2\tau^2+\omega_c^2\tau^2)S^{-1}\rangle, \tag{A3}$$

$$\sigma_{11}''=-(ne^2/m^*)\omega\langle\tau^2(1+\omega^2\tau^2-\omega_c^2\tau^2)S^{-1}\rangle, \tag{A4}$$

$$\sigma_{12}'=-(ne^2/m^*)\omega_c\langle\tau^2(1-\omega^2\tau^2+\omega_c^2\tau^2)S^{-1}\rangle, \tag{A5}$$

$$\sigma_{12}''=2(ne^2/m^*)\omega\omega_c\langle\tau^3S^{-1}\rangle, \tag{A6}$$

where

$$S\equiv(1+\omega^2\tau^2+\omega_c^2\tau^2)^2-4\omega^2\omega_c^2\tau^4.$$

Often the tensors corresponding to more complicated models are published for the steady state. These can be readily generalized to the high-frequency case by replacing τ with $\tau/(1+i\omega\tau)$, or, in the case of tensor relaxation time, by replacing each component τ_α with $\tau_\alpha/(1+i\omega\tau_\alpha)$.