Neutron Scattering by the Complex Harmonic Oscillator Potential*

J. SOKOLOFF†‡
Argonne National Laboratory, Argonne, Illinois and Northwestern University, Evanston, Illinois
(Received May 15, 1961)

A complex harmonic oscillator potential is used in an attempt to fit low-energy neutron scattering and absorption data. Differential scattering cross sections are compared with experiment for 1-Mev incident neutrons and a wide spectrum of target nuclei. Total cross sections are compared with experiment at 350 Kev, 1 Mev, and 1440 Kev for an extensive mass spectrum. For the parameters used in the calculations, the complex harmonic oscillator seems to share the shortcomings of the "best-fit" complex square well.

I. INTRODUCTION

THE "shell model"[1] has had notable success in predicting level sequences for bound states of the nucleus, a success achieved principally through the addition of a spin-orbit coupling term to either the square well or the harmonic oscillator potential. (Though there have been many refinements in the "shell model," the occurrence of all "magic numbers" can be understood completely on the basis of a square well potential or a harmonic oscillator potential, with the addition of a strong spin-orbit coupling term. No additional assumptions need be made.) Feshbach, Porter, and Weisskopf,[2] in their original treatment of the complex potential formalism for neutron scattering and absorption, used the simplest potential possible for a treatment of unbound states, the complex square well. It seemed natural to try the complex harmonic oscillator, recalling the success of the shell model with a similar potential.

The complex potential model has been extended by the addition of a spin-orbit coupling term[3] and by the use of more sophisticated central potentials, such as the Woods-Saxon[4] potential. For a limited amount of experimental data, the possibility of fitting the data with theory is very much a function of the number of free parameters available in the theory. However, the aim of the complex potential model has been to display the gross features of total cross sections, cross sections for the formation of the compound nucleus, and differential scattering cross sections, as functions of the energy of the incident neutron and the mass of the target nucleus. Given a digital computer, this still involves a great deal of tedious work. Since the possible cross sections increase geometrically with the number of free parameters, the advantage of a simple potential in attempting this type of gross "best fit" is quite apparent. With respect to the number of free parameters in a potential which has any hope of achieving the aforementioned type of "best fit," the complex square well and the complex harmonic oscillator, which have the same number, are minimal.

2. SOLUTION OF THE RADIAL SCHRÖDINGER EQUATION

For details of the scattering formalism, the reader is referred to the literature.[2,5,6] All cross sections, in the phase shift analysis of FPW, are expressed in terms of the logarithmic derivative of the radial wave function

$$f_l = R\left[\frac{du_l(r)/dr}{u_l(r)}\right]_{r=R},$$

where $u_l(r)$ is the reduced radial wave function of angular momentum quantum number l, r is the radial variable, and R is the radius of the nucleus.

The complex harmonic oscillator potential has been taken as

$$\begin{aligned} V &= (V_0 r^2/R^2 - V_0)(1+i\zeta), && r \leqslant R \\ &= 0, && r > R \end{aligned} \qquad (1)$$

in analogy to the complex square well potential of FPW.

The reduced radial Schrödinger equation is then

$$\frac{d^2u_l}{dr^2} + \left[k^2 + \frac{2m}{\hbar^2}V_0(1+i\zeta) - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2}\frac{V_0}{R^2}(1+i\zeta)r^2\right]u_l = 0, \quad r \leq R, \qquad (2)$$

where k is the wave number in the entrance channel and m is the reduced mass in the channel.

With the substitutions

$$K^2 = k^2 + \frac{2m}{\hbar^2}V_0(1+i\zeta),$$

$$\beta^2 = \frac{2m}{\hbar^2}\frac{V_0}{R^2}(1+i\zeta),$$

---

* Work done under the auspices of the U. S. Atomic Energy Commission.

† Based on a section of a thesis submitted to Northwestern University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

‡ Now with Research Laboratory, Lockheed Missiles and Space Division, Palo Alto, California.

[1] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, New York, 1955).

[2] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954); hereafter referred to as FPW.

[3] F. Bjorklund and S. Fernbach, Phys. Rev. **109**, 1295 (1958); J. M. Peterson, A. Bratenahl, and J. A. Stoering, ibid. **120**, 521 (1960).

[4] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954); M. A. Melkanoff, S. A. Moszkowski, J. Nodvik, and D. S. Saxon, ibid **101**, 507 (1956).

[5] J. Sokoloff, Argonne National Laboratory Report, ANL-5618, 1956 (unpublished).

[6] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics*, (John Wiley & Sons, New York, 1952).

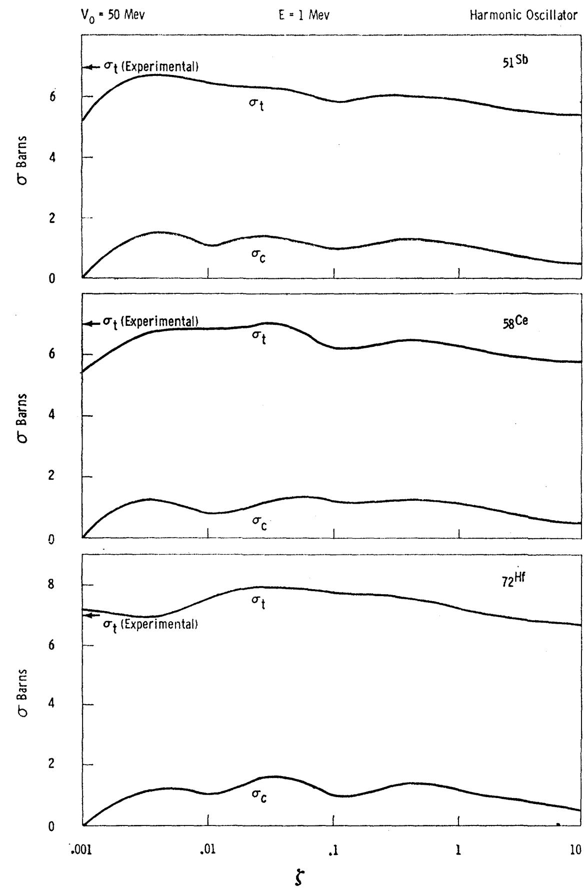

FIG. 1. Total cross section, σ_t, and cross section for the formation of the compound nucleus, σ_c vs ζ.

we have

$$\frac{d^2u_l}{dr^2}+\left[K^2-\frac{l(l+1)}{r^2}-\beta^2r^2\right]u_l=0, \quad r\leq R. \tag{3}$$

Under the change of variable

$$u_l(r)=e^{-z/2}r^{l+1}F(z), \tag{4}$$

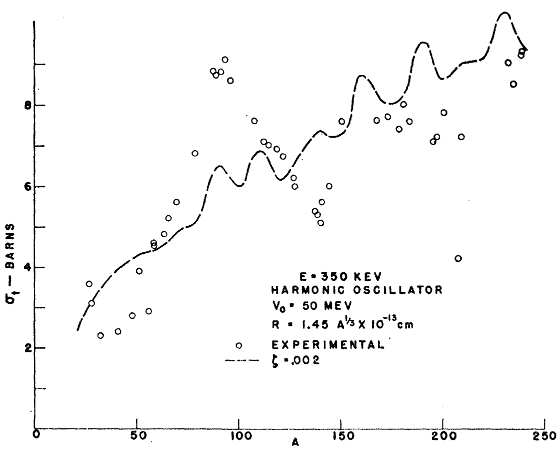

FIG. 2. Total cross section vs atomic weight.

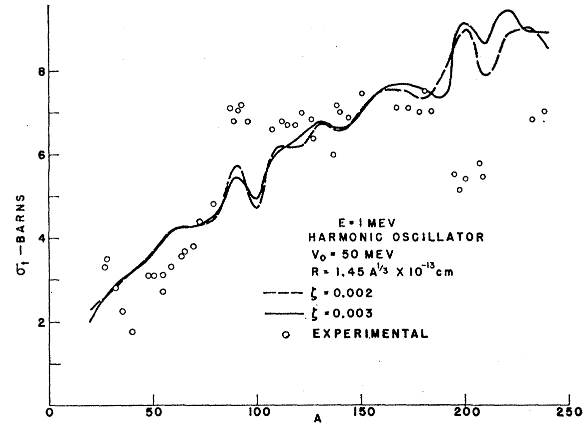

FIG. 3. Total cross section vs atomic weight.

where

$$z=\beta r^2,$$

the reduced radial equation becomes

$$zd^2F/dz^2+(l+\tfrac{3}{2}-z)dF/dz \\ -[(l+\tfrac{3}{2})/2-K^2/4\beta]F=0. \tag{5}$$

Equation (5) is the confluent hypergeometric equation[7]

$$zd^2F/dz^2+(c-z)dF/dz-pF=0, \tag{6}$$

with

$$p=(l+\tfrac{3}{2})/2-K^2/4\beta,$$
$$c=l+\tfrac{3}{2}.$$

The solution of Eq. (6) which is analytic at $z=0$ is given by the confluent hypergeometric series

$$F(p|c|z)=1+\frac{p}{c}z+\frac{1}{2!}\frac{p(p+1)}{c(c+1)}z^2 \\ +\frac{1}{3!}\frac{p(p+1)(p+2)}{c(c+1)(c+2)}z^3+\cdots. \tag{7}$$

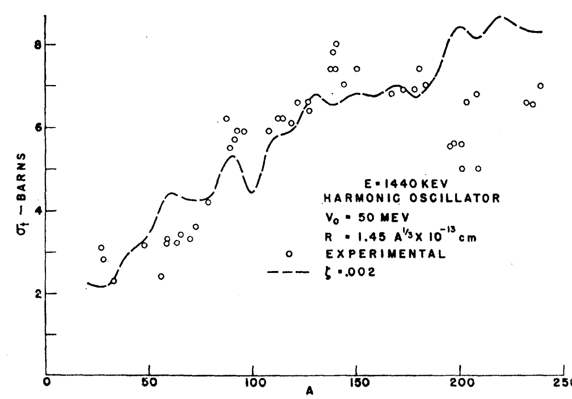

FIG. 4. Total cross section vs atomic weight.

[7] Recursion formulas for the confluent hypergeometric function, $F(p|c|z)$, may be found in S. Rushton, Sankhya **13**, 377 (1945). There is a typographical error in formula (36). For the factor $(\gamma/\alpha)e^xx^{\alpha-\gamma}$ read $[\Gamma(\gamma)/\Gamma(\alpha)]e^xx^{\alpha-\gamma}$. Tables of $F(p|c|z)$ for some of the desired half-integer values of c may be found in S. Rushton and E. D. Long, Sankhya **13**, 377 (1945).

The series converges in the entire complex plane.

The series for dF/dz is obtained easily as

$$(d/dz)F(p|c|z)=(p/c)F(p+1|c+1|z), \tag{8}$$

so that

$$(d/dr)F(p|c|z)=2\beta r(p/c)F(p+1|c+1|z). \tag{9}$$

The logarithmic derivative, f_l, may then be evaluated directly from the series for $F(p|c|z)$ and $(d/dr)F(p|c|z)$, or, alternatively, a very convenient continued fraction representation may be obtained.

From Eq. (4)

$$\frac{du_l/dr}{u_l}=\frac{l+1-\beta r^2}{r}+\frac{dF/dr}{F}, \tag{10}$$

so that

$$f_l=[l+1-z+2z(d/dz)\ln F]_{r=R}. \tag{11}$$

From Ince,[8] one has the continued fraction,

$$z\frac{d}{dz}\ln F(p|c|z)=\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots\frac{a_n}{+b_n} \tag{12}$$

where

$$a_n=(p+n-1)z,$$
$$b_n=c-z+n-1.$$

3. COMPARISON OF HARMONIC OSCILLATOR CROSS SECTION WITH EXPERIMENT

A. Total Cross Sections

Figure 1 indicates the dependence of the total cross section σ_t and the cross section for the formation of the compound nucleus σ_c on ζ for $V_0=50$ Mev and $E=1$ Mev. This is one of a series of parameter studies made to determine a "best fit." Unless specifically indicated on the drawings, $R=1.45A^{\frac{1}{3}}\times10^{-13}$ cm.

An indication of the approach to the "hard sphere" limit[5] ($\zeta\rightarrow\infty$) of the cross sections may be seen at the largest values of ζ.

Figures 2–4 illustrate harmonic oscillator total cross sections as functions of the atomic weight of the target nucleus for three energies of the incident neutron. The harmonic oscillator parameters used were those which appeared to be best on the basis of a survey, part of which is shown in the previous figure. As is true in the case of similar curves for the square well,[5] the heavy end of the mass spectrum is not fitted by theory. Nor does there appear to be enough indication of the rise in the total cross sections in the region $A\approx90$ at 350 kev.

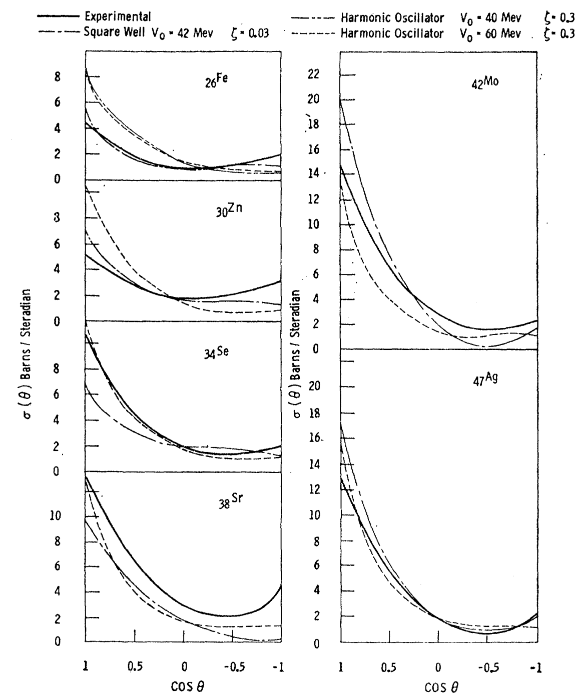

FIG. 5. Differential scattering cross sections for 1-Mev elastically scattered neutrons. The experimental data are those of Walt and Barschall.

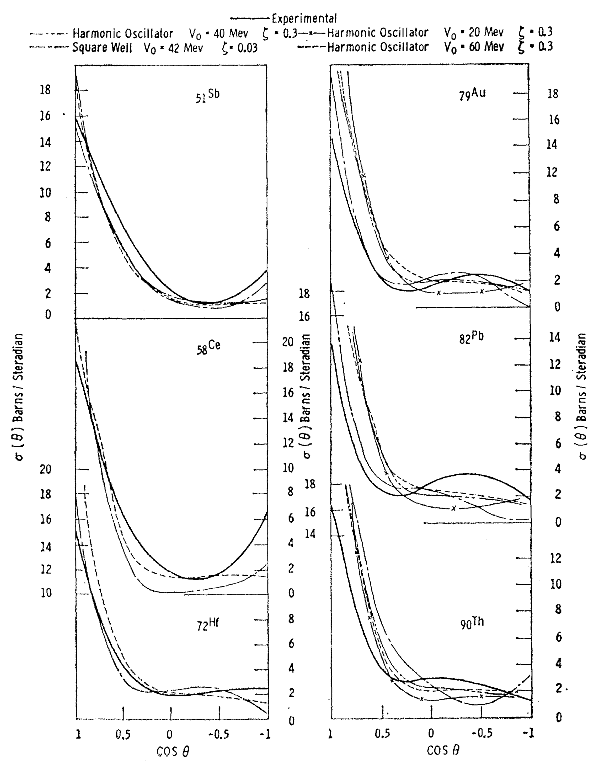

FIG. 6. Same as Fig. 5.

[8] E. L. Ince, *Ordinary Differential Equations* (Longman's Green and Company, Inc., New York, 1926), p. 181.

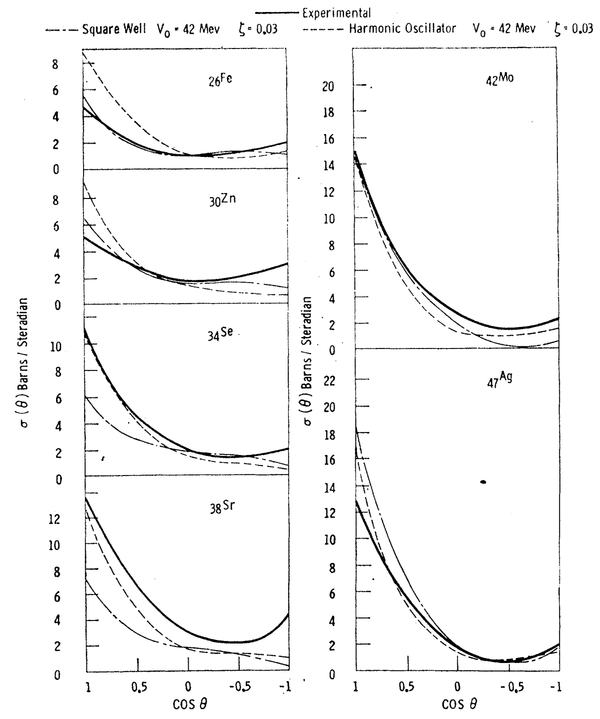

FIG. 7. Same as Fig. 5. Modified radius parameter, as indicated, used.

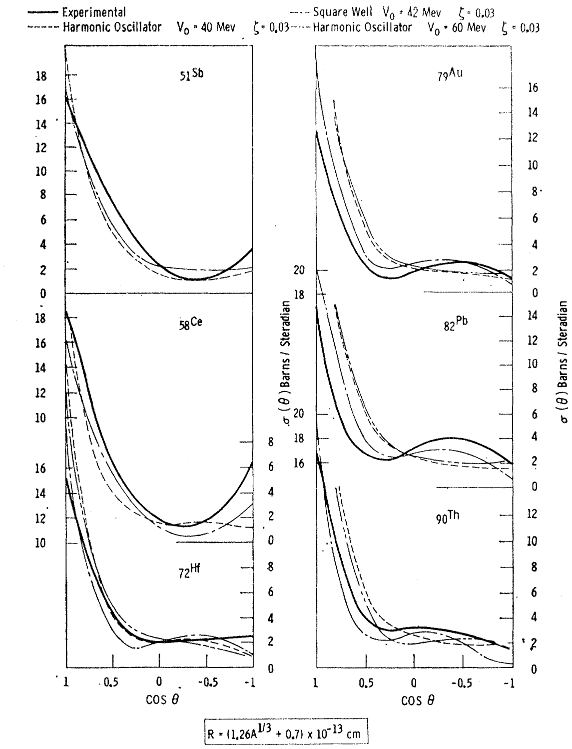

FIG. 8. Same as Fig. 7.

B. Differential Scattering Cross Sections

Figures 5–8 contain a comparison of theoretical angular distributions with the experimental angular distributions of elastically scattered neutrons at 1 Mev by Walt and Barschall.[9] A number of harmonic oscillator parameters have been used in an attempt to obtain a better fit to the heavy nuclei. These have been compared, in some cases, with the "best fit" square well parameters of FPW.

It is quite apparent that, in general, the theoretical angular distributions are excessively peaked in the forward direction. In addition the broad maximum at $\approx 90°$ in the experimental angular distribution for the heavier nuclei is only suggested by some of the theoretical curves.

Several calculations were made with a modified radius parameter.[10] The modified parameter led to a better fit at the heavy end of the spectrum, but it failed to improve the light end.

[9] M. Walt and H. H. Barschall, Phys. Rev. **93**, 1062 (1954).
[10] Private discussion with Dr. W. S. Emmerich.

CONCLUSIONS

On the basis of the work completed with the harmonic oscillator potential in fitting angular distributions of elastically scattered neutrons at 1 Mev, it is apparent that the harmonic oscillator potential shares the shortcomings of the square well potential. As mentioned previously, it gives excessive forward peaking and its success in reproducing the broad peak at $\approx 90°$ for the heavier elements is moderate, at best. The modified radius parameter improved the fit of the latter, however. The 1-Mev angular distributions are probably a severe test of the harmonic oscillator potential since the energy region has proven to be the most difficult for the square well potential to reproduce.[10]

ACKNOWLEDGMENTS

I wish to thank Dr. Morton Hamermesh for suggesting the problem and for his counsel during the course of the investigation. I am grateful to James W. Butler for the preliminary numerical analysis for the digital computer code and Miss Loretta I. Kassel for coding the problem.