Coulomb Field Effects on the Decay of Bound Polarized Muons*

W. R. JOHNSON, R. F. O'CONNELL, AND C. J. MULLIN
University of Notre Dame, Notre Dame, Indiana

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The influence of the Coulomb field on the decay of a polarized muon from the K shell of a μ atom is considered. Analytical results valid to order α^3Z^3 are presented for the total decay rate and for the electron angular distribution. Formulas valid to order α^2Z^2 are given for the energy spectrum of the emitted electron.

I. INTRODUCTION

RECENTLY several computations[1–3] of the decay rate of muons bound in the K shell of mesonic atoms have been carried out in an attempt to reconcile the existing theory[4–6] with experiment.[7–9] The calculations of Überall, based on a second Born approximation wave function for the electron and a first-order muon K-shell wave function, give the total decay rate to order α^2Z^2 and the electron spectrum (for unpolarized muon decay) accurately to order αZ. Gilinsky and Mathews, using the Sommerfeld-Maue wave function for the electron together with a modified Coulomb wave function for the muon, have determined numerically the influence of finite nuclear size as well as Coulomb field effects on the decay.

The purpose of this paper is to give accurate analytical expressions for the influence of the Coulomb field alone on various aspects of the muon decay. Since the influence of nuclear size on the decay is apparently small[10] for light nuclei, such effects presumably can be treated on perturbations on the present results.

In Sec. II we shall discuss the wave functions used in describing the electron and muon, and we shall write down the matrix element for the decay. Section III is devoted to a direct computation of the total decay rate and the corresponding electron angular distribution. The results obtained are correct to terms of relative order α^3Z^3 inclusive. Formulas are presented in Sec. IV for the electron energy spectrum. Graphs of the electron's energy spectrum are given for several values of Z.

II. THIRD-ORDER MATRIX ELEMENT FOR MUON DECAY

Gilinsky and Mathews[1] show that the decay rate for polarized negative muons may be written

$$\Lambda^{(-)}=8G^2\int\frac{d\mathbf{p}}{(2\pi)^3}\int\frac{d\mathbf{k}}{(2\pi)^3}P(\mathbf{p},\mathbf{k},\boldsymbol{\zeta}),\tag{1}$$

where

$$P(\mathbf{p},\mathbf{k},\boldsymbol{\zeta})=\sum_{s_e}\frac{1}{12\pi}N_\sigma\bar{N}_\lambda[H_\sigma H_\lambda-H^2\delta_{\sigma\lambda}],\tag{2}$$

with $H=(-\mathbf{p}-\mathbf{k},\,i(w-k))$, and

$$N_\sigma=\int d\mathbf{r}[\bar{\psi}_e(\mathbf{r})\gamma_\sigma a\psi_\mu(\mathbf{r})]e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{r}},\tag{3}$$

where $a=(1+\gamma_5)/2$. In the above formulas $\mathbf{p}$ represents the nuclear recoil momentum, $\mathbf{k}$ is the electron's momentum, w is the energy of the bound muon, $\boldsymbol{\zeta}$ is a unit vector directed along the muon spin axis, and $\sum_{s_e}$ denotes the summation over electron spins. The range of integration in Eq. (1) is limited by the energy-momentum conservation laws.

The muon is to be described by an exact relativistic Coulomb wave function. The matrix element will be expanded to terms of relative order α^3Z^3, inclusive, after its evaluation. In particular,[11]

$$\psi_\mu(\mathbf{r})=N_1(2\mu r)^{\gamma-1}e^{-\mu r}[1+i(\Lambda/2m)\boldsymbol{\alpha}\cdot\hat{r}]u_\zeta,\tag{4}$$

where u_ζ is the plane wave spinor for a free particle of zero momentum and spin direction $\boldsymbol{\zeta}$. In Eq. (4)

$$N_1=\frac{1}{(4\pi)^{\frac{1}{2}}}(2\mu)^{\frac{3}{2}}\left(\frac{1+\gamma}{2\Gamma(2\gamma+1)}\right)^{\frac{1}{2}},\quad \Lambda=2m\left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{2}},$$

$$\mu=m\alpha Z,\quad\text{and}\quad \gamma=w/m=(1-\alpha^2Z^2)^{\frac{1}{2}}.$$

The electron, which is treated as an extremely relativistic particle throughout, is described by a modification of the Sommerfeld-Maue wave function[12,13] valid to order α^3Z^3. We set

$$\bar{\psi}_e=\bar{\psi}_{\mathrm{SM}}+\bar{\psi}_c+\bar{\psi}_d,\tag{5}$$

where the Sommerfeld-Maue wave function is written as

$$\bar{\psi}_{\mathrm{SM}}=N_2{}^*\bar{u}_ke^{-i\mathbf{k}\cdot\mathbf{r}}(1-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{k}}/2r)F,\tag{6}$$

and where $\bar{\psi}_c$ and $\bar{\psi}_d$ represent corrections to $\bar{\psi}_{\mathrm{SM}}$ of orders α^2Z^2 and α^3Z^3, respectively. In Eq. (6)

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$F={}_1F_1(i\nu,\,1,\,i(kr+\mathbf{k}\cdot\mathbf{r}))$, with $\nu=\alpha Z$ and

$$|N_2|^2=(\pi\nu/\sinh\pi\nu)e^{\pi\nu}.$$

In Appendix A we shall show that $\bar{\psi}_d$ may be omitted from the computation to order α^3Z^3. We may therefore write

$$N_\sigma=N_1N_2^*\int d\mathbf{r}(2\mu r)^{\gamma-1}e^{i\mathbf{p}\cdot\mathbf{r}-\mu r}\times F\bar{u}_k\left(1-i\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_k}{2r}\right)\gamma_\sigma a\left(1+i\frac{\Lambda}{2m}\boldsymbol{\alpha}\cdot\hat{r}\right)u_\zeta+N_1\int d\mathbf{r}e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{r}-\mu r}\bar{\psi}_c(\mathbf{r})\gamma_\sigma a u_\zeta+O(\alpha^4Z^4). \quad (7)$$

Using Eq. (14) of reference 12 to evaluate the last integral in Eq. (7) to order α^3Z^3, we find

$$N_\sigma=N_1N_2^*\bar{u}_k[\gamma_\sigma a(I_0'+I_0'')+\boldsymbol{\alpha}\cdot\mathbf{I}_1\gamma_\sigma a+\gamma_\sigma a\boldsymbol{\alpha}\cdot\mathbf{I}_2+\alpha_i\gamma_\sigma a\alpha_j I_{ij}]u_\zeta, \quad (8)$$

where

$$I_0'=-(\partial/\partial\mu)I'|_{\sigma=\mu}, \quad (9)$$

$$I_0''=\lim_{\delta\to 0}\frac{\pi^2\alpha^2Z^2}{p(\mathbf{p}\cdot\mathbf{k}-i\delta)}, \quad (10)$$

(the limit is to be taken after integration over $\mathbf{p}$).

$$\mathbf{I}_1=-(i/2)\boldsymbol{\nabla}_kI, \quad (11)$$

$$\mathbf{I}_2=(\mu/2m)\boldsymbol{\nabla}_pI, \quad (12)$$

$$I_{ij}=-i\frac{\mu}{4m}\frac{\partial^2}{\partial k_i\partial p_j}\int_\mu^\infty I d\mu, \quad (13)$$

with

$$I'=\int(d\mathbf{r}/r)(2\sigma r)^{\gamma-1}e^{i\mathbf{p}\cdot\mathbf{r}-\mu r}{}_1F_1(i\nu,\,1,\,i(kr+\mathbf{k}\cdot\mathbf{r})), \quad (14)$$

and

$$I=\int(d\mathbf{r}/r)e^{i\mathbf{p}\cdot\mathbf{r}-\mu r}{}_1F_1(i\nu,\,1,\,i(kr+\mathbf{k}\cdot\mathbf{r})). \quad (15)$$

In Appendix B we shall show that to order α^3Z^3 one may write

$$I_0'=I_0\left\{1+(1-\gamma)\left[-\frac{i\pi}{2}-\psi_1(2)+\frac{1}{2}\ln\frac{p^2+\mu^2}{4\mu^2}+\frac{1}{2}\left(\frac{p}{\mu}-\frac{\mu}{p}\right)\tan^{-1}\frac{p}{\mu}\right]\right\}, \quad (16)$$

with

$$I_0=2I\{[\mu(1-i\nu)/D_1]+[i\nu(\mu-ik)/D_2]\}, \quad (17)$$

where

$$I=(4\pi/D_1)(D_1/D_2)^{i\nu}, \quad (18)$$

$\psi_1(x)$ denotes the first polygamma function, $D_1=p^2+\mu^2$, and $D_2=p^2+2\mathbf{p}\cdot\mathbf{k}-2ik\mu+\mu^2$. Performing the operations indicated in Eqs. (11), (12), and (13), one finds

$$\mathbf{I}_1=\nu I\{(-\mathbf{p}+i\mu\hat{k})/D_2\}, \quad (19)$$

$$\mathbf{I}_2=\frac{\mu}{m}I\{[-\mathbf{p}(1-i\nu)/D_1]-[i\nu(\mathbf{p}+\mathbf{k})/D_2]\}, \quad (20)$$

$$I_{ij}=-\frac{\nu\mu}{2m}I\left\{\frac{\delta_{ij}}{D_2}-\frac{2(p_ip_j-i\mu\hat{k}_ip_j)}{D_1D_2}-\frac{2(p_i-\mu\hat{k}_i)k_j}{D_2^2}\right\}+O(\alpha^4Z^4), \quad (21)$$

where $\hat{k}=\mathbf{k}/k$. The principal value of the function is to be understood in Eq. (18); therefore

$$|I|^2=\frac{16\pi^2}{D_1^2}\exp\left(-\nu\pi+2\nu\tan^{-1}\frac{p^2+2\mathbf{p}\cdot\mathbf{k}+\mu^2}{2pk}\right). \quad (22)$$

The expression $\sum_{se}N_\sigma\bar{N}_\lambda$ can be decomposed into 7 terms which will contribute to order α^3Z^3. Correspondingly $P(\mathbf{p},\mathbf{k},\boldsymbol{\zeta})$ may be written

$$P(\mathbf{p},\mathbf{k},\boldsymbol{\zeta})=\frac{1}{48\pi}|N_1|^2|N_2|^2\sum_{k=0}^{6}Q_k. \quad (23)$$

Evaluating the relevant traces and summations one finds

$$Q_0=|I_0'+I_0''|^2[(3w^2-4wk-4pk\cos\theta+2wp\cos\theta-p^2)+\cos\alpha(w^2-4wk-4pk\cos\theta-2wp\cos\theta-2p^2\cos^2\theta-p^2)], \quad (24a)$$

$$Q_1=\mathbf{I}_1\cdot\mathbf{I}_1^*(3w^2+4k^2-8wk-2wp\cos\theta-p^2)+2(w-k)[(\mathbf{p}+\mathbf{k})\cdot\mathbf{I}_1\hat{k}\cdot\mathbf{I}_1^*+\text{c.c.}]+\cos\alpha[\mathbf{I}_1\cdot\mathbf{I}_1^*(4k^2-w^2-2pw\cos\theta+8pk\cos\theta+2p^2\cos^2\theta+p^2)+2\hat{k}\cdot\mathbf{I}_1\hat{k}\cdot\mathbf{I}_1^*(2k^2+2kw-w^2+4pk\cos\theta+p^2)+(\mathbf{p}\cdot\mathbf{I}_1\hat{k}\cdot\mathbf{I}_1^*+\text{c.c.})(p\cos\theta+k)], \quad (24b)$$

$$Q_2=(\mathbf{I}_2\cdot\mathbf{I}_2^*)(3w^2-4wk-4pk\cos\theta+2pw\cos\theta-p^2)+\cos\alpha[2(w^2-4kw-4pk\cos\theta-p^2)\hat{k}\cdot\mathbf{I}_2\hat{k}\cdot\mathbf{I}_2^*-2(p\cos\theta+w)(\mathbf{p}\cdot\mathbf{I}_2^*\hat{k}\cdot\mathbf{I}_2+\text{c.c.})+\mathbf{I}_2\cdot\mathbf{I}_2^*(4wk-w^2+4pk\cos\theta+2pw\cos\theta+2p^2\cos^2\theta+p^2)], \quad (24c)$$

$$Q_3=(I_0\hat{k}\cdot\mathbf{I}_1^*+\text{c.c.})(p^2+2pk\cos\theta+4wk-3w^2)-2(w-k)(I_0\mathbf{p}\cdot\mathbf{I}_1^*+\text{c.c.})+\cos\alpha[2(p\cos\theta+k)\times(I_0\mathbf{p}\cdot\mathbf{I}_1^*+\text{c.c.})+(4wk-w^2+2pk\cos\theta+2pw\cos\theta+p^2)(I_0\hat{k}\cdot\mathbf{I}_1^*+\text{c.c.})], \quad (24d)$$

$$Q_4=2(w+p\cos\theta)(I_0\mathbf{p}\cdot\mathbf{I}_2^*+\text{c.c.})+(p^2-w^2+4kw+4pk\cos\theta)(I_0\hat{k}\cdot\mathbf{I}_2^*+\text{c.c.})+\cos\alpha[(I_0\hat{k}\cdot\mathbf{I}_2^*+\text{c.c.})\times(4wk-3w^2+4pk\cos\theta-2wp\cos\theta+p^2)], \quad (24e)$$

$$Q_5=(\mathbf{I}_1^*\cdot\mathbf{I}_2+\text{c.c.})(w^2-2wk)-2wk(\hat{k}\cdot\mathbf{I}_1^*\hat{k}\cdot\mathbf{I}_2+\text{c.c.})+\cos\alpha[(\mathbf{I}_1\cdot\mathbf{I}_2^*+\text{c.c.})(w^2-2wk)+2(\hat{k}\cdot\mathbf{I}_1\hat{k}\cdot\mathbf{I}_2^*+\text{c.c.})(w^2-2wk)], \quad (24f)$$

$$Q_6=(I_0I_{ij}^*+\text{c.c.})(-2wk\hat{k}_i\hat{k}_j-w(2k-w)\delta_{ij})+\cos\alpha(I_0I_{ij}^*+\text{c.c.})[(4w^2-6wk)\hat{k}_i\hat{k}_j+(2wk-w^2)\delta_{ij}], \quad (24g)$$

where $\cos\theta=\hat{p}\cdot\hat{k}$ and $\cos\alpha=\boldsymbol{\zeta}\cdot\hat{k}$. To simplify Eqs. (24) we have used the fact that $\mathbf{v}\cdot\boldsymbol{\zeta}$ will reduce to $\mathbf{v}\cdot\hat{k}\cos\alpha$ (provided $\mathbf{v}$ is a linear combination of $\mathbf{p}$ and $\mathbf{k}$) after the integration over the azimuthal angle of $\mathbf{p}$. In the expressions for Q_5 and Q_6 only those terms which can possibly contribute to order α^3Z^3 have been retained.[14]

III. MUON DECAY RATE AND ELECTRON ANGULAR DISTRIBUTION

Taking into account the energy-momentum conservation laws, $\Lambda^{(-)}$ may be written

$$\Lambda^{(-)}=\frac{G^2|N_1|^2|N_2|^2}{96\pi^5}\sum_{l=0}^{6}\int_{-1}^{1}d(\cos\alpha)\int_0^w p^2dp\times\int_{-1}^{1}d(\cos\theta)\int_0^{k_{\max}}k^2dkQ_l, \quad (25)$$

where

$$k_{\max}=(w^2-p^2)/2(w+p\cos\theta).$$

The angular distribution of electrons with respect to the muon spin direction is

$$d\Lambda^{(-)}=\tfrac{1}{2}(R+S\cos\alpha)d(\cos\alpha). \quad (26)$$

Performing the integrations in the order indicated in Eq. (25) gives analytical results for R and S rather simply; however, since the $\mathbf{k}$ integration is performed first one cannot determine the electron spectrum directly.

To perform the integrations indicated in Eq. (25) one expands the various Q_l in Laurent series in k and integrates term by term. Setting $k_{\max}=(w/2)(1-\lambda^2\xi^2)\times(1+\lambda\xi\cos\theta)^{-1}$, with $\xi=p/\mu$ and $\lambda=\mu/w=\alpha Z/\gamma$, one is able to construct a Taylor series in $\lambda^2\xi^2$ and in $\lambda\xi\cos\theta$. The p integration is replaced by an integration over the dimensionless variable ξ extending from 0 to $1/\lambda$. For large values of ξ each term in the integrand behaves as $\lambda^m\xi^{-n}$ for $m+n\geqslant 6$. It follows that the upper limit of the ξ integration may be replaced by $+\infty$ to order α^5Z^5. Since odd powers of p and $\cos\theta$ occur together in the expressions for Q_l and in $k_{\max}$ it follows that odd powers of αZ (up to order α^5Z^5 where the upper limit of the ξ integral becomes λ dependent) must vanish. The various elementary integrations are carried out, with results which are given in Table I. We have denoted by $\bar{R}$ the common factor

$$\bar{R}=\frac{G^2w^5}{192\pi^2}\frac{|N_1|^2|N_2|^2}{\mu^3}e^{-\pi\nu}, \quad (27)$$

TABLE I. Contributions to the decay and angular distribution parameters.

k	$R_k/\bar{R}$	$S_k/\bar{R}$
0	$1-2\lambda^2+(5/3)\nu\lambda+(\pi^2/6)\nu^2-(1-\gamma)(1+2\psi_1(2))+O(\alpha^4Z^4)$	$-\frac{1}{3}[1-(14/3)\lambda^2+3\nu\lambda+(\pi^2/6)\nu^2-(1-\gamma)(1+2\psi_1(2))]+O(\alpha^4Z^4)$
1	$O(\alpha^4Z^4)$	$O(\alpha^4Z^4)$
2	$(1/4)(\mu^2/m^2)+O(\alpha^4Z^4)$	$-\frac{1}{3}[-(1/12)(\mu^2/m^2)]+O(\alpha^4Z^4)$
3	$(10/3)\nu\lambda+O(\alpha^4Z^4)$	$-\frac{1}{3}[2\nu\lambda]+O(\alpha^4Z^4)$
4	$-\lambda(\mu/m)+O(\alpha^4Z^4)$	$-\frac{1}{3}[(5/3)\lambda(\mu/m)]+O(\alpha^4Z^4)$
5	$O(\alpha^4Z^4)$	$O(\alpha^4Z^4)$
6	$O(\alpha^4Z^4)$	$O(\alpha^4Z^4)$

which when expanded to third order in αZ gives

$$\bar{R}=R_{\text{V}}\{1-\tfrac{5}{2}\alpha^2Z^2-(\pi^2/6)\nu^2+(1-\gamma)[\tfrac{1}{2}+2\psi_1(2)]\}, \quad (28)$$

where the vacuum rate $R_{\text{V}}=G^2m^5/192\pi^3$.

Adding the terms in Table I and taking Eq. (28) into account, we find

$$R=R_{\text{V}}(1-\tfrac{1}{2}\alpha^2Z^2)+O(\alpha^4Z^4), \quad (29)$$

$$S=-\tfrac{1}{3}R_{\text{V}}(1-\tfrac{5}{6}\alpha^2Z^2)+O(\alpha^4Z^4). \quad (30)$$

The expression for R is seen to agree with the second-order expression obtained by Überall. It is of interest to note that the assertion made by Überall[15] (essentially that ψ_μ may be treated to first order, provided Q_2 is neglected in the evaluation of R) is seen to be verified. The corresponding simplification is not permissible for S as is easily seen by comparing the coefficients of $\cos\alpha$ in Q_2 and Q_0.

The rates given in Eqs. (29) and (30) have a somewhat weaker Z dependence than the corresponding numerical results of Gilinsky and Mathews. This is surprising, since qualitatively nuclear size effects may be accounted for by replacing Z by $Z_{\text{eff}}<Z$.

IV. ENERGY SPECTRA OF THE EMITTED ELECTRON

One may write

$$d^2\Lambda^{(-)}=\tfrac{1}{2}(w_R+w_S\cos\alpha)k^2dkd(\cos\alpha), \quad (31)$$

where w_R and w_S are the spectral functions associated with the decay rates R and S. Using the previously mentioned fact that odd functions of p and $\cos\theta$ occur together in the Q_k, one may show that

$$w_R+w_S\cos\alpha=\frac{G^2|N_1|^2|N_2|^2}{48\pi^5}\sum_{k=0}^{6}\int_{-(w-2k)}^{w}p^2dp\times\int_{-1}^{\cos\theta_0}d(\cos\theta)Q_k, \quad (32)$$

where $\cos\theta_0=(w^2-p^2)/2pk-w/p$.

Performing the indicated integrations and keeping

[14] An expansion in powers of k has been made for reasons explained in the discussion following Eq. (26).

[15] H. Überall, reference 2, p. 366.

only those terms which contribute to order $\alpha^2 Z^2$, we find

$$
\begin{aligned}
w_R=\frac{G^2m^2}{12\pi^4}\bigg\{\bigg[&d+\lambda\bigg(-\frac{\pi^2}{4}+2b\tan^{-1}y+(\tan^{-1}y)^2\\
&+\frac{1}{y^2+1}-\frac{4}{3}\frac{1}{(y^2+1)^2}\bigg)+\lambda^2\bigg(n\bigg(\frac{3}{4}+\frac{6}{x}\bigg)-\frac{\pi}{x}\\
&+\bigg(\frac{4}{3}\frac{1}{y^2+1}-\frac{6}{(y^2+1)^2}\bigg)\tan^{-1}y+\frac{11}{6}\frac{y}{y^2+1}\\
&+\frac{7}{18}\frac{y}{(y^2+1)^2}+\frac{\pi^3}{12}+\frac{2}{3}(\tan^{-1}y)^3+2b(\tan^{-1}y)^2\\
&-\frac{y}{x}\bigg(\frac{\pi^2}{4}-(\tan^{-1}y)^2\bigg)+\frac{1}{2}b\ln(1+y^2)-\bigg(\frac{3}{4}+\frac{\pi^2}{6}\bigg)d\\
&-b\ln 2+\frac{1}{2}\Psi(2n)\bigg)\bigg](3-2x)-\lambda^2 n\bigg(4+\frac{2x}{3}\bigg)\bigg\},
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
w_S=\frac{1-2x}{3-2x}w_R+\frac{G^2m^2}{12\pi^4}\bigg\{&\frac{8}{3}\lambda\frac{1}{(y^2+1)^2}\\
&+\lambda^2\bigg[\frac{1-2x}{3-2x}+n\bigg(4+\frac{2x}{3}\bigg)+(1-2x)\bigg(2-\frac{2}{x}\bigg)n\\
&-\frac{4}{3}n-\frac{13}{3}\frac{y}{y^2+1}-\frac{11}{6}\frac{y}{(y^2+1)^2}\bigg]\bigg\}.
\end{aligned}
\tag{34}
$$

We have introduced for simplicity the following

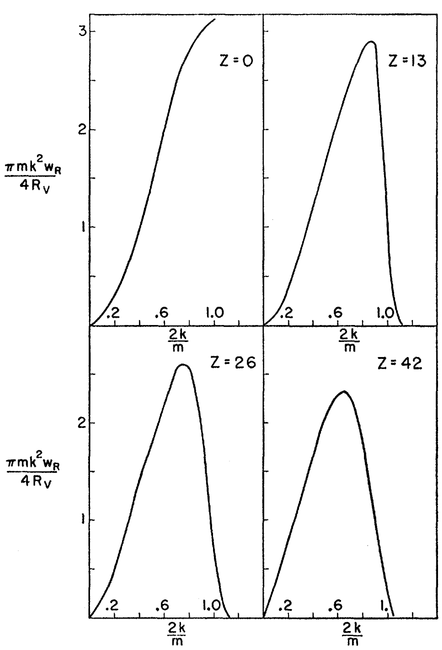

FIG. 1. The electron energy spectrum k^2w_R for unpolarized muons for several nuclei.

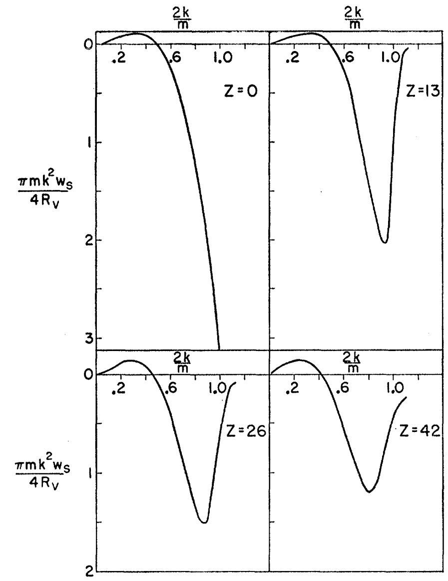

FIG. 2. The spectrum of the angular distribution parameter k^2w_S for several nuclei.

notation:

$$x=2k/w,\quad y=(w-2k)/\mu,$$

$$b=\frac{y}{(y^2+1)}\bigg(1+\frac{2}{3}\frac{1}{y^2+1}\bigg),$$

$$n=\pi/2+\tan^{-1}y,\quad d=n+b.$$

In Eq. (33) $\Psi(x)$ denotes the Spence function.[16]

The first-order term in w_R agrees with the spectrum deduced by Überall. The functions k^2w_R and k^2w_S are plotted[17] in Figs. 1 and 2. The graphs for k^2w_R and k^2w_S at $Z=0$ have been included for comparison purposes. One sees by taking the limit $\lambda\rightarrow 0$ in Eqs. (33) and (34) that the corresponding formulas are

$$w_{R0}=(G^2m/12\pi^3)(3m-4k),\tag{35}$$

$$w_{S0}=(G^2m/12\pi^3)(m-4k).\tag{36}$$

It is easily shown that

$$\binom{R}{S}=\frac{w^3\lambda}{8}\int_{-1/\lambda}^{1/\lambda}(1-\lambda y)^2\binom{w_R(y)}{w_S(y)}dy.\tag{37}$$

Performing these integrals, one verifies the expressions obtained in Eqs. (29) and (30).

[16] A. Ashour and A. Sabri, Math. Tables and Aids to Computers **10**, 57 (1956).

[17] The formulas for k^2w_R and k^2w_S became slightly negative in the region $k>1.2m$. This is explained by the fact that in this region the first- and second-order terms vanish and that some, but not all, of the third-order terms remain. If all third-order terms had been retained the spectral functions would have vanished to order α^4Z^4. In plotting the graphs we have neglected the small negative terms in this region.

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The IBM computer at Notre Dame was used to obtain the numerical results presented in the figures.

APPENDIX A

The contribution to N_σ from $\bar{\psi}_d$ may be written

$$N_1\int d\mathbf{r}e^{i(\mathbf{p}+\mathbf{k})\cdot\mathbf{r}-\mu r}\bar{\psi}_d(\mathbf{r})\gamma_\sigma a u_\zeta. \tag{A1}$$

One sees from reference 13 that $\bar{\psi}_d$ possesses an asymptotic expansion in kr of the form

$$\bar{\psi}_d=N_2{}^*\bar{u}_k\left\{\frac{f^{(3)}(\theta,\varphi)}{kr}e^{-ikr-i\alpha Z\ln 2kr}+O\left(\frac{\alpha^3Z^3}{k^2r^2}\right)\right\}, \tag{A2}$$

where $f^{(3)}$ is a term of order α^3Z^3. Since the dominant contribution from this term will occur with $k\sim w/2$, and since the factor $r^2e^{-\mu r}$ in the integrand peaks at $r=2/\mu$, the wave function $\bar{\psi}_d$ may be replaced in Eq. (A1) by

$$\bar{\psi}_d=N_2{}^*\bar{u}_ke^{-ikr}\{\alpha Zf^{(3)}(\theta,\varphi)e^{-i\alpha Z\ln\alpha Z}+O(\alpha^5Z^5)\}. \tag{A3}$$

The resulting integral is of order α^4Z^4 and therefore may be neglected.

APPENDIX B

Writing

$$I'=(2\sigma)^{\gamma-1}\int\frac{d\mathbf{r}}{r}r^{\gamma-1}e^{i\mathbf{p}\cdot\mathbf{r}-\mu r}F, \tag{B1}$$

and introducing

$$r^{\gamma-1}=\frac{\Gamma(\gamma)i^{1-\gamma}}{2\pi i}\int_{i\infty}^{0^+}x^{-\gamma}e^{ixr}dx, \tag{B2}$$

one deduces

$$I'=(2\sigma)^{\gamma-1}\frac{\Gamma(\gamma)i^{1-\gamma}}{2\pi i}\int_{i\infty}^{0^+}dx\,x^{-\gamma}\int\frac{d\mathbf{r}}{r}e^{i\mathbf{p}\cdot\mathbf{r}+ixr-\mu r}F. \tag{B3}$$

The spatial integral is carried out in the well-known manner to give[18]

$$I'=(2\sigma)^{\gamma-1}\frac{\Gamma(\gamma)i^{1-\gamma}}{2\pi i}4\pi\int_{i\infty}^{0^+}\frac{x^{-\gamma}dx}{[p^2+(\mu-ix)^2]}\times\left(\frac{p^2+(\mu-ix)^2}{p^2+2\mathbf{p}\cdot\mathbf{k}-2ik(\mu-ix)+(\mu-ix)^2}\right)^{i\nu}. \tag{B4}$$

Expanding,

$$x^{-\gamma}=(1/x)[1+(1-\gamma)\ln x]+O(\alpha^4Z^4),$$

one can rewrite I' in the form

$$I'=(2\sigma)^{\gamma-1}[\Gamma(\gamma)i^{1-\gamma}/2\pi i]4\pi[J_1+(1-\gamma)J_2], \tag{B5}$$

where

$$J_1=\int_{i\infty}^{0^+}\frac{dx}{x[p^2+(\mu-ix)^2]}\times\left(\frac{p^2+(\mu-ix)^2}{p^2+2\mathbf{p}\cdot\mathbf{k}-2ik(\mu-ix)+(\mu-ix)^2}\right)^{i\nu}=(2\pi i/D_1)(D_1/D_2)^{i\nu}, \tag{B6}$$

$$J_2=\int_{i\infty}^{0^+}\frac{dx\ln x}{x[p^2+(\mu-ix)^2]}\times\left(\frac{p^2+(\mu-ix)^2}{p^2+2\mathbf{p}\cdot\mathbf{k}-2ik(\mu-ix)+(\mu-ix)^2}\right)^{i\nu}$$
$$=2\pi i\lim_{\rho\to0}\left\{\frac{1}{D_1}\left(\frac{D_1}{D_2}\right)^{i\nu}(\ln\rho-i\pi)+\int_\rho^\infty\frac{d\zeta}{\zeta[p^2+(\mu+\zeta)^2]}\left[1+i\nu\times\ln\left(\frac{p^2+(\mu+\zeta)^2}{(\mu+\zeta)^2-2ik(\mu+\zeta)+2\mathbf{p}\cdot\mathbf{k}+p^2}\right)\right]\right\}. \tag{B7}$$

An expansion valid to order α^3Z^3 has been carried out in the last term. The scalar integrals in Eq. (B7) are evaluated and expanded in a series in k. Retaining only those terms which will result in contributions to order α^3Z^3, one obtains the result quoted in Eq. (16) of the text.

[18] A. Sommerfeld, *Atombau und Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, 1939), Vol. II, p. 461.