Dynamical Behavior of Dislocations in Anisotropic Media*

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The dynamical behavior of uniformly moving dislocations in anisotropic media is discussed for those crystal systems for which the edge and screw components can be considered separately. Expressions are obtained for the kinetic and potential energies of both edge and screw dislocations. It is found that screw dislocations behave normally at all velocities up to the limiting velocity. Edge dislocations, however, display an anomalous dynamical behavior. It appears that in general there is a range of velocities for which the shear stress on the slip plane is negative and edge dislocations of like sign attract rather than repel one another. In an isotropic material the upper limit of this velocity range is the velocity of shear sound; the lower limit is the Rayleigh wave velocity which can never be less than 0.69 the velocity of shear sound. In the anisotropic case it is possible for the limiting velocity (for a given orientation) to be less than the corresponding shear wave velocity; also the threshold velocity for the anomalous dynamical behavior can be any velocity from zero up to the shear wave velocity, depending on the elastic constants of the material and the orientation considered. An example of an edge dislocation in a hexagonal material is discussed in some detail.

I. INTRODUCTION

WEERTMAN[1] has shown that in an isotropic elastic solid there is a range of velocities for which two like edge dislocations on the same slip plane will attract rather than repel. The upper limit of this velocity range is c_s, the velocity of transverse sound; this velocity also represents the limiting velocity of the dislocation since its energy tends to infinity as c_s is approached. The lower limit of velocity c is given by c_r, the Rayleigh wave velocity. At $c=c_r$ the shear stress component of the field of the moving edge dislocation becomes zero, i.e., the repulsive force between two like edge dislocations vanishes. For further increase in velocity the shear stress is negative and the force between two like edges is then an attractive one.

A physical explanation for this anomalous behavior has been given by Weertman.[2] He concludes that dislocations of like sign will attract (and unlike repel) if the kinetic energy in the displacement field of an isolated dislocation is greater than the potential energy. Now the kinetic energy of a screw dislocation can never be greater than its potential energy; hence screw dislocations always behave normally. For edge dislocations the kinetic energy is greater than the potential energy for velocities above the Rayleigh wave velocity.

This paper sets out to answer two questions: (i) Does the same general result hold for an anisotropic medium; i.e., is there a range of velocities for which two like edge dislocations will attract? (ii) If such a range does exist, what is the lowest velocity at which the attraction will occur? It is hoped that for a certain crystal type and orientation this threshold velocity will be a small fraction of the limiting velocity for that direction; this would then represent the optimum situation for an experimental verification of this phenomenon.

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[1] J. Weertman, *Response of Metals to High-Velocity Deformation*, edited by P. G. Shewmon and V. F. Zackay (Interscience Publishers, Inc., New York, 1961).

[2] J. Weertman, Phys. Rev. **119**, 1871 (1960).

II. UNIFORMLY MOVING DISLOCATIONS

To answer these questions we consider a uniformly moving dislocation in an anisotropic medium. The problem has been treated by Bullough and Bilby[3] and this paper is based on their analysis. The equations of equilibrium for an anisotropic elastic medium are

$$F_{ijkl}u_{k\cdot jl}=\rho\ddot{u}_i, \tag{1}$$

where u_i is the displacement referred to Cartesian coordinates x_i, ρ is the density, F_{ijkl} the elastic constant tensor, and all subscripts to the right of the dot represent differentiation with respect to the appropriate space coordinate. Assume the dislocation line is parallel to x_3 and moving in the x_1 direction with velocity c. Eshelby's method[4] of solution is applied, i.e., the material is imagined cut along the x_1x_3 plane, moving tractions applied to the cut surfaces so that on rewelding a moving dislocation is obtained with no external forces or couples acting on it. The appropriate solution of (1) will correspond to a surface disturbance propagating in the $x_2=0$ plane, vanishing as $|x_2|\to\infty$, and independent of x_3. Bullough and Bilby write

$$u_j(x_1',x_2)=\sum_{n=1}^{3} C_nP_{jn}\exp\{s(-\lambda_nx_2+ix_1')\}, \tag{2}$$

where $x_1'=x_1-ct$, the P_{jn} are functions of the elastic constants and dislocation velocity, and the C_n are arbitrary complex constants. Substitution of (2) into (1) yields that λ_n is a root of the sextic equation

$$|F_{i2k2}\lambda_n^2-i\lambda_n(F_{i1k2}+F_{i2k1})-F_{i1k1}+\rho c^2\delta_{ik}|=0, \tag{3}$$

where δ_{ik} is the Kronecker delta. λ_n is in general complex; however, in order that (2) represent a surface wave we must restrict the dislocation velocity such that the real part of λ_n is greater than zero.

In the general anisotropic case both a pure screw and a pure edge dislocation involve all three components of

[3] R. Bullough and B. A. Bilby, Proc. Phys. Soc. (London) **B67**, 615 (1954).

[4] J. D. Eshelby, Proc. Phys. Soc. (London) **A62**, 307 (1949).

displacement. Rather than treat this general problem we follow Bullough and Bilby and consider the simpler one where all F_{ijkl} with an odd number of subscripts equal to 3 are zero. In the contracted notation ($ij \rightarrow i$, $i=j$; $ij \rightarrow k+3$, $i \neq j$) this reads

$$F_{14}=F_{15}=F_{24}=F_{25}=F_{46}=F_{56}=F_{34}=F_{35}=0. \quad (4)$$

This has the effect of separating the equations determining u_1 and u_2 from that determining u_3. With the stress components given by

$$p_{ij}=F_{ijkl}u_{k \cdot l}, \quad (5)$$

we find that the stresses p_{31}, p_{32} depend only on the derivatives of u_3; the other four components of stress depend only on the derivatives of u_1 and u_2. Hence the two problems $u_1=u_2=0$, $u_3 \neq 0$ and $u_1 \neq 0$, $u_2 \neq 0$, $u_3=0$ can be considered separately. Likewise the sixth-order equation in λ_n reduces to two equations: a quartic for the edge dislocation and a quadratic for the screw dislocation.

Starting with the u_i given by (2) and employing the method of Fourier transforms one may build up more general displacements U_i; the latter are determined by boundary conditions along the slip plane, i.e., discontinuity in the appropriate displacement as well as conditions on the stress components to insure that no external forces or couples are acting on the dislocation. These displacements U_i (developed in reference 3) will be used in the following discussions. The corresponding stress components σ_{ij} are given by an expression analogous to (5), i.e.,

$$\sigma_{ij}=F_{ijkl}U_{k \cdot l}. \quad (6)$$

Also the strain components ϵ_{ij} are given by the usual expression

$$\epsilon_{ij}=\tfrac{1}{2}(U_{i \cdot j}+U_{j \cdot i}). \quad (7)$$

III. SCREW DISLOCATIONS

The displacement field for a screw dislocation moving in an anisotropic medium for which (4) holds is

$$U_3=-\frac{b_0}{2\pi}\tan^{-1}\left(\frac{x_1'-bx_2}{ax_2}\right), \quad (8)$$

where

$$\lambda_3=\pm a+ib=\frac{\pm[F_{44}F_{55}-F_{45}{}^2-F_{44}\rho c^2]^{\frac{1}{2}}}{F_{44}}+i\frac{F_{45}}{F_{44}} \quad (9)$$

(the positive sign applies for $x_2>0$, the negative sign for $x_2<0$) and b_0 is the magnitude of the Burgers vector.

The nonzero stress components are

$$\sigma_{13}=\frac{ab_0}{2\pi}\left[\frac{x_1'F_{45}-x_2F_{55}}{a^2x_2{}^2+(x_1'-bx_2)^2}\right], \quad (10)$$

$$\sigma_{23}=\frac{ab_0}{2\pi}\left[\frac{x_1'F_{44}-x_2F_{45}}{a^2x_2{}^2+(x_1'-bx_2)^2}\right]. \quad (11)$$

E_p, the potential energy per unit length of the moving screw dislocation, is obtained by integrating over the crystal the strain energy stored in the stress field of the dislocation. If A represents some cross section normal to x_3, then

$$E_p=\frac{1}{2}\int_A \sigma_{ij}\epsilon_{ij}dA$$
$$=\frac{a^2b_0{}^2}{8\pi^2}\int_0^{2\pi}\int_{R_1}^{R_2}\frac{(F_{55}\sin^2\theta-F_{45}\sin 2\theta+F_{44}\cos^2\theta)}{[(a^2+b^2)\sin^2\theta-b\sin 2\theta+\cos^2\theta]^2}\times\frac{drd\theta}{r}, \quad (12)$$

where we have put

$$r^2=x_1'^2+x_2{}^2, \quad \tan\theta=x_2/x_1'. \quad (13)$$

The limits on r are those usually employed, i.e., R_2 represents a dimension of the crystal, R_1 a radius of order b_0.

Besides the potential energy there is also a kinetic energy E_k associated with the moving screw dislocation since there is motion of the medium about it.

$$E_k=\frac{1}{2}\int_A \rho(\dot{U}_3)^2dA$$
$$=\frac{\rho c^2b_0{}^2a^2}{8\pi^2}\int_0^{2\pi}\int_{R_1}^{R_2}\frac{\sin^2\theta}{[(a^2+b^2)\sin^2\theta-b\sin 2\theta+\cos^2\theta]^2}\times\frac{drd\theta}{r}. \quad (14)$$

(12) and (14) can be evaluated to give expressions of the form

$$E_p=\frac{(1-c^2/2c_\infty{}^2)E_0}{(1-c^2/c_\infty{}^2)^{\frac{1}{2}}}, \quad E_k=\frac{c^2}{2c_\infty{}^2}\cdot\frac{E_0}{(1-c^2/c_\infty{}^2)^{\frac{1}{2}}}. \quad (15)$$

The total energy $E_t=E_p+E_k$ is given by the relativistic formula

$$E_t=E_0/(1-c^2/c_\infty{}^2)^{\frac{1}{2}}. \quad (16)$$

These are the same formulas that are obtained in the isotropic case with the appropriate definition of E_0, the rest energy, and c_∞, the limiting velocity:

	Isotropic	Anisotropic	
$E_0\Big/\left(\frac{b_0{}^2}{4\pi}\ln\frac{R_2}{R_1}\right)$	μ	$[F_{44}F_{55}-F_{45}{}^2]^{\frac{1}{2}}$	(17)
$\rho c_\infty{}^2$	μ	$(F_{44}F_{55}-F_{45}{}^2)/F_{44}$	

where μ is the (isotropic) shear modulus.

The condition that $R(\lambda)>0$ demands $c<c_\infty$; c_∞ is truly a limiting velocity for the screw dislocation since

$E_t \to \infty$ as $c \to c_\infty$. Also we see that (as in the isotropic case) the kinetic energy never exceeds the potential energy; they become equal (and infinite) at $c=c_\infty$.

Note that the stress field given by (10) and (11) is well-behaved. Since a approaches zero as the velocity is increased towards c_∞, the stresses are contracted in the direction of motion. The stress components do not change sign with increasing velocity but tend to zero as the limiting velocity is approached.

In short there is no unusual dynamical behavior exhibited by a screw dislocation moving along those directions in an anisotropic medium for which (4) applies.

IV. EDGE DISLOCATIONS

A. General Formulas

Here the behavior is more complex and correspondingly the analysis, though straightforward, becomes cumbersome. In all that follows, the subscript n will take on the values 1 and 2; all indicated summations over n will likewise be for $n=1, 2$. Let us start with the attenuation parameter λ_n which is a solution of a quartic equation derived from (3). If we put $y=-i\lambda_n$ we obtain an equation

$$\sum_{\nu=0}^{4} K_\nu y^\nu = 0, \tag{18}$$

with all K_ν real. Hence λ_n is of the form $\lambda_n = \pm p_n + iq_n$, where the positive sign holds for $x_2>0$. We have

$$\begin{aligned}
K_4 &= F_{66}F_{22}-F_{26}^2,\\
K_3 &= 2(F_{26}F_{12}-F_{16}F_{22}),\\
K_2 &= (F_{11}F_{22}-F_{12}^2-2F_{12}F_{66}+2F_{16}F_{26})\\
&\qquad -(F_{22}+F_{66})\rho c^2,\\
K_1 &= 2(F_{16}F_{12}-F_{26}F_{11})+2\rho c^2(F_{16}+F_{26}),\\
K_0 &= (F_{11}-\rho c^2)(F_{66}-\rho c^2)-F_{16}^2.
\end{aligned} \tag{19}$$

In terms of the solutions of (18) we can define eight quantities (using the notation of reference 3):

$$\begin{aligned}
S_{nr} &= F_{22}(p_n^2-q_n^2)+2q_nF_{26}-F_{66}+\rho c^2,\\
S_{ni} &= 2p_n(F_{22}q_n-F_{26}),\\
R_{nr} &= F_{16}-F_{26}(p_n^2-q_n^2)-q_n(F_{12}+F_{66}),\\
R_{ni} &= p_n(F_{12}+F_{66}-2q_nF_{26}),
\end{aligned} \tag{20}$$

and two sets of variables r_n, θ_n by

$$\begin{aligned}
r_n^2 &= (x_1'-q_nx_2)^2+(p_nx_2)^2,\\
\tan\theta_n &= p_nx_2/(x_1'-q_nx_2).
\end{aligned} \tag{21}$$

We can then write the displacement field of the moving edge dislocation as

$$\begin{aligned}
U_1 &= (b_0/2\pi)\sum (\alpha_n \ln r_n - \beta_n\theta_n),\\
U_2 &= (b_0/2\pi)\sum (\delta_n \ln r_n - \epsilon_n\theta_n).
\end{aligned} \tag{22}$$

The eight constants α_n, β_n, δ_n, ϵ_n are given in terms of the four constants A_{nr}, A_{ni} of Bullough and Bilby by

$$\begin{aligned}
\alpha_n &= A_{nr}S_{nr}-A_{ni}S_{ni}, & \delta_n &= A_{nr}R_{nr}-A_{ni}R_{ni},\\
\beta_n &= A_{nr}S_{ni}+A_{ni}S_{nr}, & \epsilon_n &= A_{nr}R_{ni}+A_{ni}R_{nr}.
\end{aligned} \tag{23}$$

A_{nr} and A_{ni} are given by boundary conditions along the slip plane; these conditions can be written as

$$\begin{aligned}
&\sum [F_{66}(\alpha_np_n-\beta_nq_n)+F_{26}(\delta_np_n-\epsilon_nq_n)] = F_{16},\\
&\sum [F_{26}(\alpha_np_n-\beta_nq_n)+F_{22}(\delta_np_n-\epsilon_nq_n)] = F_{12},\\
&\sum \epsilon_n = 0, \quad \sum \beta_n = -1.
\end{aligned} \tag{24}$$

Substituting (20) and (23) into (24) and solving for A_{nr} and A_{ni}, we obtain after considerable reduction

$$\begin{aligned}
DA_{1i}(p_1,p_2,q_1,q_2) &= DA_{2i}(p_2,p_1,q_2,q_1)\\
&= 2p_1p_2(q_2-q_1)\{(F_{16}F_{26}-F_{12}F_{66})[F_{26}G_3-(F_{66}-\rho c^2)G_2]+(F_{16}F_{22}-F_{12}F_{26})(F_{16}G_2-F_{12}G_3)\}\\
&\quad +p_1p_2(F_{22}F_{66}-F_{26}^2)\{(p_2^2+q_2^2)[(p_2^2-p_1^2)+(3q_1-q_2)(q_1-q_2)](F_{66}G_1-F_{26}G_2)\\
&\quad +2[q_1p_2^2-q_2p_1^2+3q_1q_2(q_1-q_2)](F_{12}G_2-F_{16}G_1)+[(p_1^2-p_2^2)+3(q_2^2-q_1^2)](F_{12}G_3-F_{16}G_2)\},\\
DA_{1r}(p_1,p_2,q_1,q_2) &= DA_{2r}(p_2,p_1,q_2,q_1)\\
&= p_2[(p_1^2-p_2^2)-(q_1-q_2)^2]\{(F_{16}F_{26}-F_{12}F_{66})[F_{26}G_3-(F_{66}-\rho c^2)G_2]+(F_{16}F_{22}-F_{12}F_{26})\\
&\quad \times(F_{16}G_2-F_{12}G_3)\}+p_2(F_{22}F_{66}-F_{26}^2)\{[(2q_2+q_1)\langle(p_1^2-p_2^2)-(q_1-q_2)^2\rangle+2p_1^2(q_1-q_2)]\\
&\quad \times(F_{12}G_3-F_{16}G_2)+(p_2^2+q_2^2)[q_1\langle(p_1^2-p_2^2)-(q_1-q_2)^2\rangle+2p_1^2(q_1-q_2)](F_{26}G_2-F_{66}G_1)\\
&\quad +[(p_2^2+q_2^2+2q_1q_2)\langle(p_1^2-p_2^2)-(q_1-q_2)^2\rangle+4q_2p_1^2(q_1-q_2)](F_{16}G_1-F_{12}G_2)\},
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
D &= p_1p_2(F_{22}F_{66}-F_{26}^2)[(p_1+p_2)^2+(q_1-q_2)^2][(p_1-p_2)^2+(q_1-q_2)^2](G_2^2-G_1G_3),\\
G_1 &= F_{22}(F_{66}+F_{12})-2F_{26}^2,\\
G_2 &= F_{22}F_{16}+F_{26}(\rho c^2-F_{66}),\\
G_3 &= 2F_{16}F_{26}+(F_{66}+F_{12})(\rho c^2-F_{66}).
\end{aligned}$$

These expressions for A_{ni} and A_{nr} can then be used to obtain α_n, β_n, δ_n, ϵ_n and hence the displacements U_1 and U_2 are determined.

From the displacements we find that the nonzero stress components are

$$\begin{aligned}
\sigma_{11}&=\frac{b_0}{2\pi}\sum\frac{1}{r_n^2}[x_1'(F_{11}\alpha_n+F_{16}\varphi_n+F_{12}l_n)\\
&\qquad+x_2(F_{11}k_n+F_{16}\nu_n+F_{12}\omega_n)],\\
\sigma_{12}&=\frac{b_0}{2\pi}\sum\frac{1}{r_n^2}[x_1'(F_{16}\alpha_n+F_{66}\varphi_n+F_{26}l_n)\\
&\qquad+x_2(F_{16}k_n+F_{66}\nu_n+F_{26}\omega_n)],\\
\sigma_{22}&=\frac{b_0}{2\pi}\sum\frac{1}{r_n^2}[x_1'(F_{12}\alpha_n+F_{26}\varphi_n+F_{22}l_n)\\
&\qquad+x_2(F_{12}k_n+F_{26}\nu_n+F_{22}\omega_n)],\\
\sigma_{33}&=\frac{b_0}{2\pi}\sum\frac{1}{r_n^2}[x_1'(F_{13}\alpha_n+F_{36}\varphi_n+F_{32}l_n)\\
&\qquad+x_2(F_{13}k_n+F_{36}\varphi_n+F_{32}\omega_n)],
\end{aligned}\tag{26}$$

where we have made the further reductions,

$$\begin{aligned}
&k_n=\beta_np_n-\alpha_nq_n, \qquad -l_n=\delta_nq_n+\epsilon_np_n,\\
&\omega_n=\delta_n(p_n^2+q_n^2), \qquad \varphi_n=\delta_n-\alpha_nq_n-\beta_np_n,\\
&\nu_n=\epsilon_np_n-\delta_nq_n+\alpha_n(p_n^2+q_n^2).
\end{aligned}\tag{27}$$

From (26) we get the important result that the shear stress on the slip plane vanishes when

$$\sum (F_{16}\alpha_n+F_{66}\varphi_n+F_{26}l_n)=0. \tag{28}$$

If the dislocation velocity which satisfies (28) is less than the limiting velocity of the dislocation, then it represents the threshold velocity for the anomalous dynamical behavior of edge dislocations.

Using the stresses given by (26) and the strains obtained by differentiation of the displacements (22), we can obtain the strain energy per unit volume $\frac{1}{2}\sigma_{ij}\epsilon_{ij}$; integration over the crystal yields E_p, the potential energy per unit length of the dislocation:

$$E_p=\frac{b_0^2}{8\pi^2}\ln\frac{R_2}{R_1}\sum_{n,m=1}^{2}\int_0^{2\pi}\frac{(L_{nm}\cos^2\theta+M_{nm}\sin^2\theta+N_{nm}\cos\theta\sin\theta)d\theta}{[\cos^2\theta+(p_n^2+q_n^2)\sin^2\theta-q_n\sin2\theta][\cos^2\theta+(p_m^2+q_m^2)\sin^2\theta-q_m\sin2\theta]}, \tag{29}$$

where

$$\begin{aligned}
L_{nm}&=F_{11}\alpha_n\alpha_m+F_{22}l_nl_m+F_{66}\varphi_n\varphi_m+F_{12}(\alpha_nl_m+\alpha_ml_n)+F_{16}(\alpha_n\varphi_m+\alpha_m\varphi_n)+F_{26}(l_n\varphi_m+l_m\varphi_n),\\
M_{nm}&=F_{11}k_nk_m+F_{22}\omega_n\omega_m+F_{66}\nu_n\nu_m+F_{12}(k_n\omega_m+k_m\omega_n)+F_{16}(k_n\nu_m+k_m\nu_n)+F_{26}(\omega_n\nu_m+\omega_m\nu_n),\\
N_{nm}&=F_{11}(\alpha_nk_m+\alpha_mk_n)+F_{22}(l_n\omega_m+l_m\omega_n)+F_{66}(\varphi_n\nu_m+\varphi_m\nu_n)+F_{12}[(\alpha_n\omega_m+\alpha_m\omega_n)+(k_nl_m+k_ml_n)]\\
&\qquad+F_{16}[(\alpha_n\nu_m+\alpha_m\nu_n)+(k_n\varphi_m+k_m\varphi_n)]+F_{26}[(l_n\nu_m+l_m\nu_n)+(\omega_n\varphi_m+\omega_m\varphi_n)].
\end{aligned}\tag{30}$$

Using the theory of residues we can evaluate (29) to obtain

$$E_p=\frac{b_0^2}{4\pi}\ln\frac{R_2}{R_1}\sum_{n,m=1}^{2}\frac{L_{nm}\{p_n(p_m^2+q_m^2)+p_m(p_n^2+q_n^2)\}+M_{nm}(p_n+p_m)+N_{nm}(p_nq_m+p_mq_n)}{p_np_m[(p_n+p_m)^2+(q_n-q_m)^2]}. \tag{31}$$

The kinetic energy per unit volume of the displacement field of the moving edge dislocation is

$$(\rho/2)[(\dot{U}_1)^2+(\dot{U}_2)^2].$$

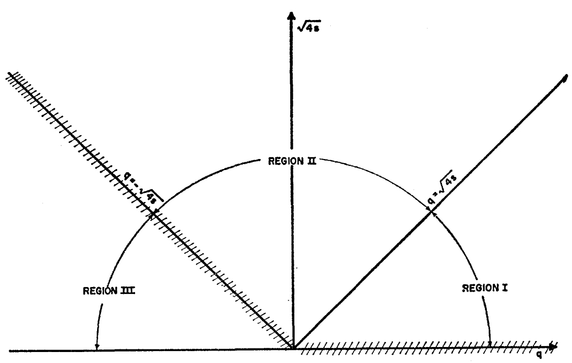

FIG. 1. The hashed line is the locus of limiting velocities for a moving edge dislocation. q and s are functions of the elastic constants and dislocation velocity defined in (36). The type of surface wave corresponding to a moving edge dislocation is different in each of the three regions shown.

When written in terms of r and θ [as given by (13)] this has the same θ dependence as the integrand of (29). The kinetic energy (per unit length) E_k is then given by an expression of the form (31) with L, M, N replaced by other constants, i.e.,

$$E_k=\rho c^2E_p[L_{mn}\to F_{mn},\, M_{nm}\to G_{nm},\, N_{nm}\to H_{nm}], \tag{32}$$

where

$$\begin{aligned}
&F_{mn}=\alpha_m\alpha_n+\delta_m\delta_n, \qquad G_{mn}=k_mk_n+\psi_m\psi_n,\\
&H_{mn}=(\alpha_mk_n+\alpha_nk_m)+(\delta_m\psi_n+\delta_n\psi_m),\\
&\psi_n=\epsilon_np_n-\delta_nq_n.
\end{aligned}\tag{33}$$

Recall that the dislocation velocity must be restricted such that $R(\lambda_n)>0$, i.e., $p_n>0$. This condition gives the true limiting velocity of the dislocation since as $p_n\to 0$, both E_p and $E_k\to\infty$. Also note that at velocities near the limiting velocity the total energy of the edge dislocation varies as $[R(\lambda)]^{-3}$; for the screw dislocation we have from (9) and (16) that the energy varies as $[R(\lambda)]^{-1}$ at high velocities. These results are analogous to those obtained by Weertman[1] for the isotropic case.

The condition that the kinetic and potential energies be equal is

$$\sum_{n,m=1}^{2}\frac{(L_{nm}-\rho c^2F_{nm})\{p_n(p_m^2+q_m^2)+p_m(p_n^2+q_n^2)\}+(M_{nm}-\rho c^2G_{nm})(p_n+p_m)+(N_{nm}-\rho c^2H_{nm})(p_nq_m+p_mq_n)}{p_np_m[(p_n+p_m)^2+(q_n-q_m)^2]}=0. \tag{34}$$

In the isotropic case the velocity at which $E_k=E_p$ is the same as that at which $\sigma_{12}(x_1',0)=0$. It is not obvious in the anisotropic case that the velocity for which (34) is satisfied is the same as that for which (28) obtains.

The results of this section have been general for those cases for which (4) holds. Our main interest in this problem is to find those velocities for which (28) and (34) are satisfied and for which $p_n=0$, i.e., the velocity at which the shear stress on the slip plane vanishes, the velocity at which the kinetic energy becomes equal to the potential energy, and the limiting velocity of the edge dislocation. To do this we require the roots of the quartic (18). Of course these roots can be obtained, but the resulting expressions as well as the solution of (28) and (34) would be rather complicated. However, if we assume that the constants F_{16} and F_{26} also vanish, then the quartic (18) reduces to a quadratic in λ_n^2 and the analysis is greatly simplified.

B. $F_{16}=F_{26}=0$

With this simplification we find from (18) that

$$2\lambda^2=q\pm(q^2-4s)^{\frac{1}{2}}=q\pm\Delta^{\frac{1}{2}}, \tag{35}$$

where

$$\begin{aligned}F_{22}F_{66}q&=(F_{11}F_{22}-F_{12}^2-2F_{12}F_{66})-(F_{22}+F_{66})\rho c^2,\\ F_{22}F_{66}s&=(F_{11}-\rho c^2)(F_{66}-\rho c^2).\end{aligned} \tag{36}$$

There are two classes of behavior depending on the sign of Δ.

(i) $\Delta>0$: Then λ_n^2 is real;

$$2\lambda_1^2=q+(q^2-4s)^{\frac{1}{2}}\quad 2\lambda_2^2=q-(q^2-4s)^{\frac{1}{2}}. \tag{37}$$

For this case we obtain from (25) that

$$\begin{aligned}A_{1r}&=A_{2r}=0,\\ A_{1i}&=-\frac{p_2^2F_{22}F_{66}+F_{12}(F_{66}-\rho c^2)}{F_{22}(F_{12}+F_{66})(F_{66}-\rho c^2)(p_1^2-p_2^2)},\\ A_{2i}&=\frac{p_1^2F_{22}F_{66}+F_{12}(F_{66}-\rho c^2)}{F_{22}(F_{12}+F_{66})(F_{66}-\rho c^2)(p_1^2-p_2^2)}.\end{aligned} \tag{38}$$

In order that the conditions $R(\lambda_n)>0$ be satisfied we must insure $\lambda_n^2>0$; these demand $q>0$, $s>0$. c_∞, the limiting velocity of the edge dislocation, is given by $\lambda_2=0$, i.e., $s=0$. In other words, ρc_∞^2 is equal to the lesser of (F_{11},F_{66}).

(ii) $\Delta<0$: In this case λ_n is complex;

$$\lambda=\pm\tfrac{1}{2}\{[(4s)^{\frac{1}{2}}+q]^{\frac{1}{2}}\pm i[(4s)^{\frac{1}{2}}-q]^{\frac{1}{2}}\}. \tag{39}$$

From (25) we obtain that

$$\begin{aligned}A_{1i}&=A_{2i}=\frac{F_{66}}{2(F_{66}+F_{12})(F_{66}-\rho c^2)},\\ A_{1r}&=-A_{2r}=\frac{F_{12}(F_{66}-\rho c^2)+F_{22}F_{66}(a^2-b^2)}{4abF_{22}(F_{66}+F_{12})(F_{66}-\rho c^2)},\end{aligned} \tag{40}$$

where

$$p_1=p_2=a,\quad q_2=-q_1=b.$$

We see from (39) that the limiting velocity is now determined from $q=-(4s)^{\frac{1}{2}}$. This equation can be satisfied by values of ρc^2 less than F_{66}.

The limiting velocity can be discussed simply for both cases with the aid of Fig. 1 in which we have a $[q,(4s)^{\frac{1}{2}}]$ plot. In Region I, both q and Δ are positive; in Region II, $\Delta<0$; in Region III, $\Delta>0$, $q<0$. We can show that there is no solution possible in Region III since the conditions $\Delta>0$, $q<0$ violate the elastic stability criterion that $F_{11}F_{22}-F_{12}^2>0$. Therefore at $c=0$ we are situated at some point in Region I or II. As c increases, both s and q decrease; hence we move down and to the left until we come to either $s=0$ or $q=-(4s)^{\frac{1}{2}}$. The hashed line represents the locus of limiting velocities for the dislocation.

To see how this compares with the isotropic case, replace F_{11} and F_{22} by $(\lambda+2\mu)$, F_{12} by λ, and F_{66} by μ in (35) and (36) where (in this paragraph) λ and μ are the Lamé constants. We find that

$$\begin{aligned}q_{\text{iso}}&=2-\frac{(\lambda+3\mu)}{\mu(\lambda+2\mu)}\rho c^2,\quad \Delta_{\text{iso}}=\left[\frac{\rho c^2(\lambda+\mu)}{\mu(\lambda+2\mu)}\right]^2,\\ s_{\text{iso}}&=\left(1-\frac{\rho c^2}{\mu}\right)\left(1-\frac{\rho c^2}{\lambda+2\mu}\right).\end{aligned} \tag{41}$$

Since $\Delta>0$ we are always in Region I of the $[q,(4s)^{\frac{1}{2}}]$ diagram and of course the limiting velocity is always given by $s=0$. Hence one of the novel features introduced by anisotropy is the possibility of limiting velocities lower than those corresponding to the shear wave velocity.

From (28) we obtain that the shear stress on the slip plane vanishes when $\sum \varphi_n(A_{ni},A_{nr})=0$. Using either (38) or (40) (i.e., for either real or complex roots) we find that the threshold velocity for the anomalous behavior of edge dislocations is given as a solution of

an equation analogous to the Rayleigh wave equation,

$$F_{22}F_{66}^2(F_{22}-F_{66})f^6-F_{22}F_{66}[F_{66}(F_{22}-F_{11})+2(F_{11}F_{22}-F_{12}^2)]f^4+(F_{11}F_{22}-F_{12}^2)\times[2F_{22}F_{66}+F_{11}F_{22}-F_{12}^2]f^2-(F_{11}F_{22}-F_{12}^2)^2=0, \quad (42)$$

where

$$f^2=\rho c^2/F_{66}.$$

If $F_{11}>F_{66}$, this equation always has a root for $f<1$, i.e., $\rho c^2<F_{66}$. Hence for the case of real roots (for which the limiting velocity is given by $\rho c_\infty{}^2=F_{66}$) there is always a range of velocities for which two like edge dislocations on the same slip plane will attract. For the case of complex roots there may or may not be such a velocity range; if the velocity which satisfies $q=-(4s)^{\frac{1}{2}}$ is less than that given by (42) then there is no anomalous behavior, i.e., the edge dislocation is "well-behaved" at all velocities up to the limiting velocity.

It should be noted that it is possible for f to take on any value from zero to unity, i.e., the threshold velocity can have any value from zero to $(F_{66}/\rho)^{\frac{1}{2}}$, depending on the elastic constants of the material. For example consider those orientations for which $F_{11}=F_{22}$; elastic stability demands $F_{11}^2>F_{12}^2$ but as F_{12}^2 more closely approximates F_{11}^2 we see from (42) that f approaches zero. f is equal to unity when $F_{11}=F_{66}$. This is to be compared with the isotropic case where it is found that the Rayleigh wave velocity can have values only between $0.69c_s$ and $0.96c_s$. This point is discussed again in the next section.

V. APPLICATION OF RESULTS

To illustrate the use of the above analysis one example[5] will be worked out in some detail—that of an edge dislocation along [0001] in a hexagonal material. The x_i axes are now the crystal axes so that the F_{ij} can be replaced by c_{ij}; also the plane of plane strain is the basal plane (which is essentially isotropic) so that a direct comparison can be made with Weertman's results for the isotropic case.

From (35) we obtain that

$$\Delta=[\rho c^2(c_{11}+c_{12})/c_{11}(c_{11}-c_{12})]^2>0,$$

i.e., we are always in Region I of the $[q,(4s)^{\frac{1}{2}}]$ diagram as in the isotropic case. The expressions for the (real) roots $\lambda_n{}^2$ are

$$\lambda_1{}^2=p_1{}^2=1-\frac{\rho c^2}{c_{66}}, \quad \lambda_2{}^2=p_2{}^2=1-\frac{\rho c^2}{c_{11}}, \qquad (43)$$

where $c_{66}=\frac{1}{2}(c_{11}-c_{12})<c_{11}$. Hence we have immediately that the limiting velocity of the edge dislocation is given by $\rho c_\infty{}^2=c_{66}$.

We obtain from (38) that

$$A_{1i}=\frac{2(1+p_1{}^2)}{p_1{}^2(1-p_1{}^2)(c_{11}+c_{12})},\qquad A_{2i}=\frac{-4}{(1-p_1{}^2)(c_{11}+c_{12})}. \qquad (44)$$

All the formulas of Sec. IV. A can now be evaluated.

The displacement field of the moving edge dislocation is given by (22) as

$$U_1=\frac{b_0}{2\pi(1-p_1{}^2)}\left[-(1+p_1{}^2)\tan^{-1}\left(\frac{p_1x_2}{x_1'}\right)+2\tan^{-1}\left(\frac{p_2x_2}{x_1'}\right)\right],$$
$$U_2=\frac{b_0}{2\pi(1-p_1{}^2)}\left[-\frac{(1+p_1{}^2)}{2p_1}\ln(x_1'^2+p_1{}^2x_2{}^2)+p_2\ln(x_1'^2+p_2{}^2x_2{}^2)\right]. \qquad (45)$$

Likewise from (26) we obtain that the stress field of the moving edge dislocation is given by

$$\sigma_{11}=\frac{b_0x_2}{2\pi(1-p_1{}^2)}\left[\frac{(c_{11}-c_{12})(1+p_1{}^2)p_1}{x_1'^2+p_1{}^2x_2{}^2}-\frac{2p_2(c_{11}-p_2{}^2c_{12})}{x_1'^2+p_2{}^2x_2{}^2}\right],$$
$$\sigma_{12}=\frac{(c_{11}-c_{12})b_0x_1'}{4\pi p_1(1-p_1{}^2)}\left[-\frac{(1+p_1{}^2)^2}{x_1'^2+p_1{}^2x_2{}^2}+\frac{4p_1p_2}{x_1'^2+p_2{}^2x_2{}^2}\right],$$
$$\sigma_{22}=\frac{b_0x_2}{2\pi(1-p_1{}^2)}\left[-\frac{(c_{11}-c_{12})(1+p_1{}^2)p_1}{x_1'^2+p_1{}^2x_2{}^2}+\frac{2p_2(c_{11}p_2{}^2-c_{12})}{x_1'^2+p_2{}^2x_2{}^2}\right],$$
$$\sigma_{33}=-\frac{c_{13}(c_{11}-c_{12})b_0p_2x_2}{2\pi c_{11}(x_1'^2+p_2{}^2x_2{}^2)}, \quad \sigma_{13}=\sigma_{23}=0. \qquad (46)$$

[5] It is planned to treat in a later paper a number of cases for which the above plane strain analysis applies and to give numerical calculations for various materials.

The potential energy per unit length is derived from (31), using (27) and (29). We find that

$$E_p=\frac{(c_{11}-c_{12})b_0^2}{8\pi(1-p_1^2)}\ln\frac{R_2}{R_1}\left[\frac{(1+p_1^2)(1-8p_1^2-p_1^4)}{2p_1^3}+\frac{2}{p_2}(1+3p_2^2)\right]. \tag{47}$$

Likewise from (32), using (33), we obtain that the kinetic energy per unit length of the moving edge dislocation is

$$E_k=\frac{(c_{11}-c_{12})b_0^2}{8\pi(1-p_1^2)}\ln\frac{R_2}{R_1}\left[\frac{(1+p_1^2)(1-6p_1^2+p_1^4)}{2p_1^3}+\frac{2}{p_2}(1+p_2^2)\right]. \tag{48}$$

Finally we can write the total energy $E_t=E_k+E_p$ as

$$E_t=\frac{(c_{11}-c_{12})b_0^2}{8\pi(1-p_1^2)}\ln\frac{R_2}{R_1}\left[\frac{(1+p_1^2)(1-7p_1^2)}{p_1^3}+\frac{4}{p_2}(1+2p_2^2)\right]. \tag{49}$$

Note that for a moving edge dislocation the total energy is not given by a simple relativistic formula as is the case for a screw dislocation.

All these results are the same as those found by Weertman for the isotropic case if c_{66} is replaced by the shear modulus and c_{11} by the compressional modulus.

The kinetic and potential energies become equal at $4p_1p_2=(1+p_1^2)^2$; at the velocity for which this is satisfied the shear stress on the slip plane vanishes. This threshold velocity is obtainable from (42) which now reads

$$f^6-8f^4+8(3-2h)f^2-16(1-h)=0. \tag{50}$$

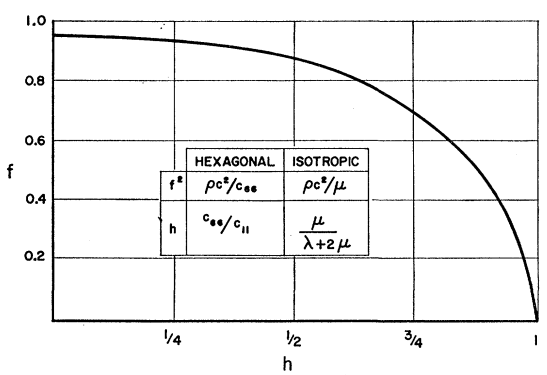

FIG. 2. The curve gives the desired root of the Rayleigh wave equation (50) for both the isotropic and hexagonal cases. For isotropic materials $0<h<\frac{3}{4}$; for the hexagonal case considered in the text $0<h<1$.

This of course is the Rayleigh wave equation and applies to both the isotropic and hexagonal (c-axis edge dislocation) cases with the following definitions:

	Hexagonal (c axis $\perp$)	Isotropic	
f^2	$\rho c^2/c_{66}$	$\rho c^2/\mu$	(51)
h	c_{66}/c_{11}	$\mu/(\lambda+2\mu)=(1-2\nu)/2(1-\nu)$,	

where λ, μ are the Lamé constants and ν is Poisson's ratio. In Fig. 2 we have plotted f vs h, where f represents the threshold velocity normalized to the corresponding limiting velocity, $(\mu/\rho)^{\frac{1}{2}}$ or $(c_{66}/\rho)^{\frac{1}{2}}$.

The analogy between the isotropic and hexagonal cases is complete except for the permissible values of h. For an isotropic material the elastic stability criteria demand that Poisson's ratio take on values only in the range $-1<\nu<\frac{1}{2}$. This in turn implies $0<h<\frac{3}{4}$ and hence $0.69<f<0.96$, i.e., the Rayleigh wave velocity is never less than 0.69 times the limiting velocity.

For a hexagonal crystal the elastic stability criteria demand $-c_{11}<c_{12}<c_{11}$ ($c_{11}>0$). This implies $0<h<1$ and hence $0<f<0.96$, i.e., the threshold velocity can take on any value down to zero. This is a direct result of the fact that c_{12} is allowed negative values down to $-c_{11}$. In other words, the threshold velocity for the anomalous behavior of a c-axis edge dislocation in a hexagonal material will be lowest (relative to the limiting velocity) in that material for which the ratio c_{12}/c_{11} most nearly approximates -1.