to 0. This gives the result

$$J_e = X_p q N_c N_A S_n \sigma_A \exp[-(E_{vc} - q\phi_A)/kT] \times [\exp(qV/kT],$$
$$J_p = X_n q N_v N_D S_p \sigma_D \exp[-(E_{vc} - q\phi_D)/kT] \times [\exp(qV/kT]. \quad (9)$$

In the last equation the roles of electrons and holes and of acceptors and donors have been permuted.

The similarity between Eq. (9) and Eq. (3) is apparent. The drift length has been replaced by the portion of the transition region where recombination occurs, and the minority carrier lifetime by a formally identical term in which the density of active recombination centers in the bulk has been replaced by their effective density in the transition region. Without a more rigorous analysis of the quantity N_D, we replace Eq. (9) by the approximate expression for electron flow into a p-type base.

$$J_j = A_j \exp[-(E_{vc} - q\phi_A - qV)/kT], \quad (4)$$
$$A_j \doteq qN_c(X/\tau),$$

with the identification that J_j is the thermal current recombining in the junction, X the distance in the junction (from the chemical junction to the Fermi level) in which recombination is occurring, and τ is the ordinary minority carrier lifetime, which determines the drift length in the field-free region. This form permits comparison of the relative importance of minority carrier injection and junction recombination, under the condition that the recombination centers lie close to the band edge. We see that for degenerate materials, junction recombination is always favored at sufficiently low temperature.

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Relaxation Equations for Two-Magnon and Magnon-Phonon Processes in Ferrimagnetic Resonance

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Equations which give the relaxation behavior of the magnetization in the case of ferrimagnetic resonance are derived from quantum mechanical rate equations for spin-wave magnons. It is shown that the concept of a unique relaxation time for a particular component of magnetization is not in general valid.

INTRODUCTION

In a previous paper[1] the relationship of linewidths to fundamental transition parameters for two-magnon and magnon-phonon processes in ferrimagnetic resonance was calculated by a method using quantum mechanical rate equations for spin-wave magnons, without recourse to a phenomenological equation of motion. This method allows one to take the fundamental two-quantum transitions into account directly instead of lumping all loss processes into an extra term in the equation of motion without regard to the processes involved. In addition, the validity of deriving relaxation times in analogy with paramagnetic resonance was discussed and it was pointed out that the paramagnetic resonance analogy is not good and care must be exercised in its use. The advent of experimental apparatus allowing the measurement of times in the mμsec range makes it feasible to do direct relaxation experiments on ferrites. It is therefore desirable to investigate more fully the relaxation behavior of the magnetization for the ferrimagnetic resonance case in order to learn what additional information may be obtained from relaxation experiments.

In a typical experiment designed to directly measure relaxation times, one may conveniently measure the behavior of the component of magnetization parallel to the dc magnetic field and the component of magnetization perpendicular to the field. We will calculate expressions for these components of the magnetization from the quantum mechanical rate equations previously derived.[1,2]

GENERAL RELAXATION EQUATIONS

The number of spin-waves present in an ellipsoid of revolution about the dc magnetic field is given by

$$M = M_0 - \sum_k \gamma\hbar n_k, \quad (1)$$

$$M_z = M - \gamma\hbar n_0, \quad (2)$$

where M_0 is the saturation magnetization, n_0 is the number of uniform precession magnons ($k=0$), and n_k is the number of spin-waves of wave number k where $k\neq 0$. From these two equations we may write

$$M_z - M_0 = -\gamma\hbar[n_0 + \sum_k n_k], \quad (3)$$

which gives the behavior of the longitudinal component

[1] P. E. Seiden, J. Phys. Chem. Solids 17, 259 (1961).

[2] H. B. Callen, J. Phys. Chem. Solids 4, 256 (1958).

of magnetization. The transverse component is obtained from

$$|M_+|^2 + M_z^2 = M^2,$$

where M_+ is the circularly polarized component of magnetization $(M_x + iM_y)$. Eliminating M and M_z by Eqs. (1) and (2), we have

$$|M_+|^2 = \gamma\hbar n_0[2M_0 - \gamma\hbar(n_0 + 2\sum_k n_k)]. \tag{4}$$

The spin-wave occupation numbers n_0 and n_k can be obtained by solving the rate equations for the occupation numbers. These equations have been previously derived[1,2] by calculating the transition probabilities from the Hamiltonian in the creation and destruction operator representation. The rate equations derived in reference 1 by this means are[3]

$$\dot{n}_0 = -\lambda_{0\sigma} n_0 - \sum_k \lambda_{0k}(n_0 - n_k), \tag{5}$$

$$\dot{n}_k = \lambda_{0k}(n_0 - n_k) - \lambda_{k\sigma} n_k, \tag{6}$$

where $\lambda_{0\sigma}$, λ_{0k}, and $\lambda_{k\sigma}$ are the parameters expressing the uniform precession–lattice, uniform precession–spin-wave, and spin-wave–lattice transitions, respectively.

These equations may be written in matrix form as follows:

$$\dot{N} = \Lambda N,$$

where N is the column matrix

$$\begin{bmatrix} n_0 \\ \vdots \\ n_k \\ \vdots \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -\lambda_{0\sigma} - \sum_k \lambda_{0k} & \cdots & \lambda_{0k} & \cdots \\ \vdots & \ddots & 0 & \\ \lambda_{0k} & & -\lambda_{0k} - \lambda_{k\sigma} & \\ \vdots & 0 & & \ddots \end{bmatrix}. \tag{7}$$

This equation may be integrated to obtain

$$N = e^{\Lambda t} N^{(0)}, \tag{8}$$

where $N^{(0)}$ is the column matrix for the magnon occupation numbers at $t=0$. In order to solve Eq. (8) for n_0 and n_k as explicit functions of the $N^{(0)}$, we must diagonalize the matrix Λ and obtain N by a similarity transformation from the eigenvectors of the matrix.

[3] These equations have been derived under the following two assumptions: A. We consider only linear processes; i.e., terms in the rate equations that are linear in spin-wave occupation numbers. B. The number of thermal phonons, n_σ, is neglected in comparison with n_0 and n_k because we assume that the sample is thermostated by the surroundings and therefore n_σ is a constant and can be accounted for by using the measured saturation magnetization at the temperature under consideration.

RELAXATION EQUATIONS FOR SMALL SPIN-WAVE AMPLITUDES

The first case we will treat is that of small spin-wave amplitudes, $n_k \ll n_0$. In this case Eq. (5) simplifies to

$$\dot{n}_0 + (\lambda_{0\sigma} + \sum_k \lambda_{0k}) n_0 = 0. \tag{9}$$

Equation (9) can be solved in a straightforward manner to obtain

$$n_0 = n_0^{(0)} \exp[-(\lambda_{0\sigma} + \sum_k \lambda_{0k})t]. \tag{10}$$

If we are to obtain some new information from relaxation measurements we must measure a quantity whose functional dependence on the transition parameters is different from that of the linewidth, a quantity which we also measure. To make the appearance of other functional dependences obvious we will substitute the linewidth into the relaxation equations obtained here. From reference 1 we find

$$\gamma\Delta H = \lambda_{0\sigma} + \sum_k \lambda_{0k}.$$

and also

$$n_k^{(0)} = n_0^{(0)} \lambda_{0k}/\lambda_{k\sigma},$$

so that Eq. (10) becomes

$$n_0 = n_0^{(0)} \exp(-\gamma\Delta H t),$$

and from Eq. (6)

$$n_k = n_0^{(0)} \lambda_{0k} \lambda_{k\sigma}^{-1} [\exp(-\gamma\Delta H t) - \gamma\Delta H \lambda_{k\sigma}^{-1} \exp(-\lambda_{k\sigma} t)].$$

Substitution into Eqs. (3) and (4) yields

$$M_z - M_0 = -\gamma\hbar n_0^{(0)} [(1 + \sum_k \lambda_{0k}\lambda_{k\sigma}^{-1} \exp(-\gamma\Delta H t) - \gamma\Delta H \sum_k \lambda_{0k}\lambda_{k\sigma}^{-2} \exp(-\lambda_{k\sigma} t)] \tag{11}$$

$$|M_+|^2 = 2\gamma\hbar M_0 n_0^{(0)} \exp(-\gamma\Delta H t). \tag{12}$$

One can in this case define a transverse relaxation time $2(\gamma\Delta H)^{-1}$, and from Eq. (11) the longitudinal component of the magnetization will be proportional to $(\gamma\Delta H)^{-1}$ unless the summation over k results in making $\sum_k \lambda_{0k}/\lambda_{k\sigma}$ large even though $\lambda_{k\sigma} \gg \lambda_{0k}$. In reference 1 it is shown that the condition of small spin-wave amplitudes implies $\lambda_{k\sigma} \gg \lambda_{0k}$; therefore, we will have other terms in the relaxation equation in addition to that of $\exp(-\gamma\Delta H t)$ if $\sum_k n_k$ is not negligible compared to n_0 even though $n_k \ll n_0$.

RELAXATION EQUATIONS FOR LARGE SPIN-WAVE AMPLITUDES

We will now consider a system where we allow one spin-wave to have a non-negligible amplitude[4] with respect to the number of uniform precession magnons. We designate this spin-wave by i and rewrite the

[4] We will not discuss here the conditions under which the finiteness of the $k \neq 0$ spin-wave amplitudes become important. This problem has been adequately discussed in reference 1 and in H. B. Callen, J. Appl. Phys. **32**, 738 (1961).

matrix, (7), as

$$\Lambda=\begin{bmatrix} -\lambda_{0\sigma}-\sum_k \lambda_{0k} & \cdots & \lambda_{0i} & \cdots & \lambda_{0k'} & \cdots \\ \vdots & \ddots & & & & \\ \lambda_{0i} & & -\lambda_{0i}-\lambda_{i\sigma} & & 0 & \\ \vdots & & & \ddots & & \\ \lambda_{0k'} & & & & -\lambda_{0k'}-\lambda_{k'\sigma} & \\ \vdots & & 0 & & & \ddots \end{bmatrix},$$

where k' refers to all k except $k=i$. Since $\lambda_{0k'}\ll\lambda_{k'\sigma}$ we will treat all the $\lambda_{0k'}$'s as a perturbation. Diagonalizing the 2×2 matrix for $k=0$ and $k=i$, we find the eigenvalues

$$-(A-B),\quad -(A+B),$$

where

$$A=\tfrac{1}{2}(\lambda_{0\sigma}+\lambda_{0i}+\lambda_{i\sigma}+\textstyle\sum_k \lambda_{0k}),$$

$$B=\tfrac{1}{2}(\lambda_{0\sigma}-\lambda_{0i}-\lambda_{i\sigma}+\textstyle\sum_k \lambda_{0k})[1+(2\lambda_{0i})^2(\lambda_{0\sigma}-\lambda_{0i}-\lambda_{i\sigma}+\sum_k \lambda_{0k})^{-2}]^{\frac{1}{2}}.$$

The diagonalized matrix is then

$$\lambda=\begin{bmatrix} -(A-B) & & & & \\ & \ddots & & 0 & \\ & & -(A+B) & & \\ & 0 & & \ddots & \\ & & & & -\lambda_{k'\sigma} \\ & & & & & \ddots \end{bmatrix},$$

where we have neglected terms of order $(\lambda_{0k'})^2$ and the solutions for the magnon occupation numbers are

$$n_0=n_0^{(0)}(2B)^{-1}\{[(A+B)-\gamma\Delta H]\exp[-(A-B)t] -[(A-B)-\gamma\Delta H]\exp[-(A+B)t]\},$$

$$n_i=n_0^{(0)}(2B)^{-1}\lambda_{0i}(\lambda_{0i}+\lambda_{i\sigma})^{-1}\{(A+B)\exp[-(A-B)t] -(A-B)\exp[-(A+B)t]\},$$

$$n_{k'}=n_0^{(0)}(2B)^{-1}\{\lambda_{0k'}[(A+B)-\gamma\Delta H][\lambda_{k'\sigma}-(A-B)]^{-1} \times\exp[-(A-B)t]-\lambda_{0k'}[(A-B)-\gamma\Delta H] \times[\lambda_{k'\sigma}-(A+B)]^{-1}\exp[-(A+B)t] +BG_{k'}\exp(-\lambda_{k'\sigma}t)\},$$

which result in the following relaxation equations:

$$M_z-M_0=-\gamma\hbar n_0^{(0)}(2B)^{-1}\{C(1+E)\exp[-(A-B)t] -D(1+F)\exp[-(A+B)t] +2B\textstyle\sum_{k'} G_{k'}\exp(-\lambda_{k'\sigma}t)\},\quad (13)$$

$$|M_+|^2=\gamma\hbar M_0B^{-1}n_0^{(0)}\{C\exp[-(A-B)t] -D\exp[-(A+B)t]\}-[\gamma\hbar n_0^{(0)}]^2(2B)^{-2} \times\{C^2(1+2E)\exp[-2(A-B)t] -D^2(1+2F)\exp[-2(A+B)t] +2CD(E-F)\exp(-2Bt)+4CB\textstyle\sum_{k'}G_{k'} \times\exp[-(\lambda_{k'\sigma}+A-B)t]+4CD\sum_{k'} G_{k'} \times\exp[-(\lambda_{k'\sigma}+A+B)t]\},\quad (14)$$

where

$$C=(A+B)-\gamma\Delta H,$$

$$D=(A-B)-\gamma\Delta H,$$

$$E=(A+B)C^{-1}\lambda_{0i}(\lambda_{0i}+\lambda_{i\sigma})^{-1} +\textstyle\sum_{k'}\lambda_{0k'}[\lambda_{k'\sigma}-(A-B)]^{-1},$$

$$F=(A-B)D^{-1}\lambda_{0i}(\lambda_{0i}+\lambda_{i\sigma})^{-1} +\textstyle\sum_{k'}\lambda_{0k'}[\lambda_{k'\sigma}-(A+B)]^{-1},$$

$$G_{k'}=\lambda_{0k'}\lambda_{k'\sigma}^{-1}\{1-[2A-\lambda_{k'\sigma}-\gamma\Delta H] \times[2A-\lambda_{k'\sigma}-\gamma\Delta H\lambda_{0i}^{-1}(\lambda_{0i}+\lambda_{i\sigma})]^{-1}\},$$

$$\gamma\Delta H=\lambda_{0\sigma}+\textstyle\sum_k \lambda_{0k}-\lambda_{0i}^2(\lambda_{0i}+\lambda_{i\sigma})^{-1}.$$

From the two relaxation equations it can be seen that there is no unique relaxation time for either component and the decay of the magnetization is a sum of many exponentials. In general, without making any assumptions about the relative magnitude of the transition parameters or about the summations over them, it is not possible to say which term is dominant or how many terms contribute to the relaxation. If we let $\lambda_{0i}\ll\lambda_{i\sigma}$ then Eqs. (13) and (14) reduce to (11) and (12) as expected; if in addition $\sum_k \lambda_{0k}/\lambda_{k\sigma}\ll1$ we will have a single exponential with the relaxation time of $(\gamma\Delta H)^{-1}$. This is the only approximation that leads to a single exponential and thereby a unique relaxation time; therefore we should expect to find deviations from a single exponential in any material where there is strong uniform precession–spin-wave scattering.

If we now wish to proceed to the general case, the diagonalization of Eq. (7), we see immediately that the problem is not tractable analytically. However, we can establish the form of the solution from the form of the matrix as being

$$M_z-M_0=\textstyle\sum_k A_k\exp(-\lambda_k t),$$

$$|M_+|^2=\textstyle\sum_k B_k\exp(-\lambda_k t)+\sum_{jk} C_k\exp[-(\lambda_j+\lambda_k)t],$$

where the λ's are the eigenvalues of the matrix and A_k, B_k, and C_k are functions of the elements of the matrix. Therefore in general (M_0-M_z) is the sum of K exponentials, where K is the total number of spin-waves (including the uniform precession), and $|M_+|^2$ is the sum of $K(K+3)/2$ exponentials.

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