Dispersion Relations for Production Amplitudes. I*

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Problems of spectral representation are investigated for the processes where a nucleon collides with a boson resulting in a single nucleon and two bosons. The kinematical system proposed by Polkinghorne, Kibble, and Logunov is considerably simplified and Lorentz-generalized. For the double Compton effect, an alternative proof is given for the dispersion relation originally derived by Logunov, Bilenkiz, and Tavkhelidze. Also for the double Compton effect, two other single-variable dispersion relations are shown to be valid.

I. INTRODUCTION

It was Polkinghorne who first heuristically proposed and discussed dispersion relations for production reactions (two particles in and three or more particles out).[1] Kibble later considered in more detail dispersion relations for the process of pion production in pion-nucleon collisions, $\pi+N\rightarrow\pi+\pi+N$.[2] He made an effort to justify his representation in a way similar to Bogoliubov's proof of the dispersion relation for scattering processes,[3] but he succeeded only in reducing the whole problem to a theorem which has not yet been proved.

Logunov and other authors have published a series of papers in which they discuss single-variable dispersion relations for the following six processes[4–6]:

$$\gamma+N\rightarrow\gamma+\gamma+N, \tag{1a}$$

$$\gamma+N\rightarrow\gamma+\pi+N, \tag{1b}$$

$$\gamma+N\rightarrow\pi+\pi+N, \tag{1c}$$

$$\pi+N\rightarrow\gamma+\gamma+N, \tag{1d}$$

$$\pi+N\rightarrow\gamma+\pi+N, \tag{1e}$$

$$\pi+N\rightarrow\pi+\pi+N. \tag{1f}$$

In one of their papers, they presented a rigorous proof of dispersion relations for the double Compton effect (1a).[6] For pion production (1f), their result does not go beyond Kibble's work.

The above six reactions constitute the class of production processes where a nucleon collides with a boson resulting in a single nucleon and two bosons, the bosons being photons and pions. For many obvious reasons these processes are of considerable interest. Throughout the present discussion of "production processes," we shall restrict ourselves to the limited class of reactions in Eqs. (1).

In any discussion of dispersion relations it is, of course, necessary to specify a choice of independent dynamical variables, one of which is to play the role of dispersion variable while the others are held fixed and physical. For the reactions under consideration the number of independent variables is five. In the various works on production reactions cited above, the authors were in each case led to consider a certain particular choice of dispersion and fixed variables. Despite the heuristic argument which made these variables appear favorable, it has turned out that one can produce, in perturbation theory, counter-examples to the dispersion relations which were conjectured for reaction (1f) and also, as one can show, for reaction (1c).[7]

To our knowledge, no essentially different set of variables, which look promising from the standpoint of leading to dispersion relations for these reactions, has yet been suggested on heuristic or other grounds. On the other hand, with the original variables it has been possible to obtain a rigorous proof of dispersion relations for reaction (1a), in the approximation of treating electromagnetic interactions to lowest order. For the remaining reactions (1b), (1d), (1e)—again treated to lowest electromagnetic order and with the same variable—a rigorous proof of dispersion relations has not yet been found. But the outlook, in perturbation theory, is favorable. We shall take this up in a subsequent paper.

Partly in preparation for this, and partly for its intrinsic interest, we shall discuss here an alternative proof of the dispersion relation already obtained by Logunov *et al.* for the double Compton effect (1a). The point is that the present treatment seems to be somewhat simpler than the earlier one; the Logunov variables are set up in a more transparent form and Lorentz-generalized; and most important, we note here that one can produce, in fact, several different sets of Logunov-like variables and thus obtain *new* dispersion relations for the double Compton effect.

In Sec. II of the present paper, the kinematics is formulated in a specific Lorentz frame where the variables which are to be chosen take on a simple meaning.

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* This paper is based on part of a thesis submitted by the author to Princeton University in partial fulfillment of the requirements for a Ph.D. degree.

[1] J. C. Polkinghorne, Nuovo cimento **4**, 216 (1956).

[2] T. W. B. Kibble, Proc. Roy. Soc. (London) **A244**, 355 (1958).

[3] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[4] A. Logunov and I. Todorov, Nuclear Phys. **10**, 552 (1958).

[5] A. Logunov and A. Tavkhelidze, Nuovo cimento **10**, 943 (1958).

[6] A. Logunov, S. Bilenkiz, and A. Tavkhelidze, Nuovo cimento **10**, 958 (1959).

[7] Y. S. Kim, Phys. Rev. Letters **6**, 313 (1961). There are numerical errors in this letter. The corrected and generalized version is given in Appendix A of the present paper.

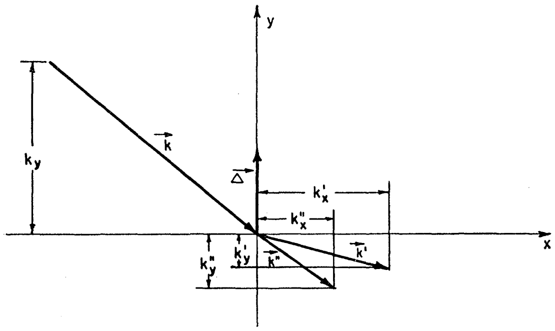

FIG. 1. The xy plane of the kinematical system introduced in Sec. II.

In Appendix A, the variables are Lorentz-generalized. In Sec. III, reduction formulas are set up, and heuristic considerations are presented for the dispersion relations. In Secs. IV and V, the dispersion relations are rigorously derived for the double Compton effect and shown to be equivalent to the earlier one. In Sec. VI, a new dispersion relation is derived by the introduction of a new set of Logunov-type variables. Throughout the entire work, spins and isotopic spins are ignored.

II. CONSTRUCTION OF THE KINEMATICAL SYSTEM

Since spins and isotopic spins are ignored, a particle involved in the production process is completely specified by its four-momentum. Let p and p' denote, respectively, the four-momenta of the initial and final nucleons; k, that of the initial boson; and k' and k'', those of the final bosons. The boson masses are defined by

$$-k'^2=\mu_1^2, \quad -k''^2=\mu_2^2, \quad -k^2=\mu_0^2, \tag{2}$$

where for the reactions under consideration, $\mu_1=0$ or μ, the latter being the pionic mass. We denote the nucleonic mass by m.

We proceed to a selection of dynamical variables in the following way. Choose, for convenience, the Lorentz frame where

$$p_0=p_0', \tag{3}$$

$$(\mathbf{p}+\mathbf{p}')\cdot\mathbf{k}'=0, \tag{4}$$

$$(\mathbf{p}+\mathbf{p}')\cdot\mathbf{k}''=0. \tag{5}$$

Since the Lorentz transformation has three degrees of freedom, the above restrictions do not reduce generality. On the basis of these, one can formulate a simple kinematics.

Consider two vectors:

$$\mathbf{Q}=\mathbf{p}+\mathbf{p}', \quad \boldsymbol{\Delta}=\mathbf{p}-\mathbf{p}'. \tag{6}$$

Then by Eq. (3),

$$\mathbf{Q}\cdot\boldsymbol{\Delta}=0. \tag{7}$$

Now we construct an orthogonal coordinate system in which the vector $\mathbf{Q}$ is in the z direction and the vector $\boldsymbol{\Delta}$ in the y. The x direction is naturally defined by the direction of $\boldsymbol{\Delta}\times\mathbf{Q}$. Now by Eqs. (4), (5), and (7) and momentum conservation, it follows that the vectors $\mathbf{k}'$, $\mathbf{k}''$, and $\mathbf{k}$ be in the xy plane. See Fig. 1.

It is clear that, in this system, the following five variables form an independent set.

$$p_0,\ k_0,\ \Delta^2=(\mathbf{p}-\mathbf{p}')^2, \tag{8}$$

and

$$t_1=-\mathbf{k}'\cdot(\mathbf{p}-\mathbf{p}'), \quad t_2=-\mathbf{k}''\cdot(\mathbf{p}-\mathbf{p}'). \tag{9}$$

p_0 and k_0 are, respectively, the incoming nucleon and boson energies; Δ is the momentum transfer between the nucleons; $-t_1/\Delta$, and $-t_2/\Delta$ are the y components of the vectors $\mathbf{k}'$ and $\mathbf{k}''$, respectively.

One can easily see that the vectors $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}''$, which are in the xy plane, depend only on the variables k_0, Δ, t_1, and t_2. In the subsequent discussion we shall study analytic properties of the transition amplitude in the variable p_0, fixing the other variables at physical values. For convenience, we denote p_0 by ω and call it the dispersion variable. Before proceeding with the discussion of analyticity, let us add a few remarks on the kinematics.

The independent variables of Eqs. (8) and (9) are Lorentz-generalized in Appendix A and are then redefined to give a final set which is simply related to scalar-product variables. The equivalence to the Logunov variables is also established there. For the proof of dispersion relations, however, it will be convenient for us to work in the Lorentz frame of Eqs. (3)–(5), with $p_0\equiv\omega$ as the dispersion variable and k_0, Δ, t_1, t_2 as the fixed variables. It will be useful for us to note here the forms which certain scalar-product variables take in our special reference frame: For the double Compton effect ($\mu_0=\mu_1=\mu_2=0$) we find

$$\begin{aligned}(p+k)^2&=-m^2-T-2\omega k_0,\\(p'-k)^2&=-m^2-T+2\omega k_0,\\(p'+k')^2&=-m^2+t_1-2\omega k_0',\\(p-k')^2&=-m^2+t_1+2\omega k_0',\\(p'+k'')^2&=-m^2+t_2-2\omega k_0'',\\(p-k'')^2&=-m^2+t_2+2\omega k_0'',\end{aligned} \tag{10}$$

where

$$T=\Delta^2+t_1+t_2.$$

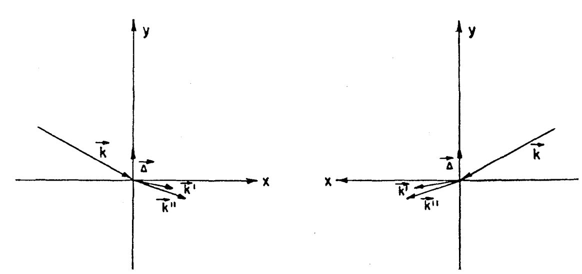

FIG. 2. The two kinematical systems with opposite directions of $\mathbf{e}$. In this figure, however, the unit vector $\mathbf{e}$, which is perpendicular to the xy plane, has the same direction in both systems.

III. REDUCTION FORMULAS

The "S" matrix element for our reaction

$$k+p\rightarrow p'+k'+k''$$

can be written in the standard way.

$$S_{fi}=(2\pi)^4 i\delta(p+k-p'-k'-k'')\times(32k_0'k_0''k_0p_0p_0')^{-\frac{1}{2}}T_R,$$

$$T_R=(16k_0'k_0''k_0p_0')^{\frac{1}{2}}\langle k'k''p'\text{ out}|\bar{f}(0)|k\rangle$$

$$=i(8k_0'k_0''k_0)^{\frac{1}{2}}\int d^4x\exp[-i(p+p')\cdot x/2]\times\langle k'k''\text{ out}|\theta(x_1)[f(x/2),\bar{f}(-x/2)]|k\rangle+R_R,\quad(11)$$

where $f(x)$ is the nucleon current source (we treat the nucleons here as spinless bosons but observe the proper selection rules). R_R is an "equal-time" commutator term which is at most a polynomial in the dispersion variable ω and thus need not concern us insofar as analytic properties of T_R are concerned; we henceforth suppress it.

We now define T_A as

$$T_A=(16p_0'k_0'k_0''k_0)^{\frac{1}{2}}\langle k'k''p'\text{ in}|\bar{f}(0)|k\rangle$$

$$=-i(8k_0'k_0''k_0)^{\frac{1}{2}}\int d^4x\exp[-i(p+p')\cdot x/2]\times\langle k'k''\text{ in}|\theta(-x_0)[f(x/2),\bar{f}(-x/2)]|k\rangle,\quad(12)$$

where again an equal-time commutator term R_A is suppressed.

We then bring out the ω-dependence by writing T_R and T_A as

$$T_R=\int d^4x\exp[i\omega x_0-i\mathbf{e}\cdot\mathbf{x}(\omega^2-\Delta^2/4-m^2)^{\frac{1}{2}}]\times F_R(x;k_0,\Delta,t_1,t_2),$$

$$T_A=\int d^4x\exp[i\omega x_0-i\mathbf{e}\cdot\mathbf{x}(\omega^2-\Delta^2/4-m^2)^{\frac{1}{2}}]\times F_A(x;k_0,\Delta,t_1,t_2),\quad(13)$$

where

$$F_R(x;k_0,\Delta,t_1,t_2)=i(8k_0'k_0''k_0)^{\frac{1}{2}}\langle k'k''\text{ out}|\theta(x_0)[f(x/2)\times\theta(x_0)[f(x/2),\bar{f}(-x/2)]|k\rangle,$$

$$F_A(x;k_0,\Delta,t_1,t_2)=-i(8k_0'k_0''k_0)^{\frac{1}{2}}\langle k'k''\text{ in}|\times\theta(-x_0)[f(x/2),\bar{f}(-x/2)]|k\rangle;$$

and $\mathbf{e}$ is a unit vector in the direction of $(\mathbf{p}+\mathbf{p}')$. Space inversion invariance guarantees that $T(\mathbf{e})=T(-\mathbf{e})$, so we henceforth take

$$T=\tfrac{1}{2}[T(\mathbf{e})+T(-\mathbf{e})].\quad(14)$$

The ω dependence of our amplitudes is now explicitly displayed in the exponentials, the quantities F_R and F_A depending only on x and on the fixed variables. Local commutatively tells us that

$$F_R(x;k_0,\Delta,t_1,t_2)=0\quad\text{unless }x\in L^{(+)},$$
$$F_A(x;k_0,\Delta,t_1,t_2)=0\quad\text{unless }x\in L^{(-)},\quad(15)$$

where $L^{(+)}$ and $L^{(-)}$ are, respectively, the forward and backward light cones.

The standard heuristic procedure is to conjecture that T_R is analytic in the upper, T_A in the lower half ω plane. If one supposes, furthermore, that $T_R=T_A$ on some finite interval along the real ω axis, one obtains the dispersion relation:

$$T_R(\omega,k_0,\Delta,t_1,t_2)=\frac{1}{2\pi i}\lim_{\epsilon\rightarrow0^+}\int_{-\infty}^{\infty}d\omega'\frac{T_R(\omega')-T_A(\omega')}{\omega'-\omega-i\epsilon},\quad(16)$$

where we are neglecting the possible need for subtractions.

In the physical region, $(T_R-T_A)=2i\,\mathrm{Im}\,T_R$ since there $T_R=T_A{}^*$. As in the case of nonforward scattering, the contour of the dispersion integral, in general, runs over some unphysical region in which the integrand has no proper meaning. One may hope to give a rigorous proof of the dispersion relation by making an analytic continuation in m^2, the standard procedure for the ordinary scattering. For the process of pion production (1f), Kibble[2] and Logunov *et al.*[4] attempted to justify their conjectures by a method similar to the one just mentioned, but neither of them succeeded in carrying out the proof. On the contrary, it has been shown from a concrete counter-example in perturbation theory that the conjectured dispersion relation is not valid for processes with a two-pion final state.[7]

But not all the production processes are subject to such a drawback. In the case of the double Compton effect, the fixed variables can be arranged so that the integrand may vanish in the unobservable region of the dispersion variable, if the electromagnetic interaction is restricted to lowest order. In this case, one can justify the spectral representation without making the analytic continuation in the parameter m^2. Let us study the double Compton effect in the following sections.

IV. CONSTRUCTION OF RETARDED AND ADVANCED FUNCTIONS

In the rest of this paper, we shall discuss only the double Compton effect restricting the electromagnetic interaction to lowest order. Furthermore, we shall assume that the quantity $-(k'+k'')^2$ is smaller than $4\mu^2$ so that

$$|k'k''\text{ out}\rangle=|k'k''\text{ in}\rangle=|k'k''\rangle.\quad(17)$$

Our analysis will be similar to the standard procedure followed by Symanzik and Zimmermann in their treatments of the forward scattering.[8] In every standard

[8] K. Symanzik, Phys. Rev. **105**, 743 (1957); W. Zimmermann, Nuovo cimento **13**, 503 (1959).

method, one encounters the intermediate-state expansion and the intermediate-mass spectrum. As a preparatory measure, let us introduce the following polynomials.

$$\begin{aligned} P_1(p_0,\mathbf{p}) &\equiv [(p_0+k_0)^2-(\mathbf{p}+\mathbf{k})^2-m^2] \\ &\quad\times[(p_0-k_0')^2-(\mathbf{p}-\mathbf{k})^2-m^2] \\ &\quad\times[(p_0-k_0'')^2-(\mathbf{p}-\mathbf{k}'')^2-m^2], \\ P_2(p_0',\mathbf{p}') &\equiv [(p_0'-k_0)^2-(\mathbf{p}'-\mathbf{k})^2-m^2] \\ &\quad\times[(p_0'+k_0')^2-(\mathbf{p}'+\mathbf{k}'')^2-m^2] \\ &\quad\times[(p_0'+k_0'')^2-(\mathbf{p}'+\mathbf{k}'')^2-m^2], \end{aligned} \tag{18}$$

and operators:

$$\begin{aligned} \bar{f}_1(-x) &\equiv P_1(p_0,\,-i\nabla_x)\bar{f}(-x), \\ f_2(x) &\equiv P_2(p_0',\,-i\nabla_x)f(x). \end{aligned} \tag{19}$$

Then

$$\begin{aligned} K_R \equiv P_1P_2T_R &= \int d^4x\, \exp[i\omega x_0 - i\mathbf{e}\cdot\mathbf{x}(\omega^2-\omega_p{}^2)^{\frac{1}{2}}] \\ &\quad\times\mathfrak{F}_R(x;\,k_0,\Delta,t_1,t_2), \\ K_A \equiv P_1P_2T_A &= \int d^4x\, \exp[i\omega x_0 - i\mathbf{e}\cdot\mathbf{x}(\omega^2-\omega_p{}^2)^{\frac{1}{2}}] \\ &\quad\times\mathfrak{F}_A(x;k_0,\Delta,t_1,t_2), \end{aligned} \tag{20}$$

where

$$\begin{aligned} &\mathfrak{F}_R(x;\,k_0,\Delta,t_1,t_2) \\ &\qquad = i\langle k'k''|\theta(x_0)[f_2(x/2),\,\bar{f}_1(-x/2)]|k\rangle, \\ &\mathfrak{F}_A(x;\,k_0,\Delta,t_1,t_2) \\ &\qquad = -i\langle k'k''|\theta(-x_0)[f_2(x/2),\,\bar{f}_1(-x/2)]|k\rangle, \\ &\qquad \omega_p{}^2=\Delta^2/4+m^2. \end{aligned}$$

Here again, $\mathfrak{F}_R(\mathfrak{F}_A)$ is local and retarded (advanced). We define also

$$\begin{aligned} &G_R(\mathbf{x};\,\omega,k_0,\Delta,t_1,t_2) \\ &\quad = \frac{\cos[\mathbf{e}\cdot\mathbf{x}(\omega^2-\omega_p{}^2)^{\frac{1}{2}}]}{P(\omega,k_0,\Delta,t_1,t_2)} \int_{|\mathbf{x}|}^{\infty} dx_0\, e^{i\omega x_0} \\ &\qquad\qquad \times\mathfrak{F}_R(x_0,\mathbf{x};\,k_0,\Delta,t_1,t_2), \\ &G_A(\mathbf{x};\,\omega,k_0,\Delta,t_1,t_2) \\ &\quad = \frac{\cos[\mathbf{e}\cdot\mathbf{x}(\omega^2-\omega_p{}^2)^{\frac{1}{2}}]}{P(\omega,k_0,\Delta,t_1,t_2)} \int_{-\infty}^{|\mathbf{x}|} dx_0\, e^{\prime\prime\omega x_0} \\ &\qquad\qquad \times\mathfrak{F}_A(x_0,\mathbf{x};\,k_0,\Delta,t_1,t_2), \end{aligned} \tag{21}$$

where

$$P(\omega,k_0,\Delta,t_1,t_2)=P_1(p_0,\mathbf{p})P_2(p_0',\mathbf{p}').$$

It is clear now that

$$\begin{aligned} T_R &= \tfrac{1}{2}[T_R(\mathbf{e})+T_R(-\mathbf{e})] \\ &= (8k_0'k_0''k_0)^{\frac{1}{2}} \int d^3\mathbf{x}\, G_R(\mathbf{x};\,\cdots), \\ T_A &= \tfrac{1}{2}[T_A(\mathbf{e})+T_A(-\mathbf{e})] \\ &= (8k_0'k_0''k_0)^{\frac{1}{2}} \int d^3\mathbf{x}\, G_A(\mathbf{x};\,\cdots). \end{aligned}$$

$G_R(\mathbf{x};\,\omega,\cdots)$ is analytic in the upper half of the complex ω plane while the same is true for $G_A(\mathbf{x},\cdots)$ in the lower half. Let us see whether one can make an analytic continuation from the lower to the upper half plane for the function $G(\mathbf{x};\,\cdots)$ defined as

$$G(\mathbf{x};\,\omega,\cdots)=\begin{cases} G_R(\mathbf{x};\,\omega,k_0,\Delta,t_1,t_2) & \text{for Im}\,\omega>0, \\ G_A(\mathbf{x};\,\omega,k_0,\Delta,t_1,t_2) & \text{for Im}\,\omega<0. \end{cases} \tag{22}$$

In order to prove that the continuation is possible, we have to show only that there exists an interval on the real axis where

$$G_R=G_A.$$

Aside from the factor $P(\omega,\cdots)$ and the symmetrization of Eq. (14), G_R (G_A) is essentially a form of the K_R (K_A) where the spatial integration is not completed. From Eqs. (20), (K_R-K_A) can be written as

$$\begin{aligned} B\equiv K_R-K_A &= i\int d^4x\, \exp[-i(p+p')\cdot x/2] \\ &\quad\times\langle k'k''|[f_1(x/2),\bar{f}_2(-x/2)]|k\rangle. \end{aligned} \tag{23}$$

Expanding in a complete set of intermediate states and completing the time integration, we obtain

$$\begin{aligned} B &= 2\pi i\int d^3x\, \exp[-i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}/2] \\ &\quad\times\Big[\sum_{n,\alpha}\langle k'k''|f(\mathbf{x}/2)|p_n,\alpha\rangle\langle p_n,\alpha|\bar{f}(-\mathbf{x}/2)|\mathbf{k}\rangle \\ &\quad\times P_1(p_0,\,\mathbf{p}_n-\mathbf{k})P_2(p_0',\,\mathbf{p}_n-\mathbf{k}'-\mathbf{k}'')\delta(p_{n0}-p_0-k_0) \\ &\quad-\sum_{n,\alpha}\langle k'k''|f(-\mathbf{x}/2)|p_n\alpha\rangle\langle p_n,\alpha|f(\mathbf{x}/2)|k\rangle \\ &\quad\times P_1(p_0,\,\mathbf{k}'+\mathbf{k}''-\mathbf{p}_n)P_2(p_0',\,\mathbf{k}-\mathbf{p}_n) \\ &\quad\times\delta(p_{n0}+p_0'-k_0)\Big], \end{aligned} \tag{24}$$

where p_n stands for the intermediate-state momentum and α all other relevant quantum numbers. Except at the points where $P(\omega,\cdots)$ is zero, G_R and G_A will coincide where B vanishes. Let us now find the interval in which B is zero.

According to the selection rules, the lowest mass intermediate state is that of a single nucleon. In lowest order electromagnetic interaction, one can ignore all intermediate photons except the following discrete states.

$$(k'),\ (k''),\ (k),\ (k,k'),\ (k,\,k''),\ (k',k''),\ (k,k',k''),$$

where the photons are represented by their four-momenta.

But it can be shown easily that for the state of a

single nucleon or a nucleon plus any of the above discrete states, either one of the matrix elements vanishes or the polynomial P_1P_2 has the same factor as the argument of the delta function in the expansion (24). The next lowest mass intermediate state is that of one nucleon and one pion. Thus

$$p_{n0}>(m+\mu), \tag{25}$$

that is, if

$$-(m+\mu-k_0)<\omega<(m+\mu-k_0),$$

then

$$G_R(\mathbf{x};\omega,\cdots)=G_A(\mathbf{x};\omega,\cdots),$$

except where $P(\omega,\cdots)=0$.

The polynomial $P(\omega,\cdots)$ vanishes for the following values of ω.

$$\omega_{\pm0}=\pm T/2k_0,\quad \omega_{\pm1}=\pm t_1/2k_0',\quad \omega_{\pm2}=\pm t_2/2k_0''. \tag{26}$$

V. DISPERSION RELATION

It has been shown that the function $G(\mathbf{x};\omega,\cdots)$ defined in Eq. (22) is analytic in the entire ω plane except possibly on the real axis where

$$|\omega|>(m+\mu-k_0),$$

and at the poles of Eq. (35). One can thus perform the standard contour integration and write the expression:

$$G(\mathbf{x};\omega,\cdots)=\frac{1}{2\pi i}\int_{-\infty}^{-(m+\mu-k_0)}d\omega'\frac{G_R(\mathbf{x};\omega',\cdots)-G_A(\mathbf{x};\omega',\cdots)}{\omega'-\omega}+\frac{1}{2\pi i}\int_{(m+\mu-k_0)}^{\infty}d\omega'\frac{G_R(\mathbf{x};\omega',\cdots)-G_A(\mathbf{x};\omega',\cdots)}{\omega'-\omega}$$
$$+\sum_{i=\pm0}^{\pm2}\lim_{\omega'\to\omega_i}\left\{\frac{(\omega'-\omega_i)}{2(\omega-\omega_i)}[G_R(\mathbf{x};\omega',\cdots)-G_A(\mathbf{x};\omega',\cdots)]\right\}. \tag{27}$$

If k_0 and Δ are so small, yet physical, that $(m+\mu-k_0)$ is larger than ω_p, the physical threshold for the variable ω, that is, if

$$\Delta^2<4\mu^2[2m(1-k_0/\mu)/\mu+(1-k_0/\mu)],\quad k_0<\mu, \tag{28}$$

then the integral in Eq. (27) is only along the physical region of the dispersion variable. Thus we can complete the spatial integration and, on the right-hand side, interchange it with the ω' integration to obtain

$$T_R(\omega,\cdots)=\frac{1}{2\pi i}\int_{-\infty}^{-(m+\mu-k_0)}d\omega'\frac{T_R(\omega',\cdots)-T_A(\omega',\cdots)}{\omega'-\omega}+\frac{1}{2\pi i}\int_{(m+\mu-k_0)}^{\infty}d\omega'\frac{T_R(\omega',\cdots)-T_A(\omega',\cdots)}{\omega'-\omega}$$
$$+\sum_{i=\pm0}^{\pm2}\lim_{\omega'\to\omega_i}\left\{\frac{(\omega'-\omega_i)}{2(\omega-\omega_i)}[T_R(\omega',\cdots)-T_A(\omega',\cdots)]\right\},\quad \mathrm{Im}\,\omega>0. \tag{29}$$

The quantity $\lim_{\omega'\to\omega_{2}^-}[T_R(\omega')-T_A(\omega')]$ depends only on the fixed variables, that is, as far as the ω dependence is concerned, it is to be regarded as a constant. The integrand in Eq. (29) vanishes if

$$-\omega_0<\omega<\omega_0$$

where

$$\omega_0=\min\left\{\frac{m\mu}{k_0}\left(1-\frac{T-\mu^2}{2m\mu}\right),\frac{m\mu}{k_0'}\left(1+\frac{t_1+\mu^2}{2m\mu}\right),\frac{m\mu}{k_0''}\left(1+\frac{t_2+\mu^2}{2m\mu}\right)\right\}. \tag{30}$$

Thus we write

$$T_R(\omega,\cdots)=\frac{1}{\pi}\int_{-\infty}^{-\omega_0}d\omega'\frac{\mathrm{Im}\,T_R(\omega',\cdots)}{\omega'-\omega}+\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega'\frac{\mathrm{Im}\,T_R(\omega',\cdots)}{\omega'-\omega}$$
$$+\sum_{i=\pm0}^{\pm2}\lim_{\omega'\to\omega_i}\left\{\frac{(\omega'-\omega_i)}{2(\omega-\omega_i)}[T_R(\omega',\cdots)-T_A(\omega',\cdots)]\right\},\quad \mathrm{Im}\,\omega>0. \tag{31}$$

Now the derivation of a dispersion relation is complete.

In the course of this derivation, the restrictions (28) have been imposed on the fixed variables. It is evident that the second condition automatically implies our earlier assumption, $-(k'+k'')^2<4\mu^2$. These conditions are met if the energy of the incoming photon and the momentum transfer of the nucleon system are sufficiently low.

It is shown in Appendix A that the dispersion variable ω is, in fact, equivalent to the one adopted by Logunov, Bilenkiz, and Tavkhelidze.[6] It can be shown by a simple algebra that the restrictions on the fixed variables, the threshold ω_0, and location of the poles are also equivalent to those in the earlier treatment. Thus, the dispersion relation (31) is another form of what has been obtained in reference 6.

VI. FURTHER REMARKS ON THE DOUBLE COMPTON EFFECT

In formulating the kinematics of the previous section, it was essential for construction of the orthogonal coordinate system that the initial and final nucleons have the same mass. We notice here that the initial and final photons have the same mass. Thus one can choose two other Lorentz frames where the role of the nucleon p is replaced by the boson k and that of the nucleon p' by one of the final bosons.

Let us consider the frame where

$$k_0=k_0',\quad (\mathbf{k}'+\mathbf{k})\cdot\mathbf{p}=0,\quad (\mathbf{k}'+\mathbf{k})\cdot\mathbf{k}''=0, \tag{32}$$

and choose the following five variables.

$$\begin{gathered}\sigma=k_0,\quad p_0,\quad \theta^2=(\mathbf{k}-\mathbf{k}')^2,\\ \tau_1=-\mathbf{p}'\cdot(\mathbf{k}-\mathbf{k}'),\quad \tau_2=-\mathbf{k}''\cdot(\mathbf{k}-\mathbf{k}').\end{gathered} \tag{33}$$

We shall study, in this section, the analytic property of the transition amplitude in the variable σ fixing the others at physical values. In Appendix B, this kinematical system is compared with the one formulated in Sec. II. It is shown that the two systems are not equivalent, that is, the variable σ is not the same as ω in view of the fact that the fixed τ_1 and p_0 are also dependent on the former dispersion variable.

In order to investigate the analytic properties, let us consider the quantity

$$W_R=-i(16k_0'k_0''p_0'p_0)^{\frac{1}{2}}\langle k'k''p'\text{ out}|j(0)|p\rangle, \tag{34}$$

where $j(x)$ is the current operator for the photon field. One obtains this formula after reducing the photon k from the initial state. By a standard algebra,

$$\begin{aligned}W_R&=(8p_0p_0'k_0'')^{\frac{1}{2}}\int d^4x\exp[-i(k+k')\cdot x/2]\\&\quad\times\langle k''p'\text{ out}|\theta(x_0)[j(x/2),\,j(-x/2)]|p\rangle, \quad (35)\\&=i(4p_0p_0'k_0'')^{\frac{1}{2}}\int d^4xd^4y\, e^{-ik'\cdot x-ik''\cdot y}\\&\quad\times\langle p'|\theta(x_0)\theta(y_0-x_0)[j(y),[j(x),j(0)]]\\&\quad+\theta(y_0)\theta(x_0-y_0)[j(x),[j(y),j(0)]]|p\rangle. \quad (36)\end{aligned}$$

We have neglected here terms due to the equal-time commutators.

The first of the above two formulas is to be used in investigating analyticity while the second is for the computation of Im W_R. In the physical region,

$$\begin{aligned}W_A&\equiv W_R^*\\&=\int d^4x\exp[-i(k+k')\cdot x/2]\\&\quad\times\langle k''p'\text{ in}|\theta(-x_0)[j(x/2),j(-x/2)]|p\rangle, \quad (37)\\&=i(4p_0p_0''k_0'')^{\frac{1}{2}}\int d^4xd^4y e^{-ik'\cdot x-ik''\cdot y}\\&\quad\times\langle p'|\theta(-x_0)\theta(x_0-y_0)[j(y),[j(x),j(0)]]\\&\quad+\theta(-y_0)\theta(y_0-x_0)[j(x),[j(y),j(0)]]|p\rangle. \quad (38)\end{aligned}$$

One can now express W_R and W_A in terms of the variables σ, p_0, θ, τ_1, and τ_2, show that the σ-dependence is entirely contained in the exponent of the integrand in Eq. (35) or (37), and construct a retarded (advanced) function after multiplying and dividing W_R (W_A) by the following three polynomials in a way similar to the procedure in the previous discussion.

$$\begin{aligned}h_0(k_0,\mathbf{k})&=[(p_0+k_0)^2-(\mathbf{p}+\mathbf{k})^2-m^2]\\&\quad\times[(p_0'-k_0)^2-(\mathbf{p}'-\mathbf{k})^2-m^2],\\h_1(k_0',\mathbf{k}')&=[(p_0-k_0')^2-(\mathbf{p}-\mathbf{k}')^2-m^2]\\&\quad\times[(p_0'+k_0')^2-(\mathbf{p}'+\mathbf{k}')^2-m^2], \quad (39)\\h_2(k_0'',\mathbf{k}'')&=[(p_0-k_0'')^2-(\mathbf{p}-\mathbf{k}'')^2-m^2]\\&\quad\times[(p_0'+k_0'')^2-(\mathbf{p}'+\mathbf{k}'')^2-m^2].\end{aligned}$$

Then we analyze the quantity $[h_0h_1h_2(W_R-W_A)]$ constructed from Eqs. (36) and (38). Performing the standard contour integration and following the standard procedure for interchanging the order of integration, we arrive at

$$\begin{aligned}W_R(\sigma,\cdots)&=\frac{1}{\pi}\int_{-\infty}^{-\sigma_0}d\sigma'\frac{\text{Im }W_R(\sigma',\cdots)}{\sigma'-\sigma}\\&\quad+\frac{1}{\pi}\int_{\sigma_0}^{\infty}d\sigma'\frac{\text{Im }W_R(\sigma',\cdots)}{\sigma'-\sigma}\\&\quad+\sum_{i=\pm 0}^{\pm 1}\lim_{\sigma'\to\sigma_i}\bigg\{\frac{(\sigma'-\sigma_i)}{2(\sigma-\sigma_i)}\\&\quad\times[W_R(\sigma',\cdots)-W_A(\sigma',\cdots)]\bigg\},\quad \text{Im }\sigma>0, \quad (40)\end{aligned}$$

where

$$\sigma_0=\min\left\{\frac{m\mu}{p_0}\left(1-\frac{\tau_1+\tau_2+\theta-\mu^2}{2m\mu}\right),\frac{m\mu}{p_0'}\left(1+\frac{\tau_1+\mu^2}{2m\mu}\right)\right\},$$

$$\sigma_{\pm 0}=\pm(\theta+\tau_1+\tau_2)/2p_0,\quad \sigma_{\pm 1}=\pm\tau_1/2p_0'$$

with the restriction on the fixed variables:

$$(\theta/2+p_0)<(m+\mu). \tag{41}$$

This completes the derivation of the second dispersion relation.

Next, one can consider another kinematical system where the role of the final boson k' is replaced by the other boson k''. It will be seen that this new system represents a set of independent variables which is not equivalent to either of the former two. Then we can follow the same procedure as before to derive another dispersion relation. Thus there are at least three single-variable dispersion relations.

CONCLUSION

A Lorentz frame has been introduced for the study of production processes. This simplifies the kinematical

formulation of the early works. In this simplified system, a heuristic consideration has been given for the spectral representation of the transition amplitude. Regarding the double Compton effect, it has been shown that the present formalism gives the same conclusion as the one derived by Logunov, Bilenkiz, and Tavkhelidze.[6] It has further been shown that there exist two additional spectral representations.

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APPENDIX A

We have chosen the five independent variables in the specific Lorentz frame defined by the restrictions (3)–(5). Let us now obtain covariant expressions for those variables.

The expressions for t_1, t_2, and Δ can be trivially written.

$$t_1=-k'\cdot(p-p'),\quad t_2=-k''\cdot(p-p'),\qquad \Delta^2=(p-p')^2. \tag{A1}$$

For the others, one finds the following

$$k_0^2=\frac{[R^2-r_1^2-r_2^2]^2-4r_1^2r_2^2}{\beta(2-\beta)(R^2-r_1^2-r_2^2)-(2-\beta)^2r_2^2-\beta^2r_1^2}, \tag{A2}$$

$$\omega=-\tfrac{1}{2}(p+p')\cdot(k'+k'')\times\left[\frac{\beta(2-\beta)(R^2-r_1^2-r_2^2)-(2-\beta)^2r_2^2-\beta^2r_1^2}{(R^2-r_1^2-r_2^2)^2-4r_1^2r_2^2}\right]^{\frac{1}{2}}, \tag{A3}$$

where

$$r_1^2=(t_1/\Delta)^2+\mu_1^2,\quad r_2^2=(t_2/\Delta)^2+\mu_2^2,$$
$$R^2=[(\Delta^2+t_1+t_2)/\Delta]^2+\mu_0^2,$$
$$\beta=1-\frac{(k'-k'')\cdot(p+p')}{(k'+k'')\cdot(p+p')}.$$

Now the five variables have been written covariantly.

From the previous discussion it is clear that we are varying the variable E defined as

$$E=-(p+p')\cdot(k'+k''), \tag{A4}$$

keeping the following fixed.

$$x_1=-k'\cdot(p-p'),\quad x_2=-k'\cdot(p-p'),$$
$$\eta=(k'-k'')\cdot(p+p')/(k'+k'')\cdot(p+p'), \tag{A5}$$
$$v=(p-p')^2.$$

It has been seen that the variable E corresponds to the nucleon energy in the frame where the restrictions (3)–(5) are satisfied.

After a simple algebra one can write E as

$$E=-(p+k)^2-m^2-\mu_0^2-v-x_1-x_2. \tag{A6}$$

It is clear from this expression that the variable E is linearly related to the square of the total energy in the center-of-mass system.

Finally in the Breit system where $\mathbf{p}+\mathbf{p}'=0$, E is, in effect, the energy of the incoming boson. The early authors formulated their kinematics in this system from a different point of view. We shall see in the following that the variable E is the same as what was adopted by Logunov *et al.* In reference 5, one chooses two four-vectors

$$A=\tfrac{1}{2}[\alpha^{-\frac{1}{2}}(1-\xi)k'+\alpha^{\frac{1}{2}}(1+\xi)k''],$$

and

$$B=\tfrac{1}{2}[\alpha^{-\frac{1}{2}}k'-\alpha^{\frac{1}{2}}k''], \tag{A7}$$

where α and ξ are defined, in the Breit system, as

$$\alpha=k_0'/k_0'',\quad \xi=-(1/2B^2)(\alpha^{-1}\mu_1^2-\alpha\mu_2^2). \tag{A8}$$

Hence $A\cdot B=0$ and $B_0=0$ in the same frame. One then chooses the following five independent variables.

$$A_0,\ A^2,\ \mathbf{B}\cdot\mathbf{p},\ \alpha,\ \text{and}\ \mathbf{p}^2,$$

A_0 being the dispersion variable.

These five quantities can be related to the variables E, η, x_1, x_2, and v by:

$$A_0=\frac{1}{4}\left(\frac{1-\eta^2}{m^2+\frac{1}{4}v}\right)^{\frac{1}{2}}E,$$
$$A^2=-\tfrac{1}{4}(1-\xi)^2\left(\frac{1-\eta}{1+\eta}\right)\mu_1^2-\tfrac{1}{4}(1+\xi)^2\left(\frac{1+\eta}{1-\eta}\right)\mu_2^2$$
$$-\tfrac{1}{4}(1-\xi^2)\{2(x_1+x_2)+v+(\mu_0^2-\mu_1^2-\mu_2^2)\}, \tag{A9}$$
$$\mathbf{B}\cdot\mathbf{p}=-\frac{1}{4}\left[\left(\frac{1-\eta}{1+\eta}\right)^{\frac{1}{2}}x_1-\left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}}x_2\right],$$
$$\alpha=(1-\eta)/(1+\eta),$$
$$\mathbf{p}^2=v/4,$$

where

$$\xi=\frac{(1-\eta)^2\mu_1^2-(1+\eta)^2\mu_2^2}{(1-\eta)^2\mu_1^2+(1+\eta)^2\mu_2^2-(1-\eta^2)\{2(x_1+x_2)+r+(\mu_0^2-\mu_1^2-\mu_2^2)\}}.$$

The quantities A^2, $\mathbf{B}\cdot\mathbf{p}$, α, and $\mathbf{p}^2$ depend only on the fixed variables while A_0 is linearly related to E. Thus the two dispersion variables E and A_0 are equivalent.

APPENDIX B

In this appendix, a covariant version of the kinematics of Sec. VI is presented and compared with the kinematics introduced in Appendix A.

It is rather obvious that the new set of variables is represented by the following five quantities.

$$E'=-(k'+k)\cdot(p'+k''),$$
$$\eta'=(p'-k'')\cdot(k+k')/(p'+k'')\cdot(k+k'),$$
$$x_1'=-p'\cdot(k-k'),$$
$$x_2'=-k''\cdot(k-k'),$$
$$v'=(k-k')^2,$$

where E' is the dispersion variable, and η', x_1', x_2', and v' are the fixed variables.

These quantities are related to the former ones by

$$E'=\tfrac{1}{2}(3+\eta)E+x+x_1,$$
$$\eta'=\frac{-\frac{1}{2}(3+\eta)E+(3x_1-x)+2v}{-\frac{1}{2}(3+\eta)E+(3x-x_1)-2v},$$
$$x_1'=\tfrac{1}{2}(1-\eta)E+\tfrac{1}{2}(x_1-x),$$
$$x_2'=x_2+\mu_2^2,$$
$$v'=\tfrac{1}{2}(-\mu_2^2-\mu_1^2-\mu_0^2+2x_2+v),$$

where $x=v+x_1+x_2$. It is seen that the two sets of variables are inequivalent.

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Low-Energy Pion-Photon Interaction: The $(2\pi,2\gamma)$ Vertex*

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In the $(2\pi,2\gamma)$ problem, the Mandelstam representation is written for the two independent gauge-invariant amplitudes. On the basis of unitarity limitations on the asymptotic behavior of these amplitudes, only a $j=1$ subtraction in the $\gamma+\pi\rightarrow\gamma+\pi$ channel and a $j=0$ subtraction in the $\gamma+\gamma\rightarrow\pi+\pi$ channel are allowed. No over-all subtraction constants are required and the Thomson limit is automatically maintained. Only the effect of 2π intermediate states is considered. The odd-j $\pi\pi$ contribution involves the amplitude for the process $\gamma+\pi\rightarrow 2\pi$ analyzed by Wong and shown to be proportional to a pseudo-elementary constant Λ. Even with a $\pi\pi$ P resonance, the correction is negligible ($\lesssim 1\%$) if we use the value of Λ estimated by Wong on the basis of π^0 decay and confirmed by Ball in connection with photopion production on nucleons. A moderately important contribution comes from the S-wave interaction if we use a recent estimate of $\pi\pi$ S-wave phase shifts obtained from crossing relations. For the pion-pion coupling constant λ of order -0.20, this effect is $\sim 10\%$ in $\gamma+\pi\rightarrow\gamma+\pi$ scattering. For $\gamma+\gamma\rightarrow\pi+\pi$, the correction for the $I=0$ state at threshold is positive and $\sim 100\%$ of the Born approximation. However, as the energy is increased, the correction quickly changes sign.

I. INTRODUCTION

In the $(2\pi,2\gamma)$ problem, both strong and electromagnetic interactions are involved. In principle, one can calculate electromagnetic interactions on the basis of perturbation theory. Our purpose here is to understand the effects of strong pion interactions on the $(2\pi,2\gamma)$ vertex.[1]

Attempts have been made in recent years to understand strong pion interactions at low energies by using the Mandelstam representation.[2,3] In particular, a P-wave pion-pion resonance has been conjectured in connection with the nucleon electromagnetic structure.[4] If such a resonance exists, one might expect its effects to be appreciable in Compton scattering on pions (e.g., $\gamma+\pi\rightarrow\gamma+\pi$). One may recall in this connection Compton scattering on protons (e.g., $\gamma+p\rightarrow\gamma+p$), where the 3-3 resonance causes a large increase in the cross section above the value given by the Klein-Nishina-type formula.[5] Pion-pion forces may also be manifested in the final-state interactions of pion pairs produced by photons (e.g., $\gamma+\gamma\rightarrow\pi+\pi$). Such final-state interactions, if they are substantial, may be observed experimentally by producing a pion pair from a high-energy photon in the Coulomb field of a nucleus.

Further, an understanding of the $(2\pi,2\gamma)$ vertex is a prerequisite for a theory of nucleon-photon scattering and, in fact, for most problems where a vertex con-

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[1] A preliminary account of this work was given at the 1960 Winter meeting of the American Physical Society, December 29–31, 1960 [Bipin R. Desai, Bull. Am. Phys. Soc. **5**, 509 (1960)]. We employ units $\hbar=c=\mu=1$, where μ is the pion mass. For the charge e we use the units $e^2\simeq 1/137$. The metric is defined so that we have $g^{0,0}=1$ and $g^{ii}=-1$, where $i=1, 2, 3$.

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