APPENDIX B

In this appendix, a covariant version of the kinematics of Sec. VI is presented and compared with the kinematics introduced in Appendix A.

It is rather obvious that the new set of variables is represented by the following five quantities.

$$E'=-(k'+k)\cdot(p'+k''),$$
$$\eta'=(p'-k'')\cdot(k+k')/(p'+k'')\cdot(k+k'),$$
$$x_1'=-p'\cdot(k-k'),$$
$$x_2'=-k''\cdot(k-k'),$$
$$v'=(k-k')^2,$$

where E' is the dispersion variable, and η', x_1', x_2', and v' are the fixed variables.

These quantities are related to the former ones by

$$E'=\tfrac{1}{2}(3+\eta)E+x+x_1,$$
$$\eta'=\frac{-\frac{1}{2}(3+\eta)E+(3x_1-x)+2v}{-\frac{1}{2}(3+\eta)E+(3x-x_1)-2v},$$
$$x_1'=\tfrac{1}{2}(1-\eta)E+\tfrac{1}{2}(x_1-x),$$
$$x_2'=x_2+\mu_2^2,$$
$$v'=\tfrac{1}{2}(-\mu_2^2-\mu_1^2-\mu_0^2+2x_2+v),$$

where $x=v+x_1+x_2$. It is seen that the two sets of variables are inequivalent.

PHYSICAL REVIEW VOLUME 124, NUMBER 4 NOVEMBER 15, 1961

Low-Energy Pion-Photon Interaction: The $(2\pi,2\gamma)$ Vertex*

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(Received June 14, 1961)

In the $(2\pi,2\gamma)$ problem, the Mandelstam representation is written for the two independent gauge-invariant amplitudes. On the basis of unitarity limitations on the asymptotic behavior of these amplitudes, only a $j=1$ subtraction in the $\gamma+\pi\rightarrow\gamma+\pi$ channel and a $j=0$ subtraction in the $\gamma+\gamma\rightarrow\pi+\pi$ channel are allowed. No over-all subtraction constants are required and the Thomson limit is automatically maintained. Only the effect of 2π intermediate states is considered. The odd-j $\pi\pi$ contribution involves the amplitude for the process $\gamma+\pi\rightarrow 2\pi$ analyzed by Wong and shown to be proportional to a pseudo-elementary constant Λ. Even with a $\pi\pi$ P resonance, the correction is negligible ($\lesssim 1\%$) if we use the value of Λ estimated by Wong on the basis of π^0 decay and confirmed by Ball in connection with photopion production on nucleons. A moderately important contribution comes from the S-wave interaction if we use a recent estimate of $\pi\pi$ S-wave phase shifts obtained from crossing relations. For the pion-pion coupling constant λ of order -0.20, this effect is $\sim 10\%$ in $\gamma+\pi\rightarrow\gamma+\pi$ scattering. For $\gamma+\gamma\rightarrow\pi+\pi$, the correction for the $I=0$ state at threshold is positive and $\sim 100\%$ of the Born approximation. However, as the energy is increased, the correction quickly changes sign.

I. INTRODUCTION

In the $(2\pi,2\gamma)$ problem, both strong and electromagnetic interactions are involved. In principle, one can calculate electromagnetic interactions on the basis of perturbation theory. Our purpose here is to understand the effects of strong pion interactions on the $(2\pi,2\gamma)$ vertex.[1]

Attempts have been made in recent years to understand strong pion interactions at low energies by using the Mandelstam representation.[2,3] In particular, a P-wave pion-pion resonance has been conjectured in connection with the nucleon electromagnetic structure.[4] If such a resonance exists, one might expect its effects to be appreciable in Compton scattering on pions (e.g., $\gamma+\pi\rightarrow\gamma+\pi$). One may recall in this connection Compton scattering on protons (e.g., $\gamma+p\rightarrow\gamma+p$), where the 3-3 resonance causes a large increase in the cross section above the value given by the Klein-Nishina-type formula.[5] Pion-pion forces may also be manifested in the final-state interactions of pion pairs produced by photons (e.g., $\gamma+\gamma\rightarrow\pi+\pi$). Such final-state interactions, if they are substantial, may be observed experimentally by producing a pion pair from a high-energy photon in the Coulomb field of a nucleus.

Further, an understanding of the $(2\pi,2\gamma)$ vertex is a prerequisite for a theory of nucleon-photon scattering and, in fact, for most problems where a vertex con-

* This work was done under the auspices of the U. S. Atomic Energy Commission.

[1] A preliminary account of this work was given at the 1960 Winter meeting of the American Physical Society, December 29–31, 1960 [Bipin R. Desai, Bull. Am. Phys. Soc. 5, 509 (1960)]. We employ units $\hbar=c=\mu=1$, where μ is the pion mass. For the charge e we use the units $e^2\simeq 1/137$. The metric is defined so that we have $g^{0,0}=1$ and $g^{ii}=-1$, where $i=1, 2, 3$.

[2] S. Mandelstam, Phys. Rev. 112, 1344 (1959); 115, 1741 and 1752 (1959).

[3] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[4] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); Phys. Rev. 117, 1603 (1960).

[5] G. F. Chew, *1958 Annual International Conference on High-Energy Physics at CERN* (CERN Scientific Information Service, Geneva, 1958).

necting strongly interacting particles with two photons is involved. For example, in the calculation of the electromagnetic mass of charged pions, one needs the pion Compton scattering amplitude for virtual photons. The information obtained here may, therefore, be helpful in understanding the mass difference between charged and neutral pions.

We shall investigate the $(2\pi,2\gamma)$ problem within the framework of double-dispersion relations proposed by Mandelstam.[2] We do not think it pertinent to go into the principles and conjectures underlying the Mandelstam representation, since we have nothing new to contribute to these general questions, which have been the subject of so many papers. Following the effective-range approximation given by Chew and Mandelstam we assume the behavior of the amplitudes to be dominated by nearby singularities.[3] Moreover, the contribution of intermediate states containing one or more photons will be neglected since, even though they correspond to near singularities, powers higher than e^2 are involved.

In the next section, we shall go into the kinematics of the problem and show that because of Lorentz and gauge invariance only two invariant amplitudes are involved. The Mandelstam representation for these amplitudes is then written in Sec. III, and the question of subtractions discussed. In Sec. IV, the helicity amplitudes of Jacob and Wick are introduced.[6] In Sec. V, we consider Compton scattering, $\gamma+\pi\rightarrow\gamma+\pi$, and discuss the effect of the $\pi\pi$ interactions. In Sec. VI, pion-pair production, $\gamma+\gamma\rightarrow\pi+\pi$, is considered and the effect of final-state $\pi\pi$ S-wave interactions discussed.

One of our main results is negative and very surprising, in view of the large enhancement of nucleon Compton scattering by the 33 resonance.[5] We find that the effect of the 2π P resonance on pion Compton scattering is negligibly small. The important matrix element here is that for $\gamma+\pi\rightarrow\pi+\pi$ and has been estimated by Wong on the basis of the π^0 lifetime, where this amplitude also plays a role.[7] Wong's estimate, confirmed in order of magnitude by Ball in connection with photoproduction of pions from nucleons,[8] is smaller by about a factor of 10 than one might naively guess. Since this matrix element appears squared in the Compton amplitude, the 2π resonance turns out to make a contribution only of the order of 1%. In Sec. V, we shall discuss the probable reason for the smallness of Wong's amplitude. We do not here consider a 3π bound state or resonance, which may play a large role in pion Compton scattering.

In the $\gamma+\gamma\rightarrow\pi+\pi$ channel only even angular-momentum states are involved because of charge-conjugation invariance. By a reasonable choice of $\pi\pi$ S-phase shifts, we find in Sec. IV that the contribution of the final-state interaction is large. For the $I=0$ state, where the interaction is strongest, the contribution at low energies is found to be positive corresponding to attraction and is of the order of 100% of the Born amplitude at threshold. As the energy is increased, however, it quickly changes sign. Such a circumstance corresponds to the fact that the pions are produced with a large relative separation ($\sim$ one pion Compton wavelength) and have, therefore, a fairly small probability of interacting with each other.

[6] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

[7] How-sen Wong, Phys. Rev. Letters **5**, 70 (1960) and Phys. Rev. **121**, 289 (1961).

[8] James S. Ball, University of California Radiation Laboratory Report UCRL-9172, 1960 (unpublished); Phys. Rev. Letters **5**, 73 (1960).

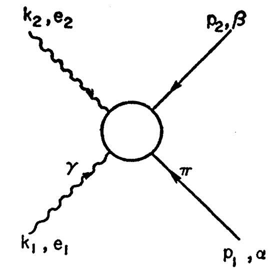

FIG. 1. The $(2\pi,2\gamma)$ vertex.

II. KINEMATICS AND INVARIANCE CONSIDERATIONS

Figure 1 describes the $(2\pi,2\gamma)$ vertex under consideration, where the wavy lines indicate photons and solid lines indicate pions. For the sake of symmetry, we shall take all the lines as incoming. Let p_1, p_2 be the four-momenta of the pions and α, β the corresponding charge indices, while k_1, k_2 are the four-momenta of the photons and e_1, e_2 the corresponding polarization vectors. We then define the three Lorentz invariants s, $\bar{s}$, and t as follows:

$$s=(k_1+p_1)^2=(k_2+p_2)^2, \tag{2.1a}$$

$$\bar{s}=(k_1+p_2)^2=(k_2+p_1)^2, \tag{2.1b}$$

$$t=(k_1+k_2)^2=(p_1+p_2)^2. \tag{2.1c}$$

From energy-momentum conservation, we have

$$s+\bar{s}+t=2.$$

Notice that s, $\bar{s}$, and t are the squares of the energies of the following three reactions in the barycentric system:

$$k_1+p_1\rightarrow -k_2-p_2, \quad (\gamma+\pi\rightarrow\gamma+\pi) \tag{2.2a}$$

$$k_1+p_2\rightarrow -k_2-p_1, \quad (\gamma+\pi\rightarrow\gamma+\pi) \tag{2.2b}$$

$$k_1+k_2\rightarrow -p_1-p_2, \quad (\gamma+\gamma\rightarrow\pi+\pi). \tag{2.2c}$$

The S matrix is defined as

$$S_{fi}=\delta_{fi}-i(2\pi)^4[16\omega(k_1)\omega(p_1)\omega(k_2)\omega(p_2)]^{-\frac{1}{2}} \times\delta(k_1+p_1+k_2+p_2)T_{fi},$$

where f and i indicate final and initial states, respectively, and the ω's indicate the energies of the different particles. For the given charge indices α and β we have

for the T matrix

$$T_{\alpha\beta}=(\delta_{\alpha\beta}-\delta_{\alpha3}\delta_{\beta3})T^c+\delta_{\alpha3}\delta_{\beta3}T^n,$$

where T^c and T^n denote the T matrices corresponding to charged and neutral pions, respectively. Henceforth we shall suppress the charged and neutral indices. We shall concentrate our attention mainly on the charged case and only comment on any alterations needed in the neutral case.

We may further write

$$T=e_{2\mu}T^{\mu\nu}e_{1\nu},$$

where $T^{\mu\nu}$ is a tensor of second rank which can be expressed in the most general form as

$$T^{\mu\nu}=Ak_1^\mu k_2^\nu+B\Delta^\mu k_2^\nu+Ck_2^\mu k_2^\nu+Dk_1^\mu k_1^\nu+E\Delta^\mu k_1^\nu +Fk_2^\mu k_1^\nu+Gk_1^\mu\Delta^\nu+H\Delta^\mu\Delta^\nu+Ik_2^\mu\Delta^\nu+Jg^{\mu\nu},$$

where $\Delta=p_1-p_2$, and $g^{\mu\nu}$ is the conventional metric tensor.[1] The amplitudes $A\cdots J$ are functions of the invariants s, $\bar{s}$, and t. Gauge invariance requires that (a) $k_{2\mu}T^{\mu\nu}=0$ and (b) $T^{\mu\nu}k_{1\nu}=0$. With the above conditions and the requirement of zero photon mass, $k_1^2=0=k_2^2$, we obtain

$$\begin{aligned}T(s,\bar{s},t)=&(e_2\cdot k_1e_1\cdot k_2-k_2\cdot k_1e_2\cdot e_1)A(s,\bar{s},t)\\&+(-e_1\cdot e_2k_2\cdot\Delta+(k_2\cdot k_1/k_2\cdot\Delta)e_2\cdot\Delta e_1\cdot\Delta\\&+e_2\cdot\Delta e_1\cdot k_2-e_2\cdot k_1e_1\cdot\Delta)B(s,\bar{s},t).\end{aligned}\quad(2.3)$$

Crossing symmetry requires

$$A(s,\bar{s},t)=A(\bar{s},s,t),\quad\text{and}\quad B(s,\bar{s},t)=-B(\bar{s},s,t).\quad(2.4)$$

The foregoing results have been obtained independently by Gourdin and Martin.[9]

III. MANDELSTAM REPRESENTATION

The Mandelstam representation for A and B can be written for charged pions as

$$\begin{aligned}A(s,\bar{s},t)=&(4\pi e^2/1-s)+(4\pi e^2/1-\bar{s})\\&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{\alpha_1(s',t')}{t'-t}\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right)\\&+\frac{1}{\pi^2}\int^\infty ds'\int_4^\infty d\bar{s}'\frac{\alpha_2(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})},\end{aligned}\quad(3.1)$$

$$\begin{aligned}B(s,\bar{s},t)=&(4\pi e^2/1-s)-(4\pi e^2/1-\bar{s})\\&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty dt'\frac{\beta_1(s',t')}{t'-t}\left(\frac{1}{s'-s}-\frac{1}{s'-\bar{s}}\right)\\&+\frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty d\bar{s}'\frac{\beta_2(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}.\end{aligned}\quad(3.2)$$

Here α_1, α_2, β_1, and β_2 are the double spectral functions. Notice that the crossing condition (2.4) is explicitly contained in Eqs. (3.1) and (3.2) for $\alpha_2(s,\bar{s})=\alpha_2(\bar{s},s)$ and $\beta_2(s,\bar{s})=-\beta_2(\bar{s},s)$. The poles at $s=1$ and $\bar{s}=1$ correspond to single-pion intermediate states in reactions (2.2a) and (2.2b), respectively. The lower limits on the above integrals correspond to the fact that the least massive intermediate states in the three channels given in reactions (2.2a)–(2.2c) are the two-pion states. For neutral pions, the only difference is that the poles are absent. Subtractions are perhaps necessary in the above dispersions relations and we shall discuss them later on.

The region in which the double spectral functions α_1, α_2, β_1, and β_2 are nonzero are given as follows: For both $\alpha_1(s,t)$ and $\beta_1(s,t)$ the region is defined by the curves

$$t=4(2s+1)^2/s(s-4)\quad(3.3a)$$

and

$$t=4(s-1)/s-9.\quad(3.3b)$$

For $\alpha_2(s,\bar{s})$ and $\beta_2(s,\bar{s})$, the curves are

$$(s-4)(\bar{s}-16)-81=0\quad(3.4a)$$

and

$$(s-16)(\bar{s}-4)-81=0.\quad(3.4b)$$

Notice that there are no anomalous thresholds involved.

By a proper choice of amplitudes, the pole terms correspond in the $\gamma+\pi\rightarrow\gamma+\pi$ channel to the Thomson amplitude, which A and B should approach in the zero-energy limit. Hence on the basis of zero-energy-limit theorems, subtractions are unnecessary. We thus differ from the observations of Gourdin and Martin,[9] who use a different set of amplitudes and are uncertain, therefore, about the number of possible subtractions. We may go farther and discuss possible subtractions on the basis of unitarity limitations on the asymptotic behavior of the A and B amplitudes. Such an analysis was first carried out by Froissart in the case of scalar particles[10] and was applied by Singh and Udgaonkar to the pion-nucleon problem.[11] We give below the results for the A and B amplitudes which are derived in Secs. V.B and VI.B.

For the $\gamma+\pi\rightarrow\gamma+\pi$ channel as s approaches infinity, we have

$$|A|\lesssim s,\quad|B|\lesssim\text{constant}\quad(3.5a)$$

for fixed t (i.e., for $\cos\theta=1$),

$$|A|\lesssim s,\quad|B|\lesssim s\quad(3.5b)$$

for fixed $\bar{s}$ (i.e., for $\cos\theta=-1$), and

$$|A|\lesssim s^{-\frac{1}{4}},\quad|B|\lesssim s^{-\frac{1}{4}}\quad(3.5c)$$

for any other value of $\cos\theta$, where θ is the scattering angle in this channel. For the $\gamma+\gamma\rightarrow\pi+\pi$ channel as

[9] M. Gourdin and A. Martin, Nuovo cimento **17**, 224 (1960). The Cini-Fubini approximate version of the Mandelstam representation has been used by these authors, but no numerical estimates have been attempted.

[10] Marcel Froissart, Phys. Rev. **123**, 1053 (1961).

[11] V. Singh and B. M. Udgaonkar, Phys. Rev. **123**, 1487 (1961).

t approaches infinity, we have

$$|A|\lesssim t,\quad |B|\lesssim t \tag{3.6a}$$

for fixed s or $\bar{s}$ (i.e., for $\cos\phi=\pm 1$) and

$$|A|\lesssim t^{-\frac{1}{2}},\quad |B|\lesssim t^{-\frac{3}{2}} \tag{3.6b}$$

for any other value of $\cos\phi$, where ϕ is this scattering angle in the channel. Since $\gamma+\gamma\to\pi+\pi$ is an inelastic channel, we may assume that the A and B amplitudes do not attain their maximum values given by expression (3.6a) in the forward or backward direction. For $\cos\phi=\pm 1$ we then have

$$|A|\lesssim t^{1-\epsilon},\quad |B|\lesssim t^{1-\epsilon}, \tag{3.6c}$$

where ϵ is any small positive number.

From the above asymptotic conditions, we observe that no arbitrary over-all subtraction constants are allowed in the A and B amplitudes since their presence violates conditions (3.5c) and (3.6b). Thus we do not anticipate that any new parameters will appear in our problem. One subtraction in t, corresponding to $j=1$ in the $\gamma+\pi\to\gamma+\pi$ channel, is allowed for both A and B amplitudes. However, further subtractions bring in powers of t larger than or equal to unity and are incompatible with the asymptotic behavior of expression (3.6c). One subtraction in s (and $\bar{s}$) is allowed for the A amplitude, corresponding to $j=0$ for the $\gamma+\gamma\to\pi+\pi$ channel, but subtractions for $j>0$, where j is even, are incompatible with expression (3.5a) since they bring in powers of s (or $\bar{s}$) larger than or equal to two. For the B amplitude, the first subtraction involves $(s-\bar{s})$ and is incompatible with relation (3.5a).

IV. HELICITY AMPLITUDES

In the present problem, we shall use the helicity amplitudes given by Jacob and Wick.[6] Thus we have a simpler connection between unitarity and analyticity than when the conventional electric- and magnetic-multipole amplitudes are employed.

In a two-body collision, we denote the helicities of the initial particles by λ_a and λ_b and of the final particles by λ_c and λ_d, respectively. The corresponding scattering amplitude is given by

$$g(\theta;\lambda_c,\lambda_d,\lambda_a,\lambda_b)=(1/p)\sum_j (j+\tfrac{1}{2}) \times\langle\lambda_c\lambda_d|T^j(E)|\lambda_a\lambda_b\rangle d_{\lambda\mu}{}^j(\theta), \tag{4.1}$$

while the differential cross section is

$$d\sigma/d\Omega=|g(\theta;\lambda_c,\lambda_d,\lambda_a,\lambda_b)|^2. \tag{4.2}$$

Here we have $\lambda=\lambda_a-\lambda_b$ and $\mu=\lambda_c-\lambda_d$; j is the total angular momentum; p, E, and θ are the barycentric momentum, energy, and scattering angle, respectively; $\langle\lambda_c\lambda_d|T^j(E)|\lambda_a\lambda_b\rangle$ is the corresponding T matrix; and $d_{\lambda\mu}{}^j(\theta)$ is the function given by Jacob and Wick.[6]

In the $(2\pi,2\gamma)$ problem, the pions have zero spin, and therefore zero helicity while the photons have helicity $+1$ or -1 depending on whether they are right or left circularly polarized.

V. COMPTON SCATTERING CHANNEL

In the barycentric system, we can write

$$k_1=(k,\mathbf{k}_1),\qquad p_1=[(k^2+1)^{\frac{1}{2}},-\mathbf{k}_1],$$
$$k_2=(-k,\mathbf{k}_2);\quad p_2=[-(k^2+1)^{\frac{1}{2}},-\mathbf{k}_2];$$

and $\mathbf{k}_1\cdot\mathbf{k}_2=-k^2\cos\theta$, where θ is the scattering angle and we define

$$s=[k+(k^2+1)^{\frac{1}{2}}]^2, \tag{5.1a}$$

$$t=-2k^2(1-\cos\theta), \tag{5.1b}$$

and

$$\bar{s}=[-k+(k^2+1)^{\frac{1}{2}}]^2-2k^2(1+\cos\theta). \tag{5.1c}$$

Here s is the square of the barycentric energy, and t the square of the corresponding momentum transfer. The differential cross section is

$$\frac{d\sigma}{d\Omega}=\left|\frac{1}{8\pi}\frac{T}{\sqrt{s}}\right|^2, \tag{5.2}$$

where T is the T matrix defined in Eq. (2.3).

A. Helicity Amplitudes

Here we have $\lambda_a=0=\lambda_d$ and therefore $\lambda_a=\lambda$, $\lambda_c=\mu$, with the λ and μ values being ± 1. If we denote the helicity amplitude by $f_{\mu\lambda}(\theta)$, we have

$$f_{\mu\lambda}(\theta)=\frac{1}{k}\sum_{j=1}^{\infty}(j+\tfrac{1}{2})T_{\mu\lambda}{}^j(s)d_{\lambda\mu}{}^j(\theta), \tag{5.3}$$

and

$$d\sigma/d\Omega=|f_{\mu\lambda}(\theta)|^2. \tag{5.4}$$

If we denote λ and μ indices by $\pm$, we have

$$T_{++}{}^j(s)=T_{--}{}^j(s),$$

and

$$T_{+-}{}^j(s)=T_{-+}{}^j(s).$$

Using Eq. (5.2) with appropriate values for the polarization vectors e_1, and e_2 and comparing it with Eq. (5.4), we obtain

$$a(s,\bar{s},t)=\frac{B(s,\bar{s},t)}{s-\bar{s}}=\frac{8\pi k}{s\bar{s}-1}\frac{s}{s-1}f_{++}(\theta), \tag{5.5a}$$

and

$$b(s,\bar{s},t)=\frac{1}{4}\left[A(s,\bar{s},t)+\frac{4-t}{s-\bar{s}}B(s,\bar{s},t)\right]$$
$$=\frac{8\pi k}{t}\frac{s}{s-1}f_{+-}(\theta), \tag{5.5b}$$

where

$$f_{++}(\theta)=\frac{1}{k}\sum_{j=1}^{\infty}(j+\tfrac{1}{2})T_{++}{}^j(s)d_{1,1}{}^j(\theta), \tag{5.6a}$$

and

$$f_{+-}(\theta)=\frac{1}{k}\sum_{j=1}^{\infty}(j+\tfrac{1}{2})T_{+-}{}^j(s)d_{1,-1}{}^j(\theta). \tag{5.6b}$$

Thus we have

$$a(s,\bar{s},t)=\frac{B(s,\bar{s},t)}{s-\bar{s}}=\frac{4\pi s}{s-1}\sum_j (2j+1)T_{++}{}^j(s)\frac{d_{1,1}{}^j(\theta)}{s\bar{s}-1}, \quad (5.7\text{a})$$

and

$$b(s,\bar{s},t)=\frac{1}{4}\left[A(s,\bar{s},t)+\frac{4-t}{s-\bar{s}}B(s,\bar{s},t)\right]=\frac{4\pi s}{s-1}\sum_j (2j+1)T_{+-}{}^j(s)\frac{d_{1,-1}{}^j(\theta)}{t}. \quad (5.7\text{b})$$

The $d^j(\theta)$ functions are given by Jacob and Wick as

$$d_{1,1}{}^j(\theta)=\frac{P_j'(\cos\theta)-P_{j-1}'(\cos\theta)+j^2P_j(\cos\theta)}{j(j+1)}, \quad (5.8\text{a})$$

and

$$d_{1,-1}{}^j(\theta)=\frac{P_j'(\cos\theta)+P_{j-1}'(\cos\theta)-j^2P_j(\cos\theta)}{j(j+1)}, \quad (5.8\text{b})$$

where the primes indicate derivatives with respect to $\cos\theta$. In Eqs. (5.7a) and (5.7b) we have $s\bar{s}-1$ and t in the denominators, and therefore we can use

$$\frac{d_{1,1}{}^j(\theta)}{1+\cos\theta}=\frac{P_{j-1}''(\cos\theta)-P_j''(\cos\theta)+jP_j'(\cos\theta)}{j(j+1)}, \quad (5.9\text{a})$$

and

$$\frac{d_{1,-1}{}^j(\theta)}{1-\cos\theta}=\frac{P_{j-1}''(\cos\theta)+P_j''(\cos\theta)+jP_j'(\cos\theta)}{j(j+1)}. \quad (5.9\text{b})$$

B. Asymptotic Behavior

Unitarity demands that

$$|T_{++}{}^j(s)|\leq 1, \quad (5.10\text{a})$$

and

$$|T_{+-}{}^j(s)|\leq 1. \quad (5.10\text{b})$$

Further, the Legendre functions and their derivatives satisfy the following relations:

$$P_j(1)=1, \quad P_j'(1)=j(j+1)/2, \quad (5.11\text{a})$$

$$P_j''(1)=(j-1)j(j+1)(j+2)/8. \quad (5.11\text{b})$$

For $\cos\theta=-1$, we use the relation

$$P_j(-\cos\theta)=(-1)^jP_j(\cos\theta).$$

For $\cos\theta\neq\pm 1$, we have for large values of j

$$P_j(\cos\theta)=j^{-\frac{1}{2}}h_0(\theta) \quad (5.12\text{a})$$

$$P_j'(\cos\theta)=j^{\frac{1}{2}}h_1(\theta), \quad (5.12\text{b})$$

and

$$P_j''(\cos\theta)=j^{\frac{3}{2}}h_2(\theta), \quad (5.12\text{c})$$

where $h_0(\theta)$, $h_1(\theta)$, and $h_2(\theta)$ are functions of θ only.

For the a and b amplitudes given in Eqs. (5.7a) and (5.7b), if we keep t fixed and let s approach infinity, then, since $\cos\theta$ approaches 1, we have from Eqs. (5.9), (5.10), and (5.11)

$$|a|\lesssim\frac{1}{s^2}\sum (j+\tfrac{1}{2})\to\frac{1}{s^2}j_{\max}{}^2=\frac{1}{s^2}(kR)^2\to\frac{1}{s}, \quad (5.13\text{a})$$

and

$$|b|\lesssim\frac{1}{s}\sum (j+\tfrac{1}{2})j^2\to\frac{1}{s}j_{\max}{}^4=\frac{1}{s}(kR)^4\to s, \quad (5.13\text{b})$$

where R is the interaction radius in the sense of Froissart's analysis[10] and is essentially a constant. Similarly, if we keep $\bar{s}$ fixed and let s approach infinity, then, since $\cos\theta$ approaches -1, we have

$$|a|\lesssim\text{constant}, \quad (5.14\text{a})$$

and

$$|b|\lesssim\text{constant}. \quad (5.14\text{b})$$

For $\cos\theta\neq\pm 1$ and $s\to\infty$, we have from Eqs. (5.12a)–(5.12c)

$$|a|\lesssim\frac{1}{s^2}\sum\frac{j+\frac{1}{2}}{j^{\frac{1}{2}}}\to s^{-5/4}, \quad (5.15\text{a})$$

and

$$|b|\lesssim\frac{1}{s}\sum\frac{j+\frac{1}{2}}{j^{\frac{1}{2}}}\to s^{-\frac{1}{4}}. \quad (5.15\text{b})$$

From these asymptotic conditions for the a and b amplitudes, we have for the A and B amplitudes as s approaches infinity

$$|A|\lesssim s, \quad |B|\lesssim\text{constant} \quad (5.16\text{a})$$

for t fixed, i.e., $\cos\theta=1$;

$$|A|\lesssim s, \quad |B|\lesssim s \quad (5.16\text{b})$$

for $\bar{s}$ fixed, i.e., $\cos\theta=-1$; and

$$|A|\lesssim s^{-\frac{1}{4}}, \quad |B|\lesssim s^{-\frac{1}{4}} \quad (5.16\text{c})$$

for $\cos\theta\neq\pm 1$.

C. Fixed Momentum Transfer Dispersion Relations

In Eqs. (5.7a) and (5.7b) we notice that since B is an odd function of $s-\bar{s}$, no new singularities are introduced in the a and b amplitudes. Moreover, we have $d_{1,1}{}^j(\pi)=0$ and $d_{1,-1}{}^j(0)=0$, corresponding to the vanishing of the forward helicity-flip and backward nonhelicity-flip amplitudes. However, these zeros are absent in the a and b amplitudes because of the presence of the factors $s\bar{s}-1$ and t in the denominators in (5.7a) and (5.7b). The a and b amplitudes have the further property that each is expressed in terms of a given type of helicity amplitude.

We shall now proceed to write dispersion relations for the a and b amplitudes rather than the A and B

amplitudes because of their simple properties given above. We shall not, however, use the Mandelstam representation in its full generality, but only the part of it obtained by keeping t (the square of the momentum transfer) fixed. In order to derive maximum benefit from the Mandelstam representation, i.e., in order to use information about the singularities of the scattering amplitude in all variables, we write down partial-wave dispersion relations. If we do so in the Compton scattering channel, the total amplitude for $\gamma+\gamma\rightarrow\pi+\pi$ is explicitly involved, corresponding to the cut $t\geq 4$. For the fixed momentum transfer dispersion relations, however, because of crossing symmetry, only the absorptive part of the $\gamma+\pi\rightarrow\gamma+\pi$ amplitude is involved except for the $j=0$ amplitude for the $\gamma+\gamma\rightarrow\pi+\pi$ channel. By making proper subtractions (see Secs. V.B and VI.B), we then have, for fixed t,

$$a(s,t)=\frac{4\pi e^2}{(1-s)(1-\bar{s})}+\frac{1}{\pi}\int_4^\infty ds'\, a_1(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right), \quad (5.17\text{a})$$

and

$$b(s,t)=\frac{4\pi e^2}{(1-s)(1-\bar{s})}+4\pi C_+^0(t)+\frac{1}{\pi}\int_4^\infty ds'\, b_1(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}-\frac{1}{2p_-q_-}\times\ln\left(\frac{s'+p_-^2+q_-^2+2p_-q_-}{s'+p_-^2+q_-^2-2p_-q_-}\right)\right], \quad (5.17\text{b})$$

where $C_+^0(t)$ is the correction term coming from the $j=0$, $\gamma+\gamma\rightarrow\pi+\pi$ amplitude continued to negative t values (see Sec. VI.C) and is allowed in b but not in a by the asymptotic conditions (5.13a) and (5.13b). The correction terms $C_+^{0,c}(t)$ and $C_+^{0,n}(t)$ for the charged and neutral case, respectively, are connected through the relation (6.10) to the correction terms $C_+^{0,I}(t)$ given in Eq. (6.16). In Eqs. (5.17a) and (5.17b) we define $a_1(s,t)$, $b_1(s,t)$, p_-, and q_- by

$$a_1(s,t)=\mathrm{Im}a(s,t)=\frac{4\pi s}{s-1}\sum_{j=1}^{\infty}(2j+1)\,\mathrm{Im}T_{++}^j(s)\frac{d_{1,1}^j(\theta)}{s\bar{s}-1}, \quad (5.18\text{a})$$

$$b_1(s,t)=\mathrm{Im}b(s,t)=\frac{4\pi s}{s-1}\sum_{j=1}^{\infty}(2j+1)\,\mathrm{Im}T_{+-}^j(s)\frac{d_{1,-1}^j(\theta)}{t}. \quad (5.18\text{b})$$

and

$$p_-=i(4-t)^{\frac{1}{2}}/2$$

$$q_-=i(-t)^{\frac{1}{2}}/2.$$

Using the unitarity of the S matrix we can express $\mathrm{Im}a$ and $\mathrm{Im}b$ in terms of a sum of the absolute squares of the amplitudes for $\gamma+\pi\rightarrow n$, where n stands for the possible intermediate states. In this preliminary calculation, motivated by the success of the analogous approach for γp scattering,[5] we neglect the contribution of all but the 2π intermediate states. If a 3π resonance or bound state exists, its contribution may be non-negligible. However, because of insufficient information about such a state, we do not consider it in the present discussion. In the above approximation, then, a knowledge of the $\gamma+\pi\rightarrow 2\pi$ amplitude is sufficient to give $\mathrm{Im}a$ and $\mathrm{Im}b$. This amplitude has recently been studied by Wong on the basis of the Mandelstam representation.[7] Only a single invariant amplitude is involved, and only odd angular momenta need be considered. We denote the helicity amplitudes $\langle\gamma\pi|T^j(E)|\pi\pi\rangle$ for a given angular momentum j and energy E in the $\gamma+\pi\rightarrow 2\pi$ reaction by $R_\pm^j(s)$, where $\pm$ indicate the photon spin parallel or antiparallel to the photon's direction of motion.[6] From unitarity, we then obtain

$$\mathrm{Im}T_\pm^j(s)=\pm\tfrac{1}{2}|R^j(s)|^2,$$

where

$$R_+^j(s)=-R_-^j(s)=R^j(s). \quad (5.19)$$

The $R^j(s)$ amplitudes are connected as follows to the amplitudes $M_j(s)$ given by Wong:

$$|R^j(s)|^2=\frac{1}{(64\pi)^2}\left[\frac{(s-1)^2(s-4)}{s}\right]^{\frac{3}{2}}\times\frac{j(j+1)}{(2j+1)^2}|M_j(s)|^2. \quad (5.20)$$

Thus from Eqs. (5.18a) and (5.18b) we obtain

$$a_1(s,t)=-\frac{1}{(32\sqrt{\pi})^2}\left[\frac{(s-4)^3}{s}\right]^{\frac{1}{2}}\times\sum_{j\,\text{odd}}\frac{j(j+1)}{2j+1}|M_j(s)|^2\frac{d_{1,1}^j(\theta)}{1+\cos\theta}, \quad (5.21\text{a})$$

and

$$b_1(s,t)=\frac{1}{(32\sqrt{\pi})^2}[s(s-4)^3]^{\frac{1}{2}}\times\sum_{j\,\text{odd}}\frac{j(j+1)}{2j+1}|M_j(s)|^2\frac{d_{1,-1}^j(\theta)}{1-\cos\theta}. \quad (5.21\text{b})$$

In Eqs. (5.21a) and (5.21b) we retain only the $j=1$ term and substitute the corresponding a_1 and b_1 in the dispersion integrals (5.17a) and (5.17b). This seems to be a good approximation, since energies under consideration are low. Furthermore, because of the assumed P-wave $\pi\pi$ resonance, the amplitude for $j=1$ is expected to be larger than the higher waves. A similar approximation has been made in proton Compton

scattering, $\gamma+p\rightarrow\gamma+p$.[5] Here only the πp intermediate state is retained, and by neglecting all but the contribution of the resonance in the $j=\frac{3}{2}$ and $T=\frac{3}{2}$ state (T being the isotopic spin), the results obtained are in good agreement with experiments.[5] Reintroducing the charged and neutral superscripts c and n, we have then the following relations:

$$a^c(s,t)=\frac{4\pi e^2}{(1-s)(1-\bar{s})}-\frac{1}{3(32\sqrt{\pi})^2}\frac{1}{\pi}\times\int_4^\infty ds'\left[\frac{(s'-4)^3}{s'}\right]^{\frac{1}{2}}|M_1(s')|^2\times\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right),\quad(5.22\text{a})$$

$$b^c(s,t)=\frac{4\pi e^2}{(1-s)(1-\bar{s})}+4\pi C_+^{0,c}(t)+\frac{1}{3(32\sqrt{\pi})^2}\frac{1}{\pi}\times\int_4^\infty ds'[s'(s'-4)^3]^{\frac{1}{2}}|M_1(s')|^2\times\left[\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}-\frac{1}{2p_-q_-}\times\ln\left(\frac{s'+p_-^2+q_-^2+2p_-q_-}{s'+p_-^2+q_-^2-2p_-q_-}\right)\right],\quad(5.22\text{b})$$

$$a^n(s,t)=-\frac{1}{3(32\sqrt{\pi})^2}\frac{1}{\pi}\int^\infty ds'\left[\frac{(s'-4)^3}{s'}\right]^{\frac{1}{2}}\times|M_1(s')|^2\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right),\quad(5.23\text{a})$$

$$b^n(s,t)=4\pi C_+^{0,n}(t)+\frac{1}{3(32\sqrt{\pi})^2}\frac{1}{\pi}\times\int_4^\infty ds'[s'(s'-4)^3]^{\frac{1}{2}}|M_1(s')|^2\times\left[\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}-\frac{1}{2p_-q_-}\times\ln\left(\frac{s'+p_-^2+q_-^2+2p_-q_-}{s'+p_-^2+q_-^2-2p_-q_-}\right)\right].\quad(5.23\text{b})$$

The $M_1(s)$ amplitude has been obtained by Wong using partial-wave dispersion relations.[7] Keeping only the contribution of the 2π, $J=1$ intermediate state, we observe that the phase of $M_1(s)$ is given by the phase of the $\pi\pi$ P wave. By replacing the left cut involved in the partial-wave dispersion relations by a single pole at a, Wong gave the $M_1(s)$ amplitude

$$M_1(s)=\Lambda(1+a)D_1(1)/(s+a)D_1(s),\quad(5.26)$$

where Λ is a pseudoelementary constant proportional to the residue of the left-cut pole, and $D_1(s)$ is the denominator function of the P-wave $\pi\pi$ system which is necessary to give $M_1(s)$ the required phase. The position a is given by the behavior of the P wave, and is larger for higher values of the P-wave resonance energy. The constant Λ is estimated by Wong on the basis of the π^0 lifetime, where it plays a role.[7] For a π^0 lifetime of $\sim 4\times 10^{-16}$ sec, he estimates Λ to be $\sim e$. With the Frazer-Fulco value for the P-wave $\pi\pi$ resonance position $s_R\simeq 10$ and width $\Gamma=0.4$, Wong found $a\simeq 5.7$. When these estimates of Λ and a are inserted into the dispersion integrals in Eqs. (5.22a) and (5.22b) for charged pions, we find by an exact calculation that their contribution is $\lesssim 1\%$. Near $s\simeq s_R$, we of course expect the imaginary parts of a and b to be important. For any s value, we have from Eq. (5.26)

$$I_a(s)=\text{Im}a^c(s,t)=\text{Im}a^n(s,t)=-\frac{\Lambda^2}{3(32\sqrt{\pi})^2}\left(\frac{1+a}{s+a}\right)^2\left[\frac{(s-4)^3}{s}\right]^{\frac{1}{2}}\left|\frac{D_1(1)}{D_1(s)}\right|^2,\quad(5.27\text{a})$$

and

$$I_b(s)=\text{Im}b^c(s,t)=\text{Im}b^n(s,t)=\frac{\Lambda^2}{3(32\sqrt{\pi})^2}\left(\frac{1+a}{s+a}\right)^2[s(s-4)^3]^{\frac{1}{2}}\left|\frac{D_1(1)}{D_1(s)}\right|^2.\quad(5.27\text{b})$$

TABLE I. Values of $[I_a(s)/B_f(s)]^2$ and $[I_b(s)/B_f(s)]^2$ for $s_R=10$ and $\Gamma=0.4$.

s	$[I_a(s)/B_f(s)]^2$	$[I_b(s)/B_f(s)]^2$
5	2×10^{-9}	6×10^{-8}
10	5×10^{-5}	5×10^{-3}
15	4×10^{-5}	9×10^{-3}
20	10^{-5}	4×10^{-3}

The ratios $[I_a(s)/B_f(s)]^2$ and $[I_b(s)/B_f(s)]^2$ are given in Table I, where $B_f(s)$ is the minimum value of the Born term in Eqs. (5.22a) and (5.22b) attained in the forward direction. We observe that the above ratios are not greater than $\sim 1\%$ near the resonance energy $s_R\simeq 10$. We have, so far, discussed the resonance contribution only for $s_R\simeq 10$, but for a higher s_R value $\simeq 20$ the situation will not qualitatively change.

The biggest correction to the Born amplitude seems to come from the $C_+^{0,c}(t)$ term and is roughly of the order $\sim 10\%$ if we take $\lambda=-0.20$ (see Sec. VI.C). The ratios of the differential cross section $d\sigma/d\Omega$ to $(d\sigma/d\Omega)_B$ is given in Table II for $\theta=90$ deg and $\theta=180$ deg, where $(d\sigma/d\Omega)_B$ is the differential cross section obtained by keeping only the Born term. For $\theta=0$ deg, the b amplitude is absent, and hence the contribution to

TABLE II. Values of $\left(\frac{d\sigma^c}{d\Omega}\right)\Big/\left(\frac{d\sigma^c}{d\Omega}\right)_B$ at $\theta=90$ and 180 deg.

s	$\theta=90$ deg	$\theta=180$ deg
5	1.06	0.93
10	1.10	0.89
15	1.12	0.88
20	1.13	0.87
25	1.14	0.86

$d\sigma/d\Omega$ comes entirely from the Born term. For the neutral case, of course, the contribution of $C_+^{0,n}(t)$ is the only important one. If the correction term for the $I=2$, $j=0$, $\gamma+\gamma\rightarrow\pi+\pi$ amplitude is neglected (see Sec. VI.C) we have from Eq. (6.10)

$$C_+^{0,n}(t)\simeq C_+^{0,c}(t).$$

The above results are in great contrast to the results in proton Compton scattering where, as described earlier, the 3-3 resonance in the intermediate pion-nucleon system increases substantially the cross section coming from the Born term.[5] The reason for the negligible contribution of the $\pi\pi$ resonance is, of course, the smallness of the $\gamma+\pi\rightarrow 2\pi$ amplitude as is seen from the factor $1/[3(32\sqrt{\pi})^2]$ in front of the integrals in (5.22a) and (5.22b). The normalization of the constant Λ introduced by Wong is evidently misleading, since $\Lambda\simeq e$ suggests a substantial magnitude for the $\gamma+\pi\rightarrow 2\pi$ amplitude. Numerical factors should be absorbed in Λ so as to make it appear small compared to e. The reason Λ should be small is probably associated with the minimal character of the electromagnetic interactions which appear in $\gamma+\pi\rightarrow 2\pi$. This amplitude is essentially the vertex joining a single photon to three pions. Now from minimality we know that a photon line can couple directly only with a charged pair, and then the coupling constant is e, the elementary charge. In the case under consideration, therefore, we need a two-particle intermediate state. A two-pion intermediate state or, in fact, any state containing an even number of pions is, however, forbidden because G conjugation does not allow an even number of pions to go into an odd number. Thus particles heavier than pions (e.g., kaons or nucleon-antinucleon pairs) must be created in intermediate states. The constant Λ should then be of the order e/M (where M is the nucleon or K-meson mass) and therefore be small. In Compton scattering, the contribution of the 2π intermediate state is proportional to Λ^2 and thus to $1/M^2$.

VI. PION-PAIR PRODUCTION CHANNEL

In the barycentric system we can write $k_1=(q,-\mathbf{q})$, $k_2=(q,\mathbf{q})$, $p_1=(-q,\mathbf{p})$, and $p_2=(-q,-\mathbf{p})$, where

$$t=4q^2=4(p^2+1), \tag{6.1a}$$

$$s=-q^2-p^2+2qp\cos\phi, \tag{6.1b}$$

and

$$\bar{s}=-q^2-p^2-2qp\cos\phi. \tag{6.1c}$$

Here ϕ is the scattering angle, t is the square of the barycentric energy, and s is the square of the momentum transfer. The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{p}{q}\left|\frac{1}{8\pi}\frac{T}{\sqrt{t}}\right|^2. \tag{6.2}$$

A. Helicity Amplitudes

If λ_a, λ_b are the helicities of the photons and λ_c, λ_d those of the pions, we have $\lambda_c=0=\lambda_d$, and $\lambda=\lambda_a-\lambda_b$. If we denote the helicity amplitudes by $F_{\lambda0}(\phi)$, we have

$$F_{\lambda0}(\phi)=(1/q)\textstyle\sum_j(j+\frac{1}{2})M_{\lambda0}{}^j(\phi)d_{\lambda0}{}^j(\phi), \tag{6.3}$$

and

$$d\sigma/d\Omega=|F_{\lambda0}(\phi)|^2. \tag{6.4}$$

If by $M_{+\pm}{}^j(t)$ we denote $M_{\lambda0}{}^j(t)$ with $\lambda_a=1$ and $\lambda_b=1$ (i.e., $\lambda=0$) and $\lambda_a=1$ and $\lambda_b=-1$ (i.e., $\lambda=2$), respectively, then we have

$$M_{++}{}^j(t)=M_{--}{}^j(t),$$

and

$$M_{+-}{}^j(t)=M_{-+}{}^j(t).$$

As in Sec. V.A by comparing Eqs. (6.1) and (6.3) for appropriate polarization vectors, we have

$$a(s,\bar{s},t)=4\pi\left(\frac{t}{t-4}\right)^{\frac{1}{2}}\sum_{j=2}^{\infty}(2j+1)M_{+-}{}^j(t)\frac{d_{2,0}{}^j(\phi)}{s\bar{s}-1}$$

$$=4\pi\sum_{j=2}^{\infty}(2j+1)h_-{}^j(t)\frac{d_{2,0}{}^j(\phi)}{1-\cos^2\phi}, \tag{6.5a}$$

and

$$b(s,\bar{s},t)=4\pi\left(\frac{t}{t-4}\right)^{\frac{1}{2}}\sum_{j=0}^{\infty}(2j+1)M_{++}{}^j(t)\frac{d_{0,0}{}^j(\phi)}{t}$$

$$=4\pi\sum_{j=0}^{\infty}(2j+1)h_+{}^j(t)d_{0,0}{}^j(\phi), \tag{6.5b}$$

where

$$h_-{}^j(t)=\frac{4}{t}\left(\frac{t}{(t-4)^3}\right)^{\frac{1}{2}}M_{+-}{}^j(t), \tag{6.6a}$$

and

$$h_+{}^j(t)=\frac{1}{t}\left(\frac{t}{t-4}\right)^{\frac{1}{2}}M_{++}{}^j(t). \tag{6.6b}$$

In terms of a and b we have

$$h_-{}^j(t)=\frac{1}{8\pi}\int_{-1}^{1}d(\cos\phi)\;\bar{d}_{2,0}{}^j(\phi)a(t,\cos\phi),$$

and

$$h_+{}^j(t)=\frac{1}{8\pi}\int_{-1}^{1}d(\cos\phi)\;d_{0,0}{}^j(\phi)b(t,\cos\phi),$$

where

$$\bar{d}_{2,0}{}^j(\phi)=(1-\cos^2\phi)d_{2,0}{}^j(\phi).$$

The $d^j(\phi)$ functions are

$$d_{2,0}{}^j(\phi)=\frac{2P_{j-1}{}'(\cos\phi)-j(j-1)P_j(\cos\phi)}{[(j-1)j(j+1)(j+2)]^{\frac{1}{2}}}, \quad (6.7)$$

and

$$d_{0,0}{}^j(\phi)=P_j(\cos\phi), \quad (6.8)$$

where primes indicate derivatives with respect to $\cos\phi$. In Eq. (6.4), we have $(1-\cos^2\phi)$ in the denominator, and we find

$$\frac{d_{2,0}{}^j(\phi)}{1-\cos^2\phi}=\frac{P_j{}''(\cos\phi)}{[(j-1)j(j+1)(j+2)]^{\frac{1}{2}}}.$$

B. Asymptotic Behavior

The general procedure for establishing asymptotic behavior is the same as in Sec. V.B. Relations (5.11) and (5.12) together with the unitarity limitations

$$|M_{+-}{}^j(t)|\leq 1, \quad (6.9a)$$

and

$$|M_{++}{}^j(t)|\leq 1, \quad (6.9b)$$

then give us for $t\to\infty$

$$|a|\lesssim\text{constant},\quad |b|\lesssim\text{constant}$$

for fixed s (or $\bar{s}$), i.e., $\cos\phi=\pm1$, and

$$|a|\lesssim t^{-5/4},\quad |b|\lesssim t^{-\frac{1}{4}}$$

for any other value of $\cos\phi$. Equivalently, we have

$$|A|\lesssim t,\quad |B|\lesssim t$$

for $\cos\phi=\pm1$ and

$$|A|\lesssim t^{-\frac{1}{4}},\quad |B|\lesssim t^{-\frac{1}{4}}$$

for $\cos\phi\neq\pm1$.

C. Partial-Wave Dispersion Relations

Knowing the singularities in the amplitudes a and b, we can write down partial-wave dispersion relations for $h_-{}^j(t)$ and $h_+{}^j(t)$. The branch cuts in a and b are, of course, the same as those in A and B. There is a branch cut $t\geq4$ for the amplitudes $h_\mp{}^j(t)$. Corresponding to the cut $s\geq4$ as well as $\bar{s}\geq4$ in A and B, there will also be a cut $t<-9/4$. We shall project out the Born term and write it explicitly as $h_{B\mp}{}^j(t)$. Before we write down partial-wave dispersion relations, however, we introduce amplitudes $h_\mp{}^{j,I}$ corresponding to a definite isotropic spin I of the final two-pion state. From charge conjugation or, equivalently, from crossing symmetry, we notice that only even angular momentum states are allowed and, therefore, only states with $I=0$ and $I=2$ need be considered. For the $I=0$ and $I=2$ states, we designate the amplitudes by the superscripts 0 and 2, respectively. An elementary calculation then gives their relations with the charged-neutral amplitudes as

$$h_+{}^{j,0}(t)=(\tfrac{1}{3})^{\frac{1}{2}}[2h_+{}^{j,c}(t)+h_+{}^{j,n}(t)], \quad (6.10a)$$

and

$$h_+{}^{j,2}(t)=(\tfrac{2}{3})^{\frac{1}{2}}[h_+{}^{j,c}(t)-h_+{}^{j,n}(t)], \quad (6.10b)$$

where $h_+{}^{j,c}(t)$ and $h_+{}^{j,n}(t)$ are the charged and neutral amplitudes, respectively. Similar relations hold for $h_-{}^j(t)$ amplitudes. We then have

$$h_\mp{}^{j,I}(t)=h_{B\mp}{}^{j,I}(t)+\frac{1}{\pi}\int_{-\infty}^{-9/4}dt'\frac{\operatorname{Im}h_\mp{}^{j,I}(t')}{t'-t}+\frac{1}{\pi}\int_4^\infty dt'\frac{\operatorname{Im}h_\mp{}^{j,I}(t')}{t'-t}, \quad (6.11)$$

where

$$h_{B-}{}^{j,I}(t)=\frac{1}{\sqrt{3}}\int_{-1}^{1}d(\cos\phi)\,\bar{d}_{2,0}{}^j(\phi)\frac{e^2}{(1-s)(1-\bar{s})}, \quad (6.12a)$$

and

$$h_{B+}{}^{j,I}(t)=\frac{1}{\sqrt{6}}\int_{-1}^{1}d(\cos\phi)\,d_{0,0}{}^j(\phi)\frac{e^2}{(1-s)(1-\bar{s})}. \quad (6.12b)$$

From Eqs. (6.6) and (6.9), we have $|h_-{}^j(t)|\lesssim1/t^2$ and $|h_+{}^j(t)|\lesssim1/t$ in the physical region. Hence, in the above relations, no subtraction constants are needed. The integral along the left cut corresponds to the correction to the Born term for the crossed channel $\gamma+\pi\to\gamma+\pi$. As we have already seen, the correction is probably small and we shall neglect it in this preliminary calculation.

For the integrals in Eq. (6.11) involving positive t values greater than four, we shall use unitarity. In the approximation of including only two-pion intermediate states, we have

$$\operatorname{Im}h_\mp{}^{j,I}(t)=h_\mp{}^{j,I}(t)A^{*j,I}(t)[(t-4)/t]^{\frac{1}{2}}, \quad (6.13)$$

where

$$A^{j,I}(t)=[t/(t-4)]^{\frac{1}{2}}\exp(i\delta_j{}^I)\sin\delta_j{}^I \quad (6.14)$$

is the pion-pion scattering amplitude defined by Chew and Mandelstam for angular momentum j and isotopic spin I; $\delta_j{}^I$ being the corresponding phase shift.[3]

At low energies, we shall neglect the integrals in Eq. (6.11) along the right-hand cut for $j\geq2$, since the pion-pion amplitude for D and higher waves is expected to be small. We then have

$$h_-{}^{j,I}(t)=h_{B-}{}^{j,I}(t),$$

since always $j\geq2$;

$$h_+{}^{j,I}(t)=h_{B+}{}^{j,I}(t)$$

for $j\geq2$; and

$$h_+{}^{0,I}(t)=h_{B+}{}^{0,I}(t)+\frac{1}{\pi}\int_4^\infty dt'\left(\frac{t'-4}{t'}\right)^{\frac{1}{2}}\times\frac{h_+{}^{0,I}(t')A^{*0,I}(t')}{t'-t}. \quad (6.15)$$

Thus we consider the $\pi\pi$ interaction correction only to

TABLE III. Values of $h_{B+}^{0,0}(t)$, $\mathrm{Re}C_+^{0,0}(t)$, and $\mathrm{Im}C_+^{0,0}(t)$ for $\lambda=-0.20$, in units of e^2.

t	$h_{B+}^{0,0}(t)$	$\mathrm{Re}C_+^{0,0}(t)$	$\mathrm{Im}C_+^{0,0}(t)$
4.0	0.289	0.395	0
4.5	0.237	0.042	0.208
5.0	0.203	−0.026	0.178
6.0	0.146	−0.059	0.093
7.0	0.109	−0.056	0.056
8.0	0.086	−0.051	0.037
12.0	0.045	−0.032	0.010
16.0	0.027	−0.022	0.002

the $j=0$ state (i.e., S state). Following Chew and Mandelstam, we write[3]

$$A^{0,I}(t)=N_0{}^I(t)/D_0{}^I(t),$$

where $N_0{}^I(t)$ and $D_0{}^I(t)$ are the numerator and denominator functions in the $\pi\pi$ S amplitudes. From a modification of the form given by Omnes,[8,12] we have

$$\begin{aligned} h_+{}^{0,I}(t)&=h_{B+}{}^{0,I}(t)+\frac{1}{\pi}\frac{1}{D_0{}^I(t)}\int_4^\infty \frac{dt'}{t'-t}\left(\frac{t'-4}{t'}\right)^{\frac{1}{2}} \\ &\qquad\times h_{B+}{}^{0,I}(t')N_0{}^I(t') \\ &=h_{B+}{}^{0,I}(t)+C_+{}^{0,I}(t), \end{aligned} \tag{6.16}$$

where $C_+{}^{0,I}(t)$ is the rescattering correction term due to the $\pi\pi$ interaction.

In an earlier work,[13] values for the S amplitudes in terms of the pion-pion coupling constant λ have been obtained from crossing relations. At present $\lambda\simeq-0.20$ seems a reasonable estimate.[13] For this λ value, we have calculated the correction $C_+{}^{0,I}(t)$ to the Born term $h_{B+}{}^{0,I}(t)$. This correction, of course, comes from the final-state $\pi\pi$ interaction in the S wave. From Tables III and IV, we find that for the $I=0$ state the correction is large at low energies corresponding to strong attraction, but for higher energies it quickly changes sign. Such a circumstance can be understood as follows: If we take the Born term $h_{B+}{}^0(t)$ to be approximately a pole at $t=-t_B$, then its slope is $\sim t_B{}^{-2}$. Since the distance at which the pairs are produced is $\sim t_B{}^{-\frac{1}{2}}$, the larger the distance, the faster is the decrease of $h_{B+}{}^0(t)$ in the integral in Eq. (6.16). We have here a case in which the pions are produced at a relatively large distance—about a pion Compton wavelength—and, therefore, $h_{B+}{}^0(t)$ decreases relatively rapidly, giving rise to a sign change in the principal part of the integral in Eq. (6.16). At higher energies this negative contribution is $\sim 70\%$ of the Born term. The ratios $\sigma_+{}^I(t)/\sigma_{B+}{}^I(t)$ of the total cross sections for a given I spin with and without the correction terms are given in Table V. Such interactions as discussed above may perhaps be detected by rather accurate experiments on pion-pair production by a photon in the Coulomb field of a nucleus.[14]

TABLE IV. Values of $h_{B+}^{0,2}(t)$, $\mathrm{Re}C_+^{0,2}(t)$, and $\mathrm{Im}C_+^{0,2}(t)$ for $\lambda=-0.20$, in units of e^2.

t	$h_{B+}^{0,2}(t)$	$\mathrm{Re}C_+^{0,2}(t)$	$\mathrm{Im}C_+^{0,2}(t)$
4.0	0.204	0.027	0
4.5	0.168	0.016	0.017
5.0	0.143	0.009	0.018
6.0	0.103	0.002	0.015
7.0	0.077	−0.001	0.012
8.0	0.061	−0.003	0.011
12.0	0.032	−0.004	0.006
16.0	0.019	−0.003	0.004

TABLE V. Values of $\sigma_+{}^0(t)/\sigma_{B+}{}^0(t)$ and $\sigma_+{}^2(t)/\sigma_{B+}{}^2(t)$ for $\lambda=-0.20$.

t	$\sigma_+{}^0(t)/\sigma_{B+}{}^0(t)$	$\sigma_+{}^2(t)/\sigma_{B+}{}^2(t)$
4.0	5.60	1.29
4.5	2.15	1.21
5.0	1.54	1.16
6.0	0.76	1.05
7.0	0.54	0.99
8.0	0.42	0.94
12.0	0.22	0.83
16.0	0.21	0.77

ACKNOWLEDGMENT

This problem was suggested as the topic of my thesis by Professor Geoffrey F. Chew whose advice and encouragement throughout the period of my work I gratefully acknowledge.

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