Elementary and Composite Particles*†

M. T. VAUGHN, R. AARON,‡ AND R. D. AMADO
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania
(Received June 7, 1961)

It is shown for two simple fixed-source theories with one stable excited state of the source that (1) a knowledge of the physical S-matrix elements of the theory is sufficient to decide if the state is associated with an elementary particle or a composite particle, and (2) the theory in which the particle is composite corresponds to the elementary-particle theory in the limit of vanishing wave function renormalization constant of the elementary particle. The distinction between elementary and composite particle is related to a generalization of Levinson's theorem, and it is made plausible that this generalization is valid in local field theory. In an Appendix, the result (2) is generalized, and an application to the dispersion-theoretic treatment of rearrangement collisions is suggested.

I. INTRODUCTION

In the Lagrangian formulation of quantum field theory, there exists a natural distinction between elementary and composite particle—an elementary particle is associated with each type of field operator appearing in the Lagrangian, and a composite particle is associated with each single-particle state for which there is no corresponding field operator. In the S-matrix formulation of the theory, on the other hand, elementary and composite particle alike appear as poles in certain matrix elements, which are supposed to be analytic functions of an appropriate set of complex variables; the residue at the pole is related to a renormalized coupling constant if the particle in question is regarded as elementary, or to the asymptotic normalization of a bound-state wave function, if the particle in question is regarded as composite.[1] It is perhaps reasonable to ask whether, within the framework of such a theory, the distinction between elementary and composite particle is more than purely semantic, i.e., whether it is possible to distinguish in principle between an elementary particle and a composite particle from a knowledge of S-matrix elements alone.[2]

It is shown in the present work that such a distinction is indeed possible in principle, at least within the framework of the two fixed-source theories which are examined in detail here. The elementary-particle versions of these theories are (1) the Lee model,[3] which describes the interaction of a scalar boson θ with a fixed fermion source (possessing two internal degrees of freedom, N and V) through the virtual processes $V \leftrightarrow N+\theta$, and (2) an extension of the Lee model in which the θ particle has a distinct antiparticle $\bar{\theta}$; the interaction with the source takes place through the virtual processes $V \leftrightarrow N+\theta$ and $N \leftrightarrow V+\bar{\theta}$ (this latter model possesses crossing symmetry which is absent in the Lee model). The corresponding composite-particle versions are (1) a separable potential model, which describes the interaction of a scalar boson θ with a fixed source N through a separable potential, and (2) a generalization of the meson pair theory of Wentzel,[4] in which the (complex) boson field θ interacts with the source N through a quadratic interaction; in these versions, the single V-particle state appears as a dynamical consequence of the basic $N-\theta$ interaction. It is shown that each composite-particle theory can be obtained as a strong-coupling limit of the corresponding elementary-particle theory, in which limit the wave function renormalization constant of the V particle vanishes.[5]

The observable distinction between the elementary and composite particle theories is found in the high-energy behavior of the $N-\theta$ scattering phase shifts, which is similar to that suggested by Levinson's theorem in potential scattering,[6] and its generalizations to certain fixed-source theories which have been discussed by other authors.[7–10] These are re-examined here in the light of the Fredholm theory of the scattering equation,[11–14] and it is made plausible that a generalized

* Supported in part by the U. S. Atomic Energy Commission and the National Science Foundation.
† Part of this work is contained in the Ph.D. dissertation of R. Aaron, University of Pennsylvania (1961).
‡ National Science Foundation Cooperative Fellow. Now at Department of Physics, University of Maryland, College Park, Maryland.
[1] It should be noted here that the particle in question need not be stable; if it is unstable, the associated bound-state wave function will be of the type which is appropriate to a radioactively decaying state, and the poles in the S-matrix elements will appear on the second (unphysical) sheet of the Riemann surface associated with the complex energy variable. The following discussion is relevant to both cases.
[2] Even in the absence of an underlying Lagrangian theory, the distinction is of theoretical interest; one hopes to calculate the parameters associated with a composite-particle pole in terms of the more fundamental parameters associated with the elementary-particle poles.
[3] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[4] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).
[5] As was noted independently for the case of the Lee model by J. Houard and B. Jouvet, Nuovo cimento **18**, 466 (1960). We are indebted to E. C. G. Sudarshan for calling this work to our attention.
[6] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).
[7] J. Polkinghorne, Proc. Cambridge Phil. Soc. **54**, 560 (1958).
[8] M. Ida, Progr. Theoret. Phys. (Kyoto) **21**, 625 (1959).
[9] G. Konisi and T. Ogimoto, Progr. Theoret. Phys. (Kyoto) **22**, 807 (1959).
[10] E. Kazes (to be published).
[11] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).
[12] A. Salam and P. T. Matthews, Phys. Rev. **90**, 690 (1953).
[13] J. Schwinger, Phys. Rev. **93**, 615 (1954); **94**, 1362 (1954).
[14] M. Baker, Ann. Phys. **4**, 271 (1958).

Levinson's theorem is valid in a fully relativistic field theory; however, a rigorous proof cannot be given.

In an Appendix, it is shown that any model in which a field interacts with a fixed source through a potential, and in which there is a bound state of the field and source, can be obtained as the limiting case of a theory in which the bound state is replaced by an elementary particle. An application of this result to the dispersion-theoretic treatment of rearrangement collisions[15] is suggested, but detailed discussion is reserved for a subsequent publication.

II. LEE MODEL AND SEPARABLE POTENTIAL MODEL

The Lee model[3] describes the interaction of a neutral scalar boson θ of mass μ with a fixed extended fermion source [having two internal degrees of freedom, denoted by N and V, with (bare) masses m and $m_V^{(0)}$, respectively] through the virtual processes $V\leftrightarrow N+\theta$; the Hamiltonian of the model is

$$H=m\psi_N^\dagger\psi_N+m_V^{(0)}\psi_V^\dagger\psi_V+\int\omega_k a^\dagger(\mathbf{k})a(\mathbf{k})d^3\mathbf{k}+H_{\text{int}},\quad(1)$$

with

$$H_{\text{int}}=\frac{g^{(0)}}{(2\pi)^{\frac{3}{2}}}\left\{\int\frac{d^3k}{(2\omega_k)^{\frac{1}{2}}}u(k)\times[\psi_V^\dagger\psi_N a(\mathbf{k})+\psi_N^\dagger\psi_V a^\dagger(\mathbf{k})]\right\}.\quad(2)$$

Here $\psi_N^\dagger$, ψ_N ($\psi_V^\dagger$, ψ_V) are the creation and annihilation operators for the N state (V state) of the source; these satisfy the usual anticommutation rules. $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$ are the creation and annihilation operators for a θ particle of momentum $\mathbf{k}$ and energy $\omega_k=(k^2+\mu^2)^{\frac{1}{2}}$; $g^{(0)}$ is the (bare) coupling constant of the theory, and $u(k)$ is a cutoff function [normalized to $u(0)=1$].

The $N-\theta$ scattering states $|N\mathbf{k}(\pm)\rangle$ associated with an incident θ particle of momentum $\mathbf{k}$ satisfy the Schrödinger equation

$$(E_k-H)|N\mathbf{k}(\pm)\rangle=0,\quad(3)$$

with $E_k=m+\omega_k$. Write $|N\mathbf{k}(\pm)\rangle=a^\dagger(\mathbf{k})|N\rangle+\chi^{(\pm)}(\mathbf{k})$; then

$$\chi^{(\pm)}(\mathbf{k})=[1/(E_k-H\pm i\eta)]j(\mathbf{k})|N\rangle,\quad(4)$$

where

$$j(\mathbf{k})\equiv[H,a^\dagger(\mathbf{k})]-\omega_k a^\dagger(\mathbf{k})=[g^{(0)}/(2\pi)^{\frac{3}{2}}][u(\mathbf{k})/(2\omega_k)^{\frac{1}{2}}]\psi_V^\dagger\psi_N\quad(5)$$

is the "current operator" for a θ particle of momentum $\mathbf{k}$.

The S matrix for $N-\theta$ scattering has elements

$$\langle N\mathbf{k}'(-)|N\mathbf{k}(+)\rangle=\delta(\mathbf{k}'-\mathbf{k})-2\pi i\delta(E_{k'}-E_k)\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle,\quad(6)$$

and the T-matrix element $\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle$ is given by[16]

$$\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle=\langle N|j^\dagger(\mathbf{k}')[1/(E_{k'}-H+i\eta)]j(\mathbf{k})|N\rangle.\quad(7)$$

Equation (7) implies that the function $h(\omega)$ defined by

$$\langle Nk'(-)|j(\mathbf{k})|N\rangle=\frac{4\pi}{(2\pi)^3}\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}h(\omega_{k'})\quad(8)$$

is the boundary value of an analytic function $h(z)$ of the complex variable z, regular in the z plane cut along the real axis from μ to ∞, except for simple poles along the real axis associated with the discrete eigenstates of H for which $\langle N|j^\dagger(\mathbf{k})|\alpha\rangle\neq0$.

For the Lee model, with Hamiltonian (1), the function $h(z)$ is given by

$$h(z)=\Gamma^{(0)}\left[z-E_V^{(0)}+\frac{\Gamma^{(0)}}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{\omega_q-z}d\omega_q\right]^{-1},\quad(9)$$

where $\Gamma^{(0)}=(g^{(0)})^2/4\pi$ and $E_V^{(0)}=m_V^{(0)}-m$. The existence of a stable V particle with (physical) mass $m_V=m+E_V$ requires the existence of a solution of

$$E_V=E_V^{(0)}-\frac{\Gamma^{(0)}}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{\omega_q-E_V}d\omega_q<\mu.\quad(10)$$

If a solution of Eq. (10) exists, $h(z)$ can be written as

$$h(z)=\frac{\Gamma^{(0)}}{z-E_V}\left[1+\frac{\Gamma^{(0)}}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{(\omega_q-z)(\omega_q-E_V)}d\omega_q\right]^{-1}\quad(9')$$

$$=\frac{\Gamma}{z-E_V}\left[1+(z-E_V)\frac{\Gamma}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{(\omega_q-z)(\omega_q-E_V)^2}d\omega_q\right]^{-1},\quad(9'')$$

where $\Gamma\equiv Z\Gamma^{(0)}$, with

$$Z^{-1}=1+\frac{\Gamma^{(0)}}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{(\omega_q-E_V)^2}d\omega_q.\quad(11)$$

The state vector of the physical V particle is given by

$$|V\rangle=Z^{\frac{1}{2}}\psi_V^\dagger|\text{vac}\rangle-\frac{g}{(2\pi)^{\frac{3}{2}}}\int d^3k\frac{u(k)}{(2\omega_k)^{\frac{1}{2}}}\frac{1}{\omega_k-E_V}a^\dagger(\mathbf{k})|N\rangle,\quad(12)$$

where the renormalized coupling constant $g\equiv Z^{\frac{1}{2}}g^{(0)}$ has been introduced; Eq. (11) implies that Z, the probability that the physical V particle is in its bare state, satisfies $0<Z<1$ (recall $\Gamma^{(0)}>0$); also

$$0<\frac{\Gamma}{\pi}\int_\mu^\infty\frac{q|u(q)|^2}{(\omega_q-E_V)^2}d\omega_q=1-Z<1,\quad(13)$$

[15] R. Amado and R. Blankenbecler (unpublished).

[16] The methods used here are standard. See, for example, G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

so that the magnitude of the renormalized coupling constant $g=(4\pi\Gamma)^{\frac{1}{2}}$ cannot exceed a certain critical value, as was noted by Pauli and Källén,[17] and there are no spurious (ghost) poles of $h(z)$.

The separable potential model to be discussed here describes the interaction of a neutral scalar boson θ of mass μ with a fixed source (denoted by N) through a separable potential; the Hamiltonian of the model is

$$H=\int \omega_k a^\dagger(\mathbf{k})a(\mathbf{k})d^3k+H_{\text{int}}, \tag{14}$$

$$H_{\text{int}}=\frac{\lambda}{(2\pi)^3}\int \frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}a^\dagger(\mathbf{k}')a(\mathbf{k})d^3k\, d^3k', \tag{15}$$

where λ is the coupling constant of the theory, and $u(k)$ is a cutoff function [normalized to $u(0)=1$]; the equivalence of the model described by the Hamiltonian (14) to the Lee model is anticipated by using the same cutoff function.

The analysis of $N-\theta$ scattering proceeds as in Eqs. (3)–(7) above, except that the T-matrix element $\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle$ is now given by

$$\begin{aligned}\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle=&\langle N|[a(\mathbf{k}'),j(k)]|N\rangle\\&+\langle N|j(\mathbf{k}')[E_{k'}-H+i\eta]^{-1}j(\mathbf{k})|N\rangle.\end{aligned} \tag{7$'$}$$

The first term on the right-hand side, which is equal to zero in the Lee model, is now given by

$$\langle N|[a(\mathbf{k}'),j(\mathbf{k})]|N\rangle=\frac{\lambda}{(2\pi)^3}\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}, \tag{16}$$

and the function $h(z)$ defined by Eq. (8) is given by

$$h(z)=\xi\left[1+\frac{\xi}{\pi}\int_\mu^\infty \frac{q|u(q)|^2}{\omega_q-z}d\omega_q\right]^{-1}, \tag{17}$$

where $\xi=\lambda/4\pi$. Now if

$$1+\frac{\xi}{\pi}\int_\mu^\infty \frac{q|u(q)|^2}{\omega_q-\mu}<0, \tag{18}$$

the denominator of $h(z)$ will vanish for $z=E_V<\mu$ given by

$$1+\frac{\xi}{\pi}\int_\mu^\infty \frac{q|u(q)|^2}{\omega_q-E_V}d\omega_q=0, \tag{19}$$

corresponding to a bound state of the $N-\theta$ system at energy E_V.

Then $h(z)$ can be written as

$$h(z)=\left[\frac{1}{\pi}(z-E_V)\int_\mu^\infty \frac{q|u(q)|^2}{(\omega_q-z)(\omega_q-E_V)}\right]^{-1} \tag{17$'$}$$

$$=\frac{\Gamma_c}{z-E_V}\left[1+(z-E_V)\frac{\Gamma_c}{\pi}\times\int_\mu^\infty \frac{q|u(q)|^2}{(\omega_q-z)(\omega_q-E_V)^2}d\omega_q\right]^{-1}, \tag{17$''$}$$

where

$$\frac{\Gamma_c}{\pi}\int_\mu^\infty \frac{q|u(q)|^2}{(\omega_q-E_V)}d\omega_q=1. \tag{20}$$

Thus the $N-\theta$ scattering amplitude [Eq. (17″)] in the separable potential model with a bound state (V particle) with energy E_V has the same structure as in the Lee model with a stable V particle (with physical mass $m_v=m+E_V$); the residue of $h(z)$ at the V-particle pole is, however, an *a priori* arbitrary (except for the restriction $0<\Gamma<\Gamma_c$) parameter in the Lee model, while in the separable potential model it is determined by Eq. (20).

Also, the bound state wave function $|V_B\rangle$ in the separable potential model is

$$|V_B\rangle=\frac{(4\pi\Gamma_c)^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}\int d^3k\frac{u(k)}{(2\omega_k)^{\frac{1}{2}}}\frac{1}{E_V-\omega_k}a^\dagger(\mathbf{k})|N\rangle \tag{21}$$

(normalized to $\langle V_B|V_B\rangle=1$) which is the limit of Eq. (12) as $Z\to 0$, $g\to g_c=(4\pi\Gamma_c)^{\frac{1}{2}}$. This reflects the fact that in the separable potential model, the "V-particle" state is a superposition of $N-\theta$ plane wave states, and no independent bare V-particle state is required.[18]

It may also be noted that Eq. (17) can be obtained as the limit of Eq. (9) as $\Gamma^{(0)}\to\infty$ and $E_V{}^{(0)}\to\infty$ in such a way that $\xi=-\Gamma^{(0)}/E_V{}^{(0)}$ remains fixed (and negative). This limiting procedure also includes the case in which the physical V particle is unstable, in which case the wave function renormalization constant is no longer a well-defined quantity.

It is interesting to examine the high-energy behavior of the $N-\theta$ amplitude in the two theories. Write

$$h(\omega)\equiv\Gamma/(\omega-E_V)D(\omega), \tag{22}$$

which serves to define $D(\omega)$. It follows from Eqs. (9″) and (13) that $D(\omega)\to Z$ as $\omega\to\infty$; thus $h(\omega)\to\Gamma^{(0)}/\omega$ for large ω in the Lee model, while in the separable potential model, $h(\omega)\to\xi$ for large ω. This behavior is reflected in the high-energy behavior of the $N-\theta$ scattering phase shift $\delta(\omega)$ which is related to $h(\omega)$ by

$$e^{i\delta(\omega)}\sin\delta(\omega)=k|u(k)|^2h(\omega). \tag{23}$$

The difference $\delta(\mu)-\delta(\infty)$ is given in the Lee model by

$$\delta(\mu)-\delta(\infty)=0 \quad \text{(stable } V \text{ particle)},$$
$$\delta(\mu)-\delta(\infty)=-\pi \quad \text{(unstable } V \text{ particle)},$$

[17] W. Pauli and G. Källén, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **30**, No. 7 (1955).

[18] This has been noted independently by Houard and Jouvet (reference 5).

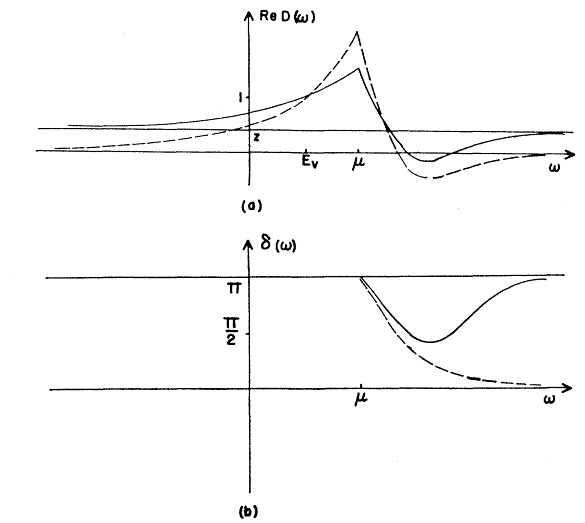

FIG. 1. (a) The real part of $D(\omega)$ defined by Eq. (22), for the cases $Z\neq0$ (———) and $Z=0$ (-----). (b) The $N-\theta$ scattering phase shift $\delta(\omega)$ shown for the case $Z\neq0$ (———) and $Z=0$ (-----).

and in the separable potential model by

$$\delta(\mu)-\delta(\infty)=\pi \quad \text{(bound state)},$$
$$\delta(\mu)-\delta(\infty)=0 \quad \text{(no bound state)}.$$

The relation between the two models in the stable V-particle case is illustrated in Fig. 1. These results, which can also be derived from the generalized versions of Levinson's theorem to be discussed in Sec. IV, show that a knowledge of the $N-\theta$ scattering phase shift $\delta(\omega)$ and the presence or absence of a single-particle bound state suffices to distinguish between the two models.

It can also be verified that the equivalence of the Lee model and the separable potential model in the limit $\Gamma\to\Gamma_c$ is valid in the other sectors as well. For example, the $V-\theta$ scattering amplitude $h_V(\omega)$ defined by

$$\langle V\mathbf{k}'(-)|j(\mathbf{k})|V\rangle=\frac{4\pi}{(2\pi)^3}\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}h_V(\omega_{k'}) \quad (24)$$

is given in the separable potential model by $h_v(\omega)=h(\omega)$, with $h(\omega)$ given by Eq. (17), since in this model, the scattering of a θ particle by the fixed N particle is not affected by the presence of an additional θ particle in a bound state; the process $V+\theta\leftrightarrow N+\theta+\theta$ is forbidden in the absence of a direct $\theta-\theta$ interaction.[19]

In the Lee model, on the other hand, $h_V(\omega)$ has the form[20]

$$h_V(\omega)=-h(\omega)\frac{D(\omega)}{1-D(\omega)+[1-J(\omega)]/[1+J(\omega)]}, \quad (25)$$

where $D(\omega)$ has been defined in Eq. (22), and $J(\omega)$ is the boundary value of an analytic function of the complex variable ω, regular in the ω plane cut along the real axis from $\omega=2\mu-E_V$ to $\omega=\infty$; its precise form will not be given here,[21] but it has the following property:

$$\lim_{\Gamma\to\Gamma_c} J(\omega)=\infty, \quad (26)$$

whence $h_V(\omega)$, defined by Eq. (25), satisfies

$$\lim_{\Gamma\to\Gamma_c} h_V(\omega)=h(\omega), \quad (27)$$

so that the Lee model approaches the separable potential model in the limit $\Gamma\to\Gamma_c$.

There is an apparent paradox in this result, since the $V-\theta$ scattering amplitude $h_V(\omega)$ has a pole at $\omega=E_V$, the residue of which is Γ according to Eq. (24) and $-\Gamma$ according to Eq. (25) $[J(E_V)=0]$. The resolution of the paradox is that the function $h_V(\omega)$ of Eq. (25) develops a second pole for $\omega<\mu$ as Γ increases from zero (this pole corresponds to a dynamical bound state of the $V-\theta$ system) and, as $\Gamma\to\Gamma_c$, the position of the pole approaches E_V, and the residue at the pole approaches 2Γ; thus Eq. (27) is consistent.

III. EXTENDED LEE MODEL AND MESON PAIR THEORY

Consider now an extended Lee model in which the θ particle has a (distinct) antiparticle $\bar{\theta}$; the interaction between the θ field and the source takes place through the virtual processes $V\leftrightarrow N+\theta$, $N\leftrightarrow V+\bar{\theta}$. The Hamiltonian of the model is

$$H=m^{(0)}\psi_N{}^\dagger\psi_N+m_V{}^{(0)}\psi_V{}^\dagger\psi_V+\int\omega_k[a^\dagger(\mathbf{k})a(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})]d^3k+H_{\text{int}} \quad (28)$$

with

$$H_{\text{int}}=\frac{g^{(0)}}{(2\pi)^{\frac{3}{2}}}\left\{\int\frac{d^3k}{(2\omega_k)^{\frac{1}{2}}}u(k)[\psi_V{}^\dagger\psi_N a(\mathbf{k})+\psi_N{}^\dagger\psi_V a^\dagger(\mathbf{k})]+\int\frac{d^3k}{(2\omega_k)^{\frac{1}{2}}}u(k)[\psi_N{}^\dagger\psi_V b(\mathbf{k})+\psi_V{}^\dagger\psi_N b^\dagger(\mathbf{k})]\right\}. \quad (29)$$

Here $m^{(0)}$, $m_V{}^{(0)}$ are the (bare) masses of the N particle and the V particle, respectively; $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$ $[b^\dagger(\mathbf{k})$ and $b(\mathbf{k})]$ are the creation and annihilation operators for a θ particle [$\bar{\theta}$ particle] of momentum k and energy $\omega_k=(k^2+\mu^2)^{\frac{1}{2}}$; $g^{(0)}$ is the (bare) coupling constant of the theory, and $u(k)$ is the cutoff function [normalized to $u(0)=1$].

The state vectors $|N\rangle$ and $|V\rangle$ associated with the physical N particle and physical V particle, respec-

[19] We are indebted to B. W. Lee for an illuminating discussion of this point.

[20] R. Amado, Phys. Rev. **122**, 696 (1961).

[21] The form of $J(\omega)$ has been given by Amado (reference 20) for the case of $m_V=m$. Its form for $m_V\neq m$ is easily derived.

tively, satisfy

$$(m-H)|N\rangle=0, \tag{30}$$

$$(m_V-H)|V\rangle=0, \tag{31}$$

where m and m_V are the physical masses of N particle and V particle, respectively (it is tacitly assumed in the following that $m<m_V<m+\mu$); formal expressions (not solutions) for the state vectors can be written down, but since they are not required in the following discussions, they will be omitted here.

The $N-\theta$ scattering states $|N\mathbf{k}'(\pm)\rangle$ associated with an incident θ particle of momentum $\mathbf{k}$ satisfy the Schrödinger equation (3), as do the $N-\bar{\theta}$ scattering states $|\bar{N}k(\pm)\rangle$ associated with an incident $\bar{\theta}$ particle of momentum $\mathbf{k}$; formally[16]

$$|N\mathbf{k}(\pm)\rangle=a^\dagger(\mathbf{k})|N\rangle+(E_k-H\pm i\eta)^{-1}j(\mathbf{k})|N\rangle, \tag{32}$$

$$|\bar{N}\mathbf{k}(\pm)\rangle=b^\dagger(\mathbf{k})|N\rangle+(E_k-H\pm i\eta)^{-1}\bar{j}(\mathbf{k})|N\rangle, \tag{33}$$

where $j(k)$ is given by Eq. (5), and

$$\bar{j}(k)=[H,b^\dagger(\mathbf{k})]-\omega_k b^\dagger(\mathbf{k})=j^\dagger(\mathbf{k}).$$

The S matrix for $N-\theta$ $(N-\bar{\theta})$ scattering has elements

$$\langle N\mathbf{k}'(-)|N\mathbf{k}(+)\rangle=\delta(\mathbf{k}-\mathbf{k}')-2\pi i\delta(E_{k'}-E_k)\times\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle, \tag{34}$$

$$\langle \bar{N}\mathbf{k}'(-)|\bar{N}\mathbf{k}(+)\rangle=\delta(\mathbf{k}-\mathbf{k}')-2\pi i\delta(E_{k'}-E_k)\times\langle \bar{N}\mathbf{k}'(-)|\bar{j}(\mathbf{k})|N\rangle, \tag{35}$$

and the T-matrix elements

$$\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle,\quad \langle \bar{N}\mathbf{k}'(-)|\bar{j}(k)|N\rangle$$

are given by

$$\begin{aligned}\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle &=\langle N|j^\dagger(\mathbf{k})(\omega_{k'}+m-H+i\eta)^{-1}j(\mathbf{k})|N\rangle\\ &\quad-\langle N|j(\mathbf{k})(\omega_{k'}-m+H)^{-1}j^\dagger(\mathbf{k}')|N\rangle,\end{aligned} \tag{36}$$

$$\begin{aligned}\langle \bar{N}\mathbf{k}'(-)|\bar{j}(\mathbf{k})|N\rangle &+\langle N|j(\mathbf{k}')(\omega_{k'}+m-H+i\eta)^{-1}j^\dagger(\mathbf{k})|N\rangle\\ &\quad-\langle N|j^\dagger(\mathbf{k})(\omega_{k'}-m+H)^{-1}j(\mathbf{k}')|N\rangle,\end{aligned} \tag{37}$$

where use has been made of the relations

$$a(\mathbf{k}')|N\rangle=-(\omega_{k'}-m+H)^{-1}j^\dagger(\mathbf{k}')|N\rangle, \tag{38}$$

$$b(\mathbf{k}')|N\rangle=-(\omega_{k'}-m+H)^{-1}j(\mathbf{k}')|N\rangle. \tag{39}$$

Consider now the analytic function of the complex variable z defined by

$$t(z)=\Gamma^{(0)}\sum_\alpha\left\{\frac{\langle N|\psi_N{}^\dagger\psi_V|\alpha\rangle\langle\alpha|\psi_V{}^\dagger\psi_N|N\rangle}{m-E_\alpha+z}+\frac{\langle N|\psi_V{}^\dagger\psi_N|\alpha\rangle\langle\alpha|\psi_N{}^\dagger\psi_V|N\rangle}{m+E_\alpha-z}\right\}, \tag{40}$$

where $\Gamma^{(0)}=(g^{(0)})^2/4\pi$; the summation extends over a complete set of eigenstates of H. $t(z)$ is regular in the z plane cut along the real axis from $-\infty$ to $-\mu$ and from μ to ∞, except for simple poles on the real axis associated with the discrete eigenstates of H for which $\langle N|\psi_N{}^\dagger\psi_V|\alpha\rangle\neq 0$ or $\langle N|\psi_V{}^\dagger\psi_N|\alpha\rangle\neq 0$.

Equations (36) and (37) imply that the functions $T(\omega)$, $\bar{T}(\omega)$ defined by

$$\langle N\mathbf{k}'(-)|j(\mathbf{k})|N\rangle=\frac{4\pi}{(2\pi)^3}\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}T(\omega_{k'}), \tag{41}$$

$$\langle \bar{N}\mathbf{k}'(-)|\bar{j}(\mathbf{k})|N\rangle=\frac{4\pi}{(2\pi)^3}\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}\bar{T}(\omega_{k'}) \tag{42}$$

are given as boundary values of $t(z)$ according to

$$T(\omega)=\lim_{\eta\to 0^+}t(\omega+i\eta), \tag{43}$$

$$\bar{T}(\omega)=\lim_{\eta\to 0^+}t(-\omega-i\eta). \tag{44}$$

The contribution to the summation in Eq. (40) from the physical V-particle state is $|\langle N|\psi_N{}^\dagger\psi_V|V\rangle|^2\equiv Z$. Define the renormalized coupling constant $g\equiv Z^{\frac{1}{2}}g^{(0)}$; including the contribution to the summation from the $N-\theta$ and $N-\bar{\theta}$ scattering states yields

$$t(z)=\frac{\Gamma}{z-E_V}-\frac{4\pi}{(2\pi)^3}\int|u(q)|^2\times\left\{\frac{|T(\omega_q)|^2}{\omega_q-z}+\frac{|\bar{T}(\omega_q)|^2}{\omega_q+z}\right\}\frac{d^3q}{2\omega_q}, \tag{45}$$

with $\Gamma=g^2/4\pi$, $E_V=m_V-m$. The contributions from the remaining states have been dropped (one-meson approximation).

The standard solution for $t(z)$ is[22]

$$t(z)=\frac{\Gamma}{(z-E_V)}\frac{1}{D(z)}, \tag{46}$$

$$D(z)=1+(z-E_V)\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{(\omega_q-z)(\omega_q-E_V)^2}+\frac{1}{(\omega_q+z)(\omega_q+E_V)^2}\right\}d\omega_q \tag{47}$$

$$=C+\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{(\omega_q-z)(\omega_q-E_V)}-\frac{1}{(\omega_q+z)(\omega_q+E_V)}\right\}d\omega_q, \tag{47'}$$

where

$$C=1-\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{(\omega_q-E_V)^2}-\frac{1}{(\omega_q+E_V)^2}\right\}d\omega_q. \tag{48}$$

[22] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956). $D(z)$ is assumed to have no poles.

It is necessary that $C>0$ in order that $t(z)$ shall have no ghost poles for complex z; thus, in the one-meson approximation at least, there is a finite upper limit on the magnitude of the renormalized coupling constant, except for the special case $E_V=0$, which is excluded here.

More generally, $t(z)$ is given by Eq. (46), with

$$D(z)=1+(z-E_V)\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{r(\omega_q)}{(\omega_q-z)(\omega_q-E_V)^2}+\frac{\bar{r}(\omega_q)}{(\omega_q+z)(\omega_q+E_V)^2}\right\}d\omega_q \quad (49)$$

$$=C'+\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{r(\omega_q)}{(\omega_q-z)(\omega_q-E_V)}-\frac{\bar{r}(\omega_q)}{(\omega_q+z)(\omega_q+E_V)}\right\}d\omega_q, \quad (49')$$

where

$$C'=1-\frac{\Gamma}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{r(\omega_q)}{(\omega_q-E_V)^2}-\frac{\bar{r}(\omega_q)}{(\omega_q+E_V)^2}\right\}d\omega_q. \quad (50)$$

Here $r(\omega)$ $[\bar{r}(\omega)]$ is the ratio of the total $N-\theta$ $[N-\bar{\theta}]$ cross section to the corresponding elastic cross section at energy ω. Again it is required that $C'>0$, but it is no longer clear that this implies a finite upper limit for Γ, since the integral in Eq. (50) may vanish or be negative. This possibility seems remote, however, and it will be seen presently that for $\Gamma\to\Gamma_c$ defined by $C=0$, $C'\to 0$ also. Note that $Z\neq C'$ in general, but is related to it by

$$Z=C'\langle N|[\psi_N{}^\dagger\psi_V,\psi_V{}^\dagger\psi_N]|N\rangle. \quad (51)$$

Finally, note that $t(z)$ can be written as

$$t(z)=\Lambda\Big/\left[z-\eta_V+\frac{\Lambda}{\pi}\int_\mu^\infty q|u(q)|^2\times\left\{\frac{r(\omega_q)}{\omega_q-z}+\frac{\bar{r}(\omega_q)}{\omega_q+z}\right\}d\omega_q\right], \quad (52)$$

where Λ is given by $\Gamma=C'\Lambda$, and

$$\eta_V=E_V+\frac{\Lambda}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{r(\omega_q)}{\omega_q-E_V}+\frac{\bar{r}(\omega_q)}{\omega_q+E_V}\right\}d\omega_q. \quad (53)$$

The relativistic meson pair theory to be considered here describes the interaction of a (complex) scalar boson θ of mass μ with a fixed source (denoted by N) through a generalized separable potential; the Hamiltonian of the model is

$$H=\int\omega_k[a^\dagger(\mathbf{k})a(\mathbf{k})+b^\dagger(\mathbf{k})b(\mathbf{k})]d^3k+H_{\text{int}}, \quad (54)$$

$$H_{\text{int}}=\frac{\lambda}{(2\pi)^3}\int\frac{u(k)u(k')}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}[a^\dagger(\mathbf{k})a(\mathbf{k}')+b^\dagger(\mathbf{k})b(\mathbf{k}')+a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}')+a(\mathbf{k})b(\mathbf{k}')]d^3kd^3k', \quad (55)$$

where λ is the coupling constant of the theory.

Since the Hamiltonian is quadratic in the field operators, the asymptotic creation and annihilation operators can be constructed[23] as linear combinations of the $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$, $b^\dagger(\mathbf{k})$, $b(\mathbf{k})$; consequently, only elastic scattering processes are allowed. Thus when the analysis of $N-\theta$ and $N-\bar{\theta}$ scattering is carried out as in Eqs. (32)–(45) [Eqs. (36) and (37) must be modified by adding to the right-hand side commutator terms $\langle N|[a(\mathbf{k}'),j(\mathbf{k})]|N\rangle$ and $\langle N|[b(\mathbf{k}'),\bar{j}(\mathbf{k})]|N$, respectively], the one-meson approximation to the function $t(z)$ defined by Eq. (40) will be exact.

Thus the exact solution for $t(z)$ is given by

$$t(z)=\xi\left\{1+\frac{\xi}{\pi}\int_\mu^\infty q|u(q)|^2\left[\frac{1}{\omega_q-z}+\frac{1}{\omega_q+z}\right]d\omega_q\right\}^{-1}, \quad (56)$$

where $\xi=\lambda/4\pi$.

Now if

$$1+\frac{\xi}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{\omega_q-\mu}+\frac{1}{\omega_q+\mu}\right\}d\omega_q<0, \quad (57)$$

the denominator of $t(z)$ will vanish for $z=E_V<\mu$ given by

$$1+\frac{\xi}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{\omega_q-E_V}+\frac{1}{\omega_q+E_V}\right\}d\omega_q=0, \quad (58)$$

corresponding to a bound state of the $N-\theta$ system at energy $E_V<\mu$.

Then $t(z)$ has the form of Eq. (46), with $D(z)$ given by

$$D(z)=\frac{\Gamma_c}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{(\omega_q-z)(\omega_q-E_V)}-\frac{1}{(\omega_q+z)(\omega_q+E_V)}\right\}d\omega_q \quad (59)$$

$$=1+(z-E_V)\frac{\Gamma_c}{\pi}\int_\mu^\infty q|u(q)|^2\times\left\{\frac{1}{(\omega_q-z)(\omega_q-E_V)^2}+\frac{1}{(\omega_q+z)(\omega_q+E_V)^2}\right\}d\omega_q, \quad (59')$$

[23] B. W. McCormick and A. Klein, Phys. Rev. 98, 1428 (1955), have constructed the solution for the neutral pair theory of Wentzel (reference 4); extension to the charged pair theory is trivial.

where

$$\frac{\Gamma_c}{\pi}\int_\mu^\infty q|u(q)|^2\left\{\frac{1}{(\omega_q-E_V)^2}-\frac{1}{(\omega_q+E_V)^2}\right\}d\omega_q=1. \quad (60)$$

Thus the extended Lee model becomes a pair theory in the limit $C\to 0$ in Eq. (48), or $Z\to 0$ in Eq. (51); this is the strong-coupling limit. It is also clear that the V-particle wave function renormalization constant Z_V vanishes in this limit, since the state vector for the V-particle bound state can be constructed from the $N-\theta$ plane wave states; as before, the distinction between the two models can be made from a knowledge of the $N-\theta$ scattering phase shift as a function of energy.

It may also be noted that if Eq. (58) is satisfied, the denominator of $t(z)$ will vanish for $z=-E_V$; this corresponds to a bound state of the $N-\bar{\theta}$ system at energy E_V. This bound state is present also in the extended Lee model: as Γ is increased from zero, there will appear a zero in $D(z)$ [Eq. (47)] which moves from $-\mu$ to $-E_V$ as Γ is increased to Γ_c.

IV. LEVINSON'S THEOREM

Levinson's theorem for potential scattering[6] is most simply demonstrated by writing the partial wave amplitude

$$T_L(E)=[e^{i\delta_L(E)}\sin\delta_L(E)]/q \quad (61)$$

for scattering in the state of angular momentum L and energy $E=q^2/2m$ in the form[24]

$$T_L(E)=N_L(E)/D_L(E), \quad (62)$$

where $D_L(E)$ is the boundary value (as E approaches the real axis from the upper half-plane) of an analytic function of the complex variable E, regular in the E plane, cut along the real axis from 0 to ∞, and satisfying $D_L(E)\to 1$ as $E\to\infty$. Also,

$$N_L(E)=\frac{1}{2iq}\lim_{\eta\to 0^+}[D_L(E+i\eta)-D_L(E-i\eta)] \quad (63)$$

is real for $0\leqslant E<\infty$. Consequently, $D_L(E)$ has the phase $-\delta_L(E)$ for $0\leqslant E<\infty$, and the S-matrix element $S_L(E)\equiv 1+2iqT_L(E)$ is given by

$$S_L(E)=e^{2i\delta_L(E)}=\frac{D_L^*(E)}{D_L(E)}=\lim_{\eta\to 0^+}\frac{D_L(E-i\eta)}{D_L(E+i\eta)}, \quad (64)$$

and the phase shift $\delta_L(E)$ is given by

$$\delta_L(E)=\frac{1}{2i}\ln S_L(E)$$
$$=\frac{1}{2i}\lim_{\eta\to 0^+}[\ln D_L(E-i\eta)-\ln D_L(E+i\eta)]. \quad (65)$$

[24] See, for example, R. Blankenbecler, M. Goldberger, N. Khuri, and S. Treiman, Ann. Phys. 10, 62 (1960).

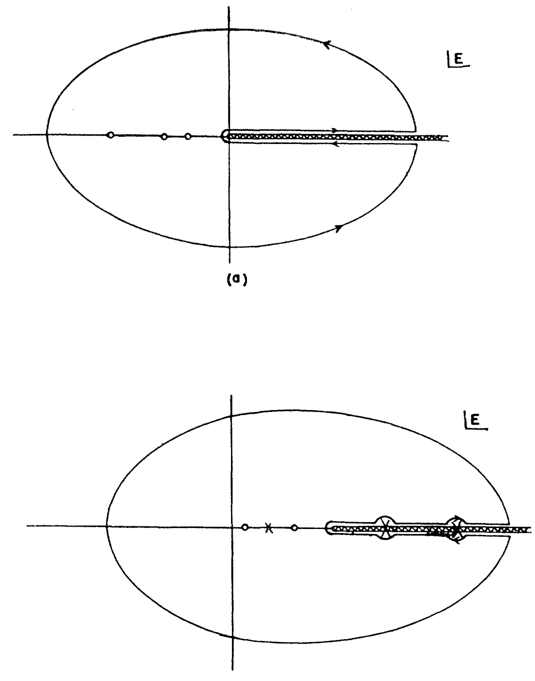

FIG. 2. (a) The contour C of Eq. (66). The circles denote zeros of $D(E)$. (b) The contour C' of Eq. (73). The circles denote zeros of $D(E)$; the crosses denote the poles of $D(E)$.

Consider now the integral

$$\frac{1}{2\pi i}\int_C\frac{D_L'(E)}{D_L(E)}dE=\frac{1}{2\pi i}\int_0^\infty\frac{d}{dE}\{\ln D_L(E+i\eta)-\ln D_L(E-i\eta)\}dE, \quad (66)$$

where the contour C is illustrated in Fig. 2(a). The value of this integral is equal to the number of zeroes of $D(E)$ enclosed by the contour C [$D(E)$ has no poles within the contour]; this is just equal to the number n_L of bound states of angular momentum L; using Eq. (65) to evaluate the right-hand side of Eq. (66) then yields the relation

$$\delta_L(0)-\delta_L(\infty)=n_L\pi, \quad (67)$$

which is Levinson's theorem.[6]

For the purpose of the subsequent discussion, it is useful to note that for the case of potential scattering, the Fredholm determinant, defined formally by

$$D(E)\equiv\det\left\|\frac{E-H}{E-H_0}\right\|, \quad (68)$$

can be expanded according to

$$D(E)=\prod_L(2L+1)D_L(E). \quad (69)$$

While the infinite product (69) diverges, the $D_L(E)$ are well-defined[14] (for a central potential), and in fact can be identified with the $D_L(E)$ defined in Eq. (62), as has been noted previously by many authors.[24]

In the Lee model, or its generalization in which there are several distinct bare V particles, the $N-\theta$ scattering amplitude $T(E)$, which is related to the phase shift $S(E)$ by Eq. (61), with $E=m+(q^2+\mu^2)^{\frac{1}{2}}$, can be written in the form

$$T(E)=N(E)/D(E), \tag{70}$$

where $D(E)$ can be chosen as the boundary value of an analytic function of the complex variable E, regular in the E plane cut along the positive real axis from $m+\mu$ to ∞ (except for simple poles at the points $z=m_1^{(0)}$, $\cdots$, $m_n^{(0)}$, where $m_1^{(0)}$, $\cdots$, $m_n^{(0)}$ are the masses of the bare V particles), and satisfying $D(E)\to 1$ as $E\to\infty$. Also,

$$N(E)=\frac{1}{2iq}\lim_{\eta\to 0^+}[D(E+i\eta)-D(E-i\eta)] \tag{71}$$

is real for $m+\mu\leqslant E<\infty$, and regular (except for simple poles at $z=m_1^{(0)}$, $\cdots$, $m_n^{(0)}$) in a neighborhood of the interval $m+\mu\leqslant E<\infty$.

An explicit representation of $D(E)$ is

$$D(E)=1-\sum_{k=1}^{n}\left(\frac{\Gamma_k^{(0)}}{m_k^{(0)}-E}\right)\frac{1}{\pi}\int_{m+\mu}^{\infty}\frac{q|u(q)|^2}{E_q-E}dE_q, \tag{72}$$

where $u(q)$ is the cutoff function of the model $[u(0)=1]$, and $\Gamma_k^{(0)}=(g_k^{(0)})^2/4\pi$ where $g_k^{(0)}$ is the (bare) $V_k-N\theta$ coupling constant. $D(E)$, as defined in this manner, can be identified with the Fredholm determinant [Eq. (68)] for the $N-\theta$ sector of the model.

Consider now the integral

$$\frac{1}{2\pi i}\left\{\int_{C'}\frac{D'(E)}{D(E)}dE+\sum_i\int_{C_j}\frac{D'(E)}{D(E)}dE\right\}$$
$$=\frac{1}{2\pi i}\int_{m+\mu}^{\infty}\frac{d}{dE}\{\ln D(E+i\eta)-\ln D(E-i\eta)\}dE, \tag{73}$$

where the contour C' is illustrated in Fig. 2(b), and the C_j are contours enclosing the poles of $D(E)$ on the real axis for $m+\mu\leqslant E<\infty$. The left-hand side of Eq. (73) is given by $n-n_0$, where n is the number of zeroes of $D(E)$ (which all lie within the contour C') and n_0 is the number of poles of $D(E)$ (which lie within either C' or one of the C_j); by virtue of Eq. (68), n is the number of discrete eigenstates of H (stable V particles), and n_0 is the number of discrete eigenstates of H_0 (bare V particles); using Eqs. (64) and (65) to evaluate the right-hand side of Eq. (73) then leads to the relation

$$\delta(m+\mu)-\delta(\infty)=(n-n_0)\pi. \tag{74}$$

This result has been known previously,[7,8] and is valid rigorously for any single-channel theory in which the relationship between the Fredholm determinant [Eq. (68)] and the S-matrix element $S(E)$ is given by Eq. (64).

The possibility of generalizing Levinson's theorem to more complex systems is elucidated by examining a generalization of the Lee model[25] in which the distinct θ fields θ_1, $\cdots$, θ_n interact with a fixed source, whose internal states are N; V_1, $\cdots$, V_{n_0}, through the virtual processes $V_j\leftrightarrow N+\theta_\alpha$. The S matrix for the process $N+\theta_\alpha\leftrightarrow N+\theta_\beta$ is related to a certain analytic function $t_{\beta\alpha}(E)$ defined in the same way as the function $h(\omega)$ of Eq. (8); the $t_{\beta\alpha}(E)$ are related to the functions $D_{\beta\alpha}(E)$ defined by

$$D_{\beta\alpha}(E)=\delta_{\beta\alpha}+\sum_i\left(\frac{g_{i\alpha}^{(0)}g_{i\beta}^{(0)}}{4\pi}\right)\left(\frac{1}{E-m_i^{(0)}}\right)\frac{1}{\pi}$$
$$\times\int_{m+\mu_\beta}^{\infty}\frac{q|u_\beta(\omega_{q\beta})|^2}{E_{q\beta}-E}dE_{q\beta} \tag{75}$$

[where $m_i^{(0)}$ is the (bare) mass of V_i, $g_{i\alpha}^{(0)}$ is the (bare) $V_iN\theta_\alpha$ coupling constant, $u_\beta(\omega_{q\beta})$ is the cutoff function associated with the θ_β field, $E_{q\beta}=m+\omega_{q\beta}$] according to[26]

$$t_{\beta\alpha}(E)=[1/D(E)]\textstyle\sum_\gamma N_{\beta\gamma}(E)d_{\gamma\alpha}(E), \tag{76}$$

where $D(E)\equiv\det\|D_{\beta\alpha}(E)\|$ is regular in the complex E plane cut along the positive real axis from $m+\mu$ to ∞ ($\mu=\min\{\mu_\alpha\}$) except for simple poles at $z=m_j^{(0)}$ ($j=1$, $\cdots$, n_0), $d_{\gamma\alpha}(E)$ is the cofactor of $D_{\alpha\gamma}(E)$ in $D(E)$, and

$$N_{\beta\gamma}(E)=\sum_i\left(\frac{g_{i\beta}^{(0)}g_{i\gamma}^{(0)}}{4\pi}\right)\left(\frac{1}{m_i^{(0)}-E}\right). \tag{77}$$

$D(E)$ as defined here can again be identified with the Fredholm determinant [Eq. (68)] for the $N-\theta$ sector of the model.

The S matrix $\mathbf{S}(E)$ can be written as

$$\mathbf{S}(E)=\mathbf{D}^*(E)\mathbf{D}^{-1}(E), \tag{78}$$

and

$$\det\mathbf{S}(E)=\lim_{\eta\to 0^+}\frac{D(E-i\eta)}{D(E+i\eta)}. \tag{79}$$

Then also

$$\sum_\gamma\delta_\gamma(E)=\frac{1}{2i}\lim_{\eta\to 0^+}[\ln D(E+i\eta)-\ln D(E-i\eta)], \tag{80}$$

where the $S_\gamma(E)$ are the eigenphase shifts. Evaluation of the integral

$$\frac{1}{2\pi i}\left\{\int_{C'}\frac{D'(\xi)}{D(\xi)}d\xi+\sum_i\int_{C_j}\frac{D'(\xi)}{D(\xi)}d\xi\right\}$$
$$=\frac{1}{2\pi i}\int_{m+\mu}^{\infty}\frac{d}{dE}\{\ln D(E+i\eta)-\ln D(E-i\eta)\}dE, \tag{81}$$

where the contours C' and C_j have been defined pre-

[25] F. J. Dyson, Phys. Rev. **106**, 157 (1957).

[26] The reader will recognize this as the generalized N/D solution of J. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

viously, then leads to the relation[27]

$$\sum_\gamma[\delta_\gamma(M+\mu_\gamma)-\delta_\gamma(\infty)]=(n-n_0)\pi, \tag{82}$$

where n is the number of stable V-particle states and n_0 is the number of bare V particles.

Furthermore, $D(E)$ can be written as

$$D(E)=\prod_{\gamma=1}^{m} d_\gamma(E), \tag{83}$$

where the $d_\gamma(E)$ are the eigenvalues of $D(E)$. It seems plausible that the $d_\gamma(E)$ can be ordered so that $d_\gamma(E)$ is regular in the E plane cut along the real axis from $m+\mu_\gamma$ to ∞, except perhaps for simple poles at $z=m_j^{(0)}$ $(j=1,\cdots,n^{(0)})$. However, the pole at $z=m_j^{(0)}$ will occur in one and only one of the $d_\gamma(E)$ (and generally in the one whose threshold lies lowest). The eigenphase shift $\delta_\gamma(E)$ is given by

$$\delta_\gamma(E)=\frac{1}{2i}\lim_{\eta\to 0^+}[\ln d_\gamma(E+i\eta)-\ln d_\gamma(E-i\eta)]. \tag{84}$$

Carrying out the usual contour integral yields

$$\delta_\gamma(m+\mu_\gamma)-\delta_\gamma(\infty)=(n_\gamma-n_{0\gamma})\pi, \tag{85}$$

where n_γ is the number of stable V particles associated with the V channel and $n_{0\gamma}$ is the number of bare V particles associated with the γ channel; in this model, it is possible to have $n_\gamma>n_{0\gamma}$.

In a fully relativistic field theory, the Fredholm determinant Eq. (68) does not exist, nor do the partial wave determinants defined by Eq. (69) exist. It may nonetheless be possible to factor the $D_L(E)$ according to

$$D_L(E)=\prod_\gamma c_\gamma d_{L\gamma}(E), \tag{86}$$

in such a way that $d_{L\gamma}(E)$ are well-defined, and possess the expected analyticity properties: regularity in the E plane with a cut from $E_\gamma^{(0)}$ to ∞, except for simple poles associated with the bare particles coupled to the γ channel. Then the eigenphase shifts $S_{L\gamma}(E)$ should be related to the $d_{L\gamma}(E)$ by Eq. (84), and the relation Eq. (85) provides a distinction (in principle) between elementary and composite particle.

V. DISCUSSION AND CONCLUSIONS

The purpose of this work has been to make it plausible (1) that a meaningful distinction between elementary and composite particles exists, even within the framework of an S-matrix theory, and (2) that a theory with a composite particle can be obtained as the limiting case of a theory with an elementary particle in which the wave-function renormalization constant of the particle tends to zero. While many of the results of this work have been discovered or conjectured previously, the present work serves to generalize these results, and emphasize their relevance to (1) and (2).

The distinction between elementary and composite particle is based on an extension of Levinson's theorem to field theory which is suggested by the discussion of Sec. IV. A rigorous proof for the relativistic case cannot be given on the basis of present knowledge; however, the validity of the theorem for the eigenphase shifts seems very plausible, since it depends only on the existence of suitable analytic functions $d_\gamma(E)$ from which the eigenphase shifts can be obtained by Eq. (84). It seems reasonable to hope that such functions $d_\gamma(E)$ with the desired analyticity properties exist[28]; perhaps it may even be true that the S matrix in field theory can be diagonalized by a simple transformation, and the $d_\gamma(E)$ may be determined from the requirements of analyticity, unitarity, and crossing symmetry.[29]

The conjectured extension of Levinson's theorem to the eigenphase shifts may prove a useful guide to understanding of certain problems in strong interaction physics. For example, if the 3–3 resonance in pion-nuclear scattering is a dynamical consequence of a fundamental interaction between pions and nucleons, the 3–3 eigenphase shift must approach zero at high energies. Is the apparent structure of the $T=\frac{3}{2}$ pion-nucleon cross section above 1 Bev related to a decrease of this phase shift through 90°? Are the Λ, Σ, Y^*,Y_0^* elementary particles, or the dynamical consequences of certain more fundamental interactions,[30] and what are the implications of the answers to these questions for the low-energy behavior of the coupled $\Lambda\pi-\Sigma\pi-\bar{K}N$ system? Is the p-wave resonance in pion-pion scattering[31] the dynamical consequence of fundamental interactions between the presently known elementary particles,[32] or the kinematical consequence of the existence of an elementary unstable vector boson?[33] These and similar questions have been asked before; the discussion of the present paper suggests strongly that there may be experimentally meaningful answers to them.[34]

[27] A similar relation has been conjectured by Kazes (reference 10); his proof by operator methods, however, requires the assumption of uniform convergence of certain integrals whose uniform convergences is precisely the point in question. See also reference 8.

[28] As was kindly pointed out to the authors by B. W. Lee, the analytic properties of the eigenvalues of the S matrix were discussed for a nonrelativistic problem with two coupled channels by Ning Hu, Phys. Rev. **74**, 131 (1948). A more recent treatment of the nonrelativistic multichannel S matrix, which unfortunately omits a discussion of the eigenamplitudes, is found in R. Newton, J. Math. Phys. **2**, 188 (1961).

[29] The spirit of this approach is exemplified in a recent review by G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289 (unpublished). The present authors would like to suggest that a study of the eigenamplitudes $S_j(E)$ will be fruitful.

[30] Such as those suggested by J. Sakurai, Ann. Phys. **11**, 1 (1960).

[31] J. Anderson *et al.*, Phys. Rev. Letters **6**, 365 (1961); W. D. Walker *et al.*, Bull. Am. Phys. Soc. **6**, 311 (1961).

[32] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); see also J. Belinfante, Princeton University thesis, 1961; and Phys. Rev. **123**, 306 (1961).

[33] J. Sakurai, reference 30; B. W. Lee and M. T. Vaughn, Phys. Rev. Letters **4**, 578 (1960).

[34] It has also been suggested by Chew (reference 29); however, it is not sufficient merely to look for zeroes of the T matrix since one of the zeroes associated with the coupling to an elementary particle may, and in general does, occur at infinite energy.

Finally, the equivalence of a theory with a composite particle to the limiting case of a theory in which the composite particle is treated as elementary can be used to remove one of the difficulties encountered in the attempt to apply dispersion relation methods to rearrangement collisions.[15]

Note added in proof. Chew [Lawrence Radiation Laboratory Report (UCRL-9701)] has recently suggested that the requirements of analyticity and unitarity, together with the new postulate of maximal coupling for the strong interactions, may suffice to determine the masses and coupling constants of all the strongly interacting particles, and that, in this sense, none of these particles should be regarded as elementary. The discussion of this paper shows that the Lee model and its generalizations (which are prototypes of relativistic partial-wave dispersion relations) provide concrete examples in which the postulate of maximal coupling consistent with unitarity has an explicit meaning, and leads to observable consequences, e.g., in the Lee model, the V particle is not elementary.

We also note that even if some of the strongly interacting particles are not elementary, it is permissible to regard them as such in treating weak and electromagnetic interactions by standard field theory, since the discussion given here indicates that a composite particle is equivalent to an elementary particle with vanishing wave function renormalization constant.

APPENDIX

Scattering and reaction processes involving bound states have been studied extensively using the techniques of formal scattering theory.[35] There are, however, a number of shortcomings in this approach; for example, no convergent iteration procedure seems to exist.[36] Also, there are difficulties connected with the symmetry between particles in projectile and target, which can be overcome for computational purposes by devices such as the distorted wave approximation,[37] but which remain as an obstacle to the formal understanding of the theory.

Similar problems have been met and partially solved in elementary-particle physics by the use of dispersion relation methods, but application of these techniques to nonrelativistic problems[15] is hindered by certain unpleasant facts. For example, in the study of deuteron stripping reactions by these methods, one is led to consider a matrix element of the form $\langle \text{vac}|J_d|np(+)\rangle$, where J_d is the current operator associated with the deuteron field, and $|np(+)\rangle$ is a neutron-proton scattering state. However, if the deuteron is regarded as the bound state of a neutron and a proton interacting through a nonrelativistic potential, this matrix element vanishes identically.[38,39] On the other hand, it does not vanish if the deuteron is treated as an elementary particle coupled to the neutron and proton by a Yukawa-type interaction, even in the limit when the wave function renormalization constant of the deuteron vanishes. This suggests that a suitable method for handling this problem, and in fact a large class of problems involving bound states (e.g., rearrangement collisions), is to treat the bound state as an elementary particle, perform the calculations of interest, and take the limit of vanishing wave function renormalization at the end of the calculation. The method is elucidated here by an example, but the application of the method to problems of physical interest is reserved for a subsequent publication.

Consider the models I and II of a scalar boson θ interacting with a fixed source N; the models are defined by the Hamiltonians

$$H(\mathrm{I})=H_S+V, \tag{A1}$$

$$H(\mathrm{II})=H_L+V, \tag{A2}$$

where H_S is the Hamiltonian of the separable potential model [Eq. (14)] and H_L is the Lee-model Hamiltonian [Eq. (1)]; V is a direct $N-\theta$ interaction of the form

$$V=\frac{1}{(2\pi)^3}\int \frac{V(\mathbf{k}',\mathbf{k})}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}a^\dagger(\mathbf{k}')a(\mathbf{k})d^3k d^3k', \tag{A3}$$

where $V(\mathbf{k}',\mathbf{k})=V(k,k')$ is a scalar function of $\mathbf{k}$, $\mathbf{k}'$.

Now let[40]

$$H_1=\int \omega_k a^\dagger(\mathbf{k})a(\mathbf{k})d^3k+V, \tag{A4}$$

and let $|Nk(\pm))_1$ be the scattering eigenstates of H_1 [with outgoing $(+)$ or incoming $(-)$ wave boundary conditions] associated with an incident θ particle of momentum $\mathbf{k}$; these states satisfy

$$|N\mathbf{k}(\pm))_1=a^\dagger(\mathbf{k})|N\rangle+\frac{1}{E_k-H_1\pm i\epsilon}j_1(\mathbf{k})|N\rangle, \tag{A5}$$

[35] See, for example, a recent review by S. Sunakawa, Progr. Theoret. Phys. (Kyoto) **24**, 980 (1960).

[36] R. Aaron, R. Amado, and B. Lee, Phys. Rev. **121**, 319 (1961).

[37] See, for example, reference 35.

[38] P. Redmond and J. Uretsky, Ann. Phys. **9**, 106 (1960). The creation operator for the deuteron has the form

$$\Psi_d(\mathbf{X},t)=\int \phi_d(\boldsymbol{\rho})\psi_n^\dagger(\mathbf{X}+\tfrac{1}{2}\boldsymbol{\rho},t)\psi_p^\dagger(\mathbf{X}-\tfrac{1}{2}\boldsymbol{\rho},t)d^3\rho,$$

where $\psi_n^\dagger$, $\psi_p^\dagger$ are neutron and proton creation operators, and $\phi_d(\boldsymbol{\rho})$ is the internal wave function of the deuteron. The deuteron current operator $J_d^\dagger(X,t)$ defined by

$$J_d^\dagger(\mathbf{X},t)=[i\partial/\partial t+\tfrac{1}{2}\nabla_X^2-E_d]\Psi_d^\dagger(\mathbf{X},t)$$

consists of a sum of terms, each of which has at its right a ψ_p or a ψ_n; since there are no antinucleons in the theory, these operators annihilate the vacuum, and thus the matrix element $\langle np(+)|J_d^\dagger|\text{vac}\rangle=0$. It is of course possible that this matrix element vanishes in a relativistic theory, but the present argument does not suffice to show it.

[39] Of course the matrix element $\langle n|j_p|d\rangle$, where j_p is the proton current, does not vanish; this matrix element can in fact be used to define the wave function. See R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

[40] The procedure followed here is essentially that of S. D. Drell and F. Zachariasen, Phys. Rev. **105**, 1407 (1957).

where $j_1(\mathbf{k})\equiv[H,a^\dagger(\mathbf{k})]-\omega_k a^\dagger(\mathbf{k})$. Also, the creation and annihilation operators $b^{(\pm)\dagger}(\mathbf{k})$, $b^{(\pm)}(\mathbf{k})$ defined by

$$b^{(\pm)\dagger}(\mathbf{k})|N\rangle=|N\mathbf{k}(\pm))_1;\quad b^{(\pm)}(\mathbf{k})|N\rangle=0 \quad \text{(A6)}$$

satisfy the usual commutation rules (it is tacitly assumed here and below that H_1 has no discrete eigenstates). The configuration space wave function $\psi^{(\pm)}(\mathbf{k},\mathbf{x})$ associated with the scattering state $|N\mathbf{k}(\pm))_1$ is defined by

$$\psi^{(\pm)}(\mathbf{k},\mathbf{x})=[\phi(\mathbf{x}),b^{(\pm)}(\mathbf{k})], \quad \text{(A7)}$$

where

$$\phi(\mathbf{x})=(2\pi)^{-\frac{3}{2}}\int[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}]\times(2\omega_k)^{-\frac{1}{2}}d^3k \quad \text{(A8)}$$

is the θ-particle field operator.

$H(\mathrm{I})$ and $H(\mathrm{II})$ can now be written as

$$H(\mathrm{I})=\int\omega_k b^{(-)\dagger}(\mathbf{k})b^{(-)}(\mathbf{k})d^3k+\frac{\lambda}{(2\pi)^3}\int\frac{\rho^*(\mathbf{k}')\rho(\mathbf{k})}{(4\omega_k\omega_{k'})^{\frac{1}{2}}}b^{(-)\dagger}(\mathbf{k}')b^{(-)}(\mathbf{k})d^3\mathbf{k}, \quad \text{(A9)}$$

$$H(\mathrm{II})=m\psi_N{}^\dagger\psi_N+m_V{}^{(0)}\psi_V{}^\dagger\psi_V+\int\omega_k b^{(-)\dagger}(\mathbf{k})b^{(-)}(\mathbf{k})d^3k+\frac{g^{(0)}}{(2\pi)^{\frac{3}{2}}}\int[\rho(\mathbf{k})\psi_V{}^\dagger\psi_N b^{(-)}(\mathbf{k})+\rho^*(\mathbf{k})\psi_V\psi_N{}^\dagger b^{(-)\dagger}(\mathbf{k})]\frac{d^3k}{(2\omega_k)^{\frac{1}{2}}}, \quad \text{(A10)}$$

with

$$\rho(\mathbf{k})\equiv\frac{1}{(2\pi)^{\frac{3}{2}}}\int u(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\psi^{(-)}(\mathbf{k},\mathbf{x})d^3x d^3q, \quad \text{(A11)}$$

where $u(q)$ is the cutoff function associated with the Hamiltonian (A1) or (A2); if $u(\mathbf{q})$ is spherically symmetric, $\rho(\mathbf{k})$ is also spherically symmetric, and that is supposed to be the case here.

The scattering states $|N\mathbf{k}(\pm)\rangle_X$ of $H(X)$ ($X=\mathrm{I}$, II) can be written as[40]

$$|N\mathbf{k}(\pm)\rangle_X=b^{(\pm)}(\mathbf{k})|N\rangle+[E_k-H(X)\pm i\epsilon]^{-1}j_X{}^{(\pm)}(\mathbf{k})|N\rangle, \quad \text{(A12)}$$

where

$$j_X{}^{(\pm)}(\mathbf{k})\equiv[H(X),b^{(\pm)}(\mathbf{k})]-\omega_k b^{(\pm)}(\mathbf{k}),$$

and the S-matrix elements for $N-\theta$ scattering can be written as

$$\langle N\mathbf{k}'(-)|N\mathbf{k}(+)\rangle_X={}_1(N\mathbf{k}'(-)|N\mathbf{k}(+))_1-2\pi i\delta(E_{k'}-E_k)\langle N\mathbf{k}'(-)|j_X{}^{(+)}(\mathbf{k})|N\rangle. \quad \text{(A13)}$$

Now

$$\langle N\mathbf{k}'(-)|j_X{}^{(+)}(\mathbf{k})|N\rangle=\int\langle N\mathbf{k}'(-)|j_X{}^{(-)}(\mathbf{k}'')|N\rangle S_1(\mathbf{k}'',\mathbf{k})d^3k'', \quad \text{(A14)}$$

where $S_1(\mathbf{k}'',\mathbf{k})_1\equiv{}_1(N\mathbf{k}'(-)|N\mathbf{k}(+))_1$; applying the methods of Sec. II shows that

$$\langle N\mathbf{k}'(-)|j_X{}^{(-)}(\mathbf{k}'')|N\rangle=\frac{4\pi}{(2\pi)^3}\frac{\rho^*(k')\rho(k'')}{(4\omega_{k'}\omega_{k''})^{\frac{1}{2}}}h_X(\omega_{k'}), \quad \text{(A15)}$$

where $h_X(\omega_{k'})$ is the boundary value of an analytic function $h_X(z)$ of the complex variable z with the now familiar analyticity properties; the functions $h_{\mathrm{I}}(z)$ and $h_{\mathrm{II}}(z)$ are given by Eq. (17) and Eq. (9), respectively, with $u(k)$ replaced by $\rho(k)$. The discussion of Sec. II shows that $h_{\mathrm{II}}(z)\rightarrow h_{\mathrm{I}}(z)$ is the limit of vanishing V-particle wave function renormalization constant; hence also

$$\langle N\mathbf{k}'(-)|N\mathbf{k}(+)\rangle_{\mathrm{II}}\rightarrow\langle N\mathbf{k}'(-)|N\mathbf{k}(+)\rangle_{\mathrm{I}} \quad \text{(A16)}$$

in the same limit [the integrand of Eq. (A14) contains a δ function of the energy, so $h(z)$ need not converge uniformly in z].

Now suppose a scalar boson interacts with a fixed source through a local potential $V(r)$; let $V(\mathbf{k}-\mathbf{k}')$ be the Fourier transform of the potential. Suppose there exists one s-wave bound state with energy E_B, and momentum space wave function $\varphi(k)$. Let $V_S(\mathbf{k}',\mathbf{k})$ be the separable potential which has a bound state with the same energy and wave function; Eq. (21) shows that this can be arranged by taking the cutoff function associated with the separable potential to be

$$u(k)=N(\omega_k-E_B)\varphi(k), \quad \text{(A17)}$$

where N is a normalization constant.

The corresponding models I and II have Hamiltonians $H(\mathrm{I})$ and $H(\mathrm{II})$ given by Eqs. (A1)–(A3), with

$$V(\mathbf{k}',\mathbf{k})=V(\mathbf{k}-\mathbf{k}')-V_S(\mathbf{k}',\mathbf{k}). \quad \text{(A18)}$$

The preceding discussion justifies the replacement of $H(\mathrm{I})$ by $H(\mathrm{II})$ for calculational purposes, provided the limit of vanishing wave function renormalization is taken at the end of the calculation. Note that the intermediate Hamiltonian, Eq. (A4), of this model has no discrete eigenstates, since that would imply the existence of a pole in the S matrix associated with H_1; however, the bound-state pole is known to be contained in the function $h(z)$.

The results of this Appendix can easily be generalized to include bound states in other partial waves, theories with spin, etc.