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Behavior of Nucleon-Nucleon Singlet Phase Shifts*

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By applying a conformal mapping to the partial wave scattering amplitudes, it is possible to confine the entire unphysical branch cut in the amplitude to a compact region. This allows us to introduce a set of expansion functions for approximating the behavior of the amplitude in the physical region. With these functions we can correlate the observed energy dependence of the phase shifts with that of the imaginary part of the amplitude along the unphysical cut. In this way we show that the $^1S_0$ energy dependence seems to require a short-range repulsive core. The attractive long-range forces tend to give a positive shape parameter and the hard core a negative one, so the actual sign can go either way. We present a two-parameter $^1S_0$ fit up to 300 Mev, and for the $^1D_2$ phase shift a one-parameter fit accurate between 40 and 250 Mev. We argue that inelastic scattering is small below 250 Mev, and also that a good fit to scattering data up to 250 Mev does not accurately determine the behavior of the amplitude in the unphysical region.

THE advent of double dispersion relations[1] has given a new impetus to work on the nucleon-nucleon problem. This approach leads to integral equations for the partial wave scattering amplitudes, which can be treated either exactly,[2] leading to complicated numerical calculations, or extremely simply. In the simple approach the singularities in the amplitude in the unphysical region are replaced by one or several poles,[3] leading to an approximate qualitative description of the behavior in the physical region.

In this work we propose an intermediate alternative, which preserves the analytic features of the amplitude and at the same time provides a "model" by which the system can be treated, and which shows a direct correlation between the phenomenological features of the scattering one wishes to explain and the associated behavior of the amplitudes in the unphysical region. The model is also capable of providing a quantitative fit to the nucleon-nucleon scattering data.

In the Mandelstam representation the partial wave amplitudes $h^{(l)}(\nu)$ are analytic in the $\nu=q^2$ (in units of the pion rest mass) plane except for two branch lines along the real axis, one for $\nu\geqslant 0$ and the other for $\nu\leqslant -\frac{1}{4}$ (see Fig. 1a). One can write[4] $h^{(l)}(\nu)=N^{(l)}(\nu)/D^{(l)}(\nu)$, where $N^{(l)}(\nu)$ is analytic except for the left-hand cut $\nu\leqslant -\frac{1}{4}$, and $D^{(l)}(\nu)$ is analytic except for the right-hand cut $\nu\geqslant 0$. One also knows that $\mathrm{Im}h^{(l)}(\nu)=-\nu^{\frac{1}{2}}R^{(l)}(\nu)/m$ for $\nu\geqslant 0$, where $R^{(l)}(\nu)=\sigma_{\mathrm{tot}}(\nu)/\sigma_{\mathrm{el}}(\nu)$ and is equal to unity below the pion production threshold (m is the nucleon mass, ≈ 6.7). We shall work nonrelativistically, although it is a simple matter to make this approach relativistic.[5]

Assuming that $N^{(l)}(\nu)\rightarrow 1/\nu$, while $D^{(l)}(\nu)$ is bounded, Cauchy's theorem then states that

$$N^{(l)}(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\frac{1}{4}} d\nu' \frac{f^{(l)}(\nu')D^{(l)}(\nu')}{\nu'-\nu}, \tag{1}$$

$$D^{(l)}(\nu)=1-\frac{\nu}{\pi m}\int_{0}^{\infty} d\nu' \,(\nu')^{\frac{1}{2}}\frac{N^{(l)}(\nu')R^{(l)}(\nu')}{\nu'(\nu'-\nu)}, \tag{2}$$

where $f^{(l)}(\nu)=\mathrm{Im}h^{(l)}(\nu)$, $\nu\leqslant -\frac{1}{4}$, and a subtraction at $\nu=0$ has been made in $D^{(l)}(\nu)$, which has been normalized to $D^{(l)}(0)=1$. In the following we shall ignore inelastic scattering[5] and assume $R^{(l)}(\nu)=1$. These formulas ignore any coupling between channels[6] and we shall apply them only to singlet states.

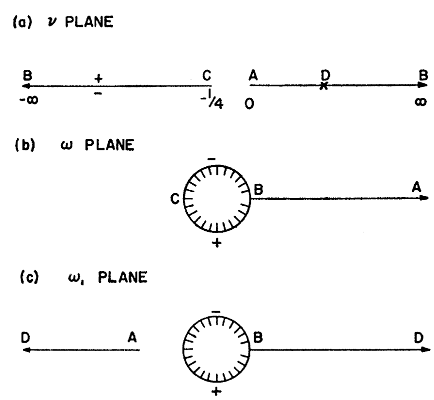

FIG. 1. Singularities in the partial-wave amplitudes in the ν, ω, and ω_1 planes.

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 and 1752 (1959). The discussion in G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289, 1960 (unpublished) is adequate for the purposes of this paper.

[2] M. L. Goldberger, M. Grisaru, S. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960). D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68 (1960); **18**, 409 (1960); *ibid.* p. 458. M. Cini and S. Fubini, Ann. Phys. **10**, 352 (1960).

[3] G. Chew, reference 1. The one-meson contribution can be treated exactly and higher singularities replaced by poles. See H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959). H. P. Noyes, Phys. Rev. **119**, 1736 (1960).

[4] S. Mandelstam (unpublished).

[5] In the energy region we will be considering, inelastic and relativistic effects are small. Upper limits have been established by D. Y. Wong (unpublished), according to H. P. Noyes.

[6] For the coupled amplitudes, it is the helicity amplitudes which have only the singularities prescribed by unitarity (Goldberger *et al.*[2]).

Our approach to the solution of these equations is to exploit an earlier suggestion[7,8] to perform a conformal mapping of the ν plane with the purpose of condensing the extended singularity in $N^{(l)}(\nu)$ into a compact region so that one might then develop a systematic approximation scheme for $N^{(l)}(\nu)$. Also, by approximating $N^{(l)}(\nu)$ rather than $f^{(l)}(\nu)$, we have only to integrate Eq. (2), rather than solve the coupled integral equations.

CONFORMAL MAPPING

Consider the successive mappings $\nu'=(\nu+\frac{1}{4})^{\frac{1}{2}}$, $-\pi<\arg\nu<\pi$, and $\omega=1/(\nu'-\frac{1}{2})$. In the ω plane, the left-hand cut has been transformed into a circle of radius 1, centered at $\omega=-1$ [Fig. 1(b)], while the low-energy physical region has been pushed out to $\omega=+\infty$. With this geometry $N^{(l)}(\omega)$ is singular only on the circle, and the procedure that suggests itself is to approximate $N^{(l)}(\omega)$ by a series of poles at the center of the circle. Dropping the superscript (l), we write

$$N(\omega)=\Gamma\frac{\omega^2}{(\omega+1)^2}\left[1+\frac{a_1}{\omega+1}+\cdots+\frac{a_k}{(\omega+1)^k}+\cdots\right]$$
$$=\Gamma[N_0(\omega)+a_1N_1(\omega)+\cdots+a_kN_k(\omega)+\cdots]. \quad (3)$$

This choice makes each $N_k(\omega)$ vanish as ω^2 at the origin [i.e., $N_l(\nu)\to 1/\nu$ as $\nu\to\infty$], while only the first term contributes at $\omega=\infty$ (or $\nu=0$). This map has the advantages of preserving the analytic properties of the amplitude and of removing the low-energy physical region to very far from the unphysical cut. There is no special reason for placing the poles at the center of the circle, except extreme convenience in performing the the integrals, and it does not affect our qualitative arguments. In fact, if we had placed them closer to the origin, the series would converge faster, for reasons to be discussed later.

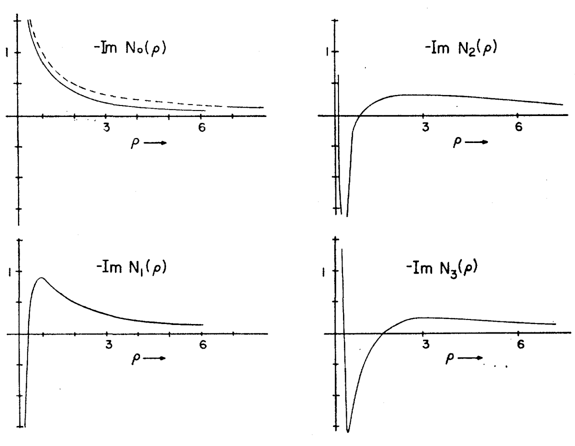

FIG. 2. Behavior of the functions $\mathrm{Im}N_k(\rho)$, $\rho=-\nu$, along the unphysical cut. The dashed curve is the exact one-meson contribution. For each k, the curve crosses the axis k times, and the low-energy oscillations have been left out.

From Eq. (2) it follows that

$$\mathrm{Re}D(\nu)=1-\frac{1}{\pi}\frac{\Gamma}{m}\sum_{k=0}^{\infty}a_kI_k(\nu), \quad a_0=1,$$
$$I_k(\nu)=\nu\mathrm{P}\int_0^{\infty}\frac{d\nu'}{(\nu')^{\frac{1}{2}}}\frac{N_k(\nu')}{\nu'-\nu}, \quad \text{for} \quad \nu>0, \quad (4)$$

where P denotes the principal value of the integral. The first few integrals are given below:

$$I_0=-8/3-2/\nu+\varphi(\nu),$$
$$I_1=-16/15-20/3\nu-2/\nu^2+(2+1/\nu)\varphi(\nu),$$
$$I_2=-24/35-62/5\nu-32/3\nu^2-2/\nu^3+(3+4/\nu+1/\nu^2)\varphi(\nu), \quad (5)$$
$$I_3=-32/63-1976/105\nu-476/15\nu^2-44/3\nu^3-2/\nu^4+(4+10/\nu+6/\nu^2+1/\nu^3)\varphi(\nu),$$

where

$$\varphi(\nu)=\frac{2(\nu+\frac{1}{4})^{\frac{1}{2}}}{\nu^{\frac{3}{2}}}\{\ln[(\nu+\tfrac{1}{4})^{\frac{1}{2}}+\tfrac{1}{2}+\sqrt{\nu}]-\ln[(\nu+\tfrac{1}{4})^{\frac{1}{2}}+\tfrac{1}{2}-\sqrt{\nu}]\}, \quad \nu>0.$$

At very low energy,

$$I_k(\nu)\to 2\left(\frac{1}{2k-1}-\frac{1}{2k+1}-\frac{1}{2k+3}+\frac{1}{2k+5}\right)\nu. \quad (6)$$

This formula is also valid for $k=0$.

The discontinuity in $N_k(\rho)$ in the unphysical region $\rho\geqslant\frac{1}{4}$, where $\rho=-\nu$, is given by

$$\mathrm{Im}N_k(\rho)=(-)^{k+1}\sin[2(k+1)\theta]/\rho,$$
$$\tan\theta=(4\rho-1)^{\frac{1}{2}}. \quad (7)$$

As ρ varies from $\frac{1}{4}$ to ∞, θ goes from 0 to $\pi/2$, so that $\mathrm{Im}N_k(\rho)$ changes sign $(k-1)$ times. In the unphysical region, the contribution of each $I_k(\rho)$ to $D(\rho)$ is slowly varying and monotonic, as can be seen from Eq. (4), since $N_k(\nu)>0$ for $\nu>0$. Therefore the behavior of $f(\rho)=D(\rho)\,\mathrm{Im}N(\rho)$ is determined almost exclusively by the $N_k(\rho)$. The functions $\mathrm{Im}N_k(\rho)$, for the few lowest values of k, are plotted in Fig. 2. Note that $\mathrm{Im}N_0(\rho)$ closely resembles $1/\rho$, the one-meson contribution to the amplitude, except that it falls off as $1/\rho^{\frac{3}{2}}$.

The phase shift δ_0 for $\nu>0$ is given by

$$k\cot\delta_0=\mathrm{Re}(D^{(0)}/N^{(0)})$$
$$=\left[\frac{m}{\Gamma}-\frac{1}{\pi}\sum_{k=0}^{\infty}a_kI_k(\nu)\right]\Big/\left[\sum_{k=0}^{\infty}a_kN_k(\nu)\right], \quad a_0=1. \quad (8)$$

At low energies this can be approximated by

$$k\cot\delta_0=1/a+\tfrac{1}{2}r_0k^2-P_0r_0^3k^4+\cdots. \quad (9)$$

[7] D. M. Greenberger and B. Margolis, Phys. Rev. Letters **6**, 310 (1960).

[8] W. Frazer, Phys. Rev. **123**, 2180 (1961). His mapping, while superficially very different from ours, is actually rather similar.

For higher partial waves, $N^{(l)}(\nu)$ behaves as ν^l for low energies, while the functions $N_k(\nu)$ behave as ν^k, so that the lowest contributing function to $N^{(l)}(\nu)$ will be $N_l(\nu)$. Thus

$$k\cot\delta_l=\left[\frac{m}{\Gamma}-\frac{1}{\pi}\sum_{k=l}^{\infty}a_kI_k(\nu)\right]\Big/\left[\sum_{k=l}^{\infty}a_kN_k(\nu)\right],\quad a_l=1. \qquad (10)$$

${}^1S_0$ PHASE SHIFT

If we try to fit the ${}^1S_0$ phase shift with only the first function $N_0(\nu)$, and choose Γ to give the right scattering length ($1/a=m/\Gamma\sim0.07$), we find too large an effective range, about 4 rather than 1.9. So we next try a two-parameter fit

$$N^{(0)}=\Gamma(N_0+a_1N_1). \qquad (11)$$

Now the effective range is determined in our approach primarily by the shape of $I_k(\nu)$ near $\nu=0$, as given by Eq. (6), and the slope of $I_0(\nu)$ has the opposite sign to that of all the other $I_k(\nu)$, $k>0$, from which it follows that a_1 must be positive in order to decrease r_0. This has the interesting consequence, if one looks at $\mathrm{Im}N^{(0)}(\rho)$ (where $\rho=-\nu$) along the unphysical cut, that the second term tends to cancel the first near $\rho=\frac{1}{4}$, but enhances it further out. This tends to support the common view that the two-pion contribution plays the dominant role[9] in the low-energy ${}^1S_0$ behavior in the sense that it determines the strength of the potential, while the one-meson contribution primarily determines the long-range tail. $\mathrm{Im}N^{(0)}(\rho)$ for this two-parameter fit is plotted in Fig. 3(a), and $k\cot\delta_0$ is plotted[10] in Fig. 4(a). We find $a_1=1.19$ in Eq. (11).

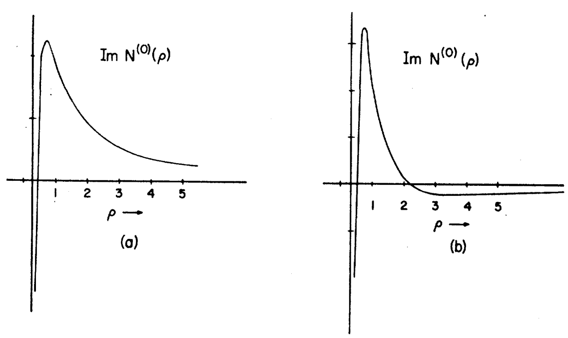

FIG. 3. Imaginary part of ${}^1S_0$ scattering amplitude along the unphysical cut ($\rho=-\nu$): (a) for two-parameter fit, (b) for three-parameter fit. The ordinate is drawn with arbitrary units.

Because of the nature of our functions, the tendency to minimize the importance of the very near unphysical region, $\rho\sim\frac{1}{4}$, is represented by rather wild oscillations in this region when only a few functions are used. If the one-pion contribution were truly most important, then we should have found $a_1\ll1$. In fact, our functions are useful primarily because they oscillate more rapidly for small ρ than for large ρ. Otherwise, their decreased magnitude at large ρ would prevent any significant large-ρ contribution to the physical region. Frazer's functions[8] have this same property, and we believe any other reasonable set of functions would also.

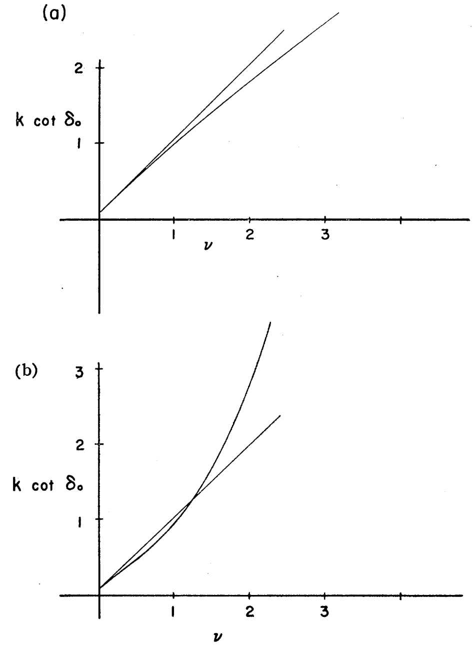

FIG. 4. Plot of $k\cot\delta_0$: (a) for two-parameter fit, (b) for three-parameter fit. The straight line is the effective range approximation.

Notice that the shape parameter determined by the two-parameter fit is positive, as found by all double dispersion relation calculations to date,[11] and also by all calculations on potentials with a tail,[12] and it would seem to be an unambiguous conclusion, if there were no high-curvature short-range effects or hard core.

Next we add a third parameter to help reproduce the high-energy behavior of δ_0. The prominent experimental feature is that δ_0 decreases monotonically and passes through zero at about 240 Mev, or $\nu=6$ (see Fig. 5). This is equivalent to $k\cot\delta_0\rightarrow\infty$ at $\nu=6$. Thus our third parameter can be determined by the requirement that $N^{(0)}$ develops a zero at $\nu=6$. It is rather drastic to expect one extra parameter to completely determine the high-energy behavior; however certain qualitative features are brought out by the attempt. We thus write

$$N^{(0)}=\Gamma(N_0+a_1N_1+a_nN_n). \qquad (12)$$

[9] It is true in perturbation calculations of meson potentials. See, for example, S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

[10] Our phase shift data are taken from the articles by M. MacGregor, M. Moravcsik, and H. Stapp, chapter in *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1960), Vol. 10; and M. MacGregor, M. Moravcsik, and H. P. Noyes, University of California Radiation Laboratory Report UCRL-6375-*T*, 1961 (unpublished).

[11] M. Cini, S. Fubini, and A. Stanghellini, Phys. Rev. **114**, 1633 (1959); Wong and Noyes, reference 3; and references 7 and 8.

[12] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, Chap. I.

Since for all k, $N_k(\nu)>0$ for $\nu>0$, a_k must be negative to give $N^{(0)}(6)=0$. If we put $n=2$ in Eq. (12), then $k\cot\delta_0$ develops a large hump at low energy, a completely wrong qualitative behavior. However, for $n=3$, we find $a_1=1.36$, $a_3=-6.44$, and for this case $k\cot\delta_0$ is plotted in Fig. 4(b) and $\mathrm{Im}N^{(0)}$ in Fig. 3(b).

Looking at Fig. 3(b), we see that after the oscillations near $\rho\sim\frac{1}{4}$, which tend to cancel, there is a further enhancement of the peak near $\rho=1$, and now a negative tail has developed in the high-ρ region. Thus by forcing the phase shift to go to zero at $\nu=6$, the $\mathrm{Im}N^{(0)}$ displays a behavior analogous to that of a hard core in potential theory. The maximum contribution to the "core" comes at about $\rho=5$, which would correspond to a potential core of about 0.3 fermi, using the naive rule of thumb[1] that the range of interaction is $1/(4\rho)^{\frac{1}{2}}$. Thus the wrong behavior for $n=2$ can be explained by its short-range attractive core, while the correct qualitative behavior for $n=3$ comes from its repulsive core.

The contribution of $N_3(\nu)$ tends to decrease the shape parameter at low energy, and had it come in a little more strongly, it would have made P_0 negative. A further result that agrees with potential calculations is that we could have chosen any higher odd value of n, with the result that each higher value would correspond to a shorter range core and at the same time have a smaller effect on the shape parameter. Thus while increasing the size or strength of the core tends to make the shape parameter more negative, the existence of a core does not seem to prohibit a positive shape parameter.[13]

We are inclined to conclude from these results that one can get a good approximate over-all behavior for the $^1S_0$ phase shift and yet learn nothing about the shape parameter, which seems to depend critically on the balancing of short- and long-range forces. Its most important general feature seems to be that it is small, and because of this, various small non-nuclear effects may have as strong a bearing on its actual value as specifically nuclear ones.

ALTERNATIVE MAPPING

The above mapping has been used because its mathematical convenience makes it possible to follow the effect of varying the parameters involved. To illustrate the point that it is possible to obtain a mapping which leads to a more rapidly converging series, we choose a map that puts the point $\nu=6$ at $\omega=\infty$. Then letting $N(\omega)\to 0$ for $\omega\to\infty$ will ensure that the phase shift goes through zero at that point. The map is

$$\omega_1=1/[(\nu+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{5}{2}], \tag{13}$$

and we choose

$$N_0(\omega_1)=\Gamma\frac{\omega_1^2}{(\omega_1+\frac{1}{5})^3}\left(1+\frac{a}{\omega_1+\frac{1}{5}}\right). \tag{14}$$

[13] This result agrees with that of J. K. Perring and R. J. N. Phillips, Nuclear Phys. 23, 153 (1961).

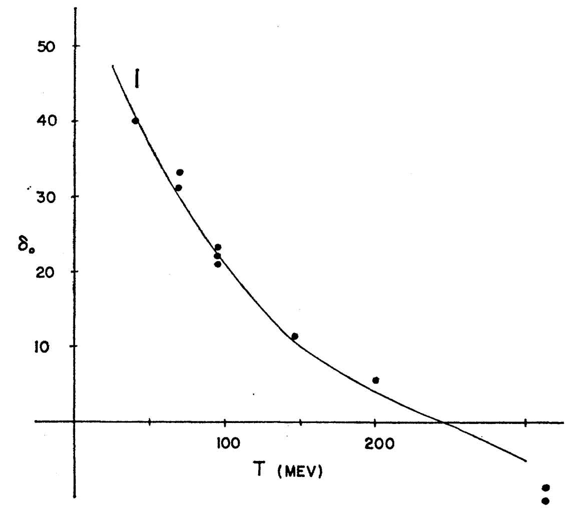

FIG. 5. Plot[10] of δ_0, as obtained from the alternate mapping ω_1.

The singularities in the ω_1 plane are plotted in Fig. 1(c). Again the poles are placed at the center of the circle $(\omega_1=-\frac{1}{5})$. With the value $a=0.15$, this two-parameter function gives an accurate fit over the entire range $0\leqslant\nu\leqslant 6$, as can be seen from Fig. 5. This mapping gives a very small but negative shape parameter, which we do not consider significant, and one can check that the same qualitative statements can be made as for our previous mapping, though the calculations are tedious and will not be presented here. The reason for the more rapid convergence of this series is that the imaginary part of the first function has an attractive maximum in the two-pion exchange region, $1<\rho\lesssim 2$, $\rho=-\nu$, and a repulsive maximum at around $\rho=10$, and therefore subsequent terms serve only as perturbations on this. A consequence is that the large oscillations of the original mapping do not appear. This mapping is equivalent to the original one with the poles located not at the center of the circle, but closer to the origin.

$^1D_2$ PHASE SHIFT

To discuss D waves we shall return to the original mapping. According to the discussion following Eq. (9), we can expect that

$$N^{(l)}(\omega)=\Gamma[N_l(\omega)+a_{l+1}N_{l+1}(\omega)+\cdots]. \tag{15}$$

The first term in this equation goes as ν^l for $\nu<\frac{1}{4}$. Above $\nu=\frac{1}{4}$, the term N_{2l} takes over the behavior ν^l. Since there are very few available data below $\nu\sim 1$ (about 40 Mev), we propose to try the one-parameter fit

$$N^{(2)}(\omega)=\Gamma N_4(\omega). \tag{16}$$

This function predominates over $N_2(\omega)$ in the region $\nu>1$, and it will only be at lower energies that $N_2(\omega)$ would be felt, and indeed be dominant.

A plot of δ_2 is drawn in Fig. 6. Notice that this one-parameter function gives an almost perfect fit up to

$\nu=6$ ($\sim$240 Mev). There are two further interesting consequences of this fit. First the integral determining $D^{(2)}(\omega)$ is very small and contributes less than 10% of the constant term in Eq. (4). Thus in the region $1<\nu<6$, $k\cot\delta_2$ has approximately the simple functional form

$$k\cot\delta_2 \simeq (m/\Gamma)/N_4(\omega). \tag{17}$$

Since the integral $I_4(\omega)$ contains all the inelastic contributions to the amplitude, and since this integral contributes very little to the amplitude (and the calculated form fits the data), we believe that this furnishes independent evidence for the statement that inelastic scattering is unimportant[5] up to at least 240 Mev.

Incidentally, we have found that a high order pole approximation, namely

$$N^{(2)}(\nu)=\Gamma\nu^2/(\nu+\nu_0)^3, \tag{18}$$

gives as good results in this energy range, though it is a two-parameter fit (with $\nu_0\sim 2.7$). In the unphysical region this function, Eq. (18), gives $\mathrm{Im}N^{(2)}(\nu)\propto\delta''(\nu+\nu_0)$ which perhaps represents the fact that the D-wave amplitude must oscillate there.

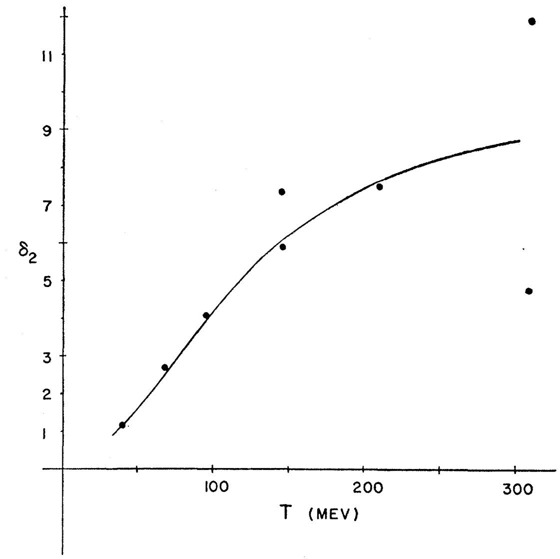

FIG. 6. Plot[10] of the $^1D_2$ phase shift δ_2, calculated from the one-parameter function Eq. (16).

CONCLUSIONS

We have shown that by performing a conformal mapping of the partial-wave amplitudes which preserves the unphysical cut, it is possible to introduce a set of functions to represent these amplitudes. These functions have the property that when chosen to produce a desired behavior for the amplitude in the physical region, they tend also to produce a reasonable behavior along the unphysical cut.

In particular, we have shown that for S waves one can qualitatively see one- and two-meson attractive effects, and the effect of a hard core. For D waves, it is possible in the region $1<\nu<6$ to find a one-parameter fit to the phase shift over the entire region, and to argue that inelastic contributions are small below 250 Mev.

It might be thought that a knowledge of the phase shift would yield the imaginary part of the scattering amplitude along the unphysical cut, which in turn could be used to construct[14] a nuclear potential; however we have been able to construct different functions which closely approximate the behavior of the phase shifts over fairly extensive regions, but whose imaginary parts have quite different detailed behavior, though they show the same general characteristics. Thus the same problems that plague the potential approach are mirrored in a corresponding form in this approach.

ACKNOWLEDGMENTS

We would like to thank Dr. H. P. Noyes and Dr. Wm. Frazer for making some results of theirs available to us prior to publication.

[14] See, for example, A. Martin, Nuovo cimento **19**, 1257 (1961).