Generalization of Singer's Formula for the General Relativistic Red Shift to Elliptic Satellite Orbits

SJUR REFSDAL
Institute for Theoretical Physics, University of Oslo, Blindern, Norway
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The average rate difference between a clock on the earth and a clock in an "elliptic satellite" is found to be dependent only on the average specific kinetic energy $\bar{T}$ of the satellite and the average gravitational potential of the satellite, $\bar{\chi}_s$. By means of the virial theorem, $\bar{T}$ and $\bar{\chi}_s$ are shown to be dependent only on the major axis, $2a$, of the satellite orbit, giving the result:

$$\bar{\Delta}\equiv(\tau_s-\tau_E)/\tau_E=6.96\times10^{-10}(1-3r_E/2a),$$

where τ_s and τ_E are the periodic times read on the satellite clock and the earth clock, respectively, and r_E is the radius of the earth.

WINTERBERG[1] and Singer[2] have shown that, according to the general theory of relativity, the rate of a clock on the earth should be different from the rate of a similar clock on an artificial satellite. Assuming a circular orbit for the satellite, Singer[3] obtains

$$\Delta\equiv(dt_s-dt_E)/dt_E=6.96\times10^{-10}(1-3r_E/2r), \quad (1)$$

t_s and t_E being the times read on the satellite clock and the earth clock, respectively, r being the orbit radius. For elliptic orbits Δ is not constant. Nararai and Ueno[4] have found the dependence of Δ on the eccentricity and the eccentric anomaly of the ellipse. Because of errors in the position determination (about 30 m), it should, however, be necessary to measure over such a long time that Δ cannot be considered as constant during this time. The only measurable quantity should therefore be the time average of Δ over a certain time. Measuring over one period for a near satellite ($\tau\approx5000$ sec), the uncertainty in $\bar{\Delta}$ should be about $2\times30/3\times10^8\times5000=4\times10^{-11}$. The best atomic clocks available today are said to be accurate to better than 1 part in 10^{12}. This should indicate a measuring time of about 40 periods. It may therefore be of some interest to calculate $\bar{\Delta}$.[5]

In the following we assume the earth to be spherically symmetric, and we neglect the diurnal rotation of the earth, the friction in the atmosphere, the influence of the sun, moon, and planets, and the relativistic corrections to the satellite motion. The satellite orbit is then the classical Kepler ellipse, and the motion is periodic. The Schwarzschild line element for the gravitational field of the earth is then

$$ds^2=c^2dt^2-[(1-2GM_E/c^2r)^{-1}dr^2+r^2d\theta^2+r^2\sin\theta d\phi^2+(2GM_E/c^2r)dt^2]. \quad (2)$$

We have $g_{i4}=0$ ($i=1, 2, 3$). The system of coordinates is then time-orthogonal and we have[6]

$$dt_s=(1+2\chi_s/c^2-u^2/c^2)^{\frac{1}{2}}dt, \quad (3)$$

$$dt_E=(1+2\chi_E/c^2)^{\frac{1}{2}}dt, \quad (4)$$

χ_s and χ_E are the gravitational potentials on the satellite and on the earth, respectively, u is the velocity of the satellite relative to the earth, and r_s is the distance of the satellite from the center of the earth. $\chi_s=-GM_E/r_s$; $\chi_E=-GM_E/r_E$. Neglecting terms of order $(\chi/c^2)^2$ and $(u/c)^4$, we obtain from Eqs. (3) and (4)

$$dt_s=[1+(\chi_s-\chi_E)/c^2-u^2/2c^2]dt_E. \quad (5)$$

Then

$$\Delta\equiv(dt_s-dt_E)/dt_E=c^{-2}(-T+\chi_s-\chi_E), \quad (6)$$

T being the specific kinetic energy of the satellite. Integration of Eq. (5) over one period leads to

$$\tau_s-\tau_E=c^{-2}(-\bar{T}+\bar{\chi}_s-\chi_E)\tau_E,$$

$\bar{T}$ and $\bar{\chi}_s$ being the time averages of T and χ_s, respectively, over one period:

$$\bar{T}\equiv\frac{1}{\tau_E}\int_0^{\tau_E}Tdt_E.$$

Then

$$\bar{\Delta}=c^{-2}(-\bar{T}+\bar{\chi}_s-\chi_E). \quad (7)$$

As the satellite motion is periodic, we have from the virial theorem

$$\bar{T}=\tfrac{1}{2}\langle(\partial\chi_s/\partial r_s)r_s\rangle_{\text{av}}.$$

Introducing $\chi_s=-GM_E/r_s$, we obtain $\bar{T}=-\frac{1}{2}\bar{\chi}_s$. Due to the conservation of energy, we have $\bar{T}+\bar{\chi}_s=E=-GM_E/2a$. We then obtain $\bar{T}=-E=GM_E/2a$, $\bar{\chi}_s=2E=-GM_E/a$, and Eq. (7) becomes

$$\bar{\Delta}=(GM_E/c^2r_E)(1-3r_E/2a)=6.96\times10^{-10}(1-3r_E/2a). \quad (8)$$

It is to be noticed that $\bar{\Delta}$ is independent of the eccentricity of the ellipse.

[1] F. Winterberg, Astronautica Acta 2, 25 (1956).
[2] S. F. Singer, Phys. Rev. 104, 11 (1956).
[3] Singer has opposite sign, but the reason for this seems to be that he defines his t_s and t_E in another way.
[4] H. Nararai and Y. Ueno, Progr. Theoret. Phys. (Kyoto) 20, 703, (1958).
[5] *Note added in proof.* After this article was submitted for publication, Professor C. Möller informed me that he has calculated $\bar{\Delta}$ for an elliptic orbit. [Nuovo cimento Suppl. 6, 393 (1957)]. It seems to me however that the present method is the simplest one.
[6] C. Möller, *The Theory of Relativity* (Oxford University Press, New York, 1952), p. 247.