Landau Damping to all Orders*

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Recent results for an asymptotic series expansion of the nonlinear equations for an electronic plasma in the "collisionless" regime are extended. Substantially, the result is that for a wide class of initial disturbances, the presence of the so-called "Landau damping" in first order indicates its persistence to all orders in perturbation theory. Similarly, exponentially growing solutions in first order imply their existence in all higher orders. The problem is considered both from the Laplace transform point of view, and by normal modes.

I. INTRODUCTION

LONGITUDINAL oscillations in an unbounded, rarified, electronic plasma are governed by a pair of coupled, nonlinear, differentio-integral equations; Poisson's equation and the "collisionless" Boltzmann equation. Recently, one of us has described a method[1] by which the solution to this system can be given, in principle, to all orders in powers of the wave amplitude. The Fourier-Laplace transforms of the field variables were shown, in the nth order, to be expressible as convolutions involving terms up to order $n-1$ only. It was shown that systems which are "stable" in the first order are also "stable" in second order, and that first-order "instabilities" imply the existence of second-order "instabilities." It was also remarked that a solution in terms of normal modes could equally well be imposed on the same series.

It is the purpose of this work to extend and elaborate these results, and to apply them to what we believe to be an important phenomenon—the persistence of the so-called "Landau damping"[2] to all orders in powers of the disturbance, indicating the eventual smoothing-out of a very wide class of initial non-uniformities.

In Sec. II, we extend some results of our earlier work[1] for second order to all higher orders, and show that the singularities of the Laplace transform of the nth-order electric field lie to the left of the imaginary p axis, if they all lie in the left half-plane in first order. Sections III–V present a detailed account of the problem from the point of view of normal modes. Section III briefly reviews Van Kampen's[3] analysis; Sec. IV extends it to nth order. Section V shows that all higher order terms in the expansion of the electric field are also damped for infinitely-differentiable, square-integrable disturbances in a Maxwellian plasma. Section VI is a discussion of the results and a comparison with other work in the literature.

II. HIGHER-ORDER TERMS BY LAPLACE TRANSFORMS

We limit ourselves to plane-wave disturbances in one direction only, in an electron (charge $-e$, mass m) plasma which is sufficiently rarified to be in the "collisionless" regime. The equations governing the development of the electron distribution $f(x,v,t)$ and the electric field $E(x,t)$ are:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{e}{m}E\frac{\partial f}{\partial v}=0, \tag{1}$$

$$\frac{\partial E}{\partial x}=4\pi e\left(N_0-\int f dv\right), \tag{2}$$

where N_0 represents a uniform positive background charge, assumed immobile. The notation is the same as reference 1, except that since we discuss only one-dimensional motions, we may use the electric field E instead of the scalar potential.

Let us write the solution to (1) and (2) formally as

$$E=\sum_{n=1}^{\infty}E^{(n)};\quad f=\sum_{n=0}^{\infty}f^{(n)}, \tag{3}$$

where $f^{(0)}=f_0(v)$ is the equilibrium distribution, and $N_0=\int f_0 dv$.

There is no unique way to split the disturbance among the various orders, but the most convenient way appears to be to put it all in $f^{(1)}$ initially, assuming

$$f^{(n)}(x,v,0)=0,\quad n>1. \tag{4}$$

Substitution of (3) into (1) and (2) gives

$$\frac{\partial f^{(n)}}{\partial t}+v\frac{\partial f^{(n)}}{\partial x}-\frac{e}{m}E^{(n)}f_0'(v)=\frac{e}{m}S^{(n)}, \tag{5}$$

and

$$\frac{\partial E^{(n)}}{\partial x}=-4\pi e\int f^{(n)}dv,\quad n\geq 1, \tag{6}$$

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[1] D. Montgomery, Phys. Rev. **123**, 1077 (1961).
[2] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).
[3] N. G. van Kampen, Physica **21**, 949 (1955). Van Kampen's work has been generalized by K. M. Case, in Ann. Phys. **7**, 349 (1959), but we shall make no use of his generalizations.

where

$$S^{(n)} \equiv \sum_{j=1}^{n-1} E^{(n-j)} \frac{\partial f^{(j)}}{\partial v}, \quad n>1, \qquad S^{(1)} \equiv 0. \tag{7}$$

If we take Fourier transforms in x (indicated by subscripts k) and Laplace transforms in time (indicated by subscripts p), we get from (5) and (6) the results of Landau[2] for $n=1$. For $n>1$, we get[1]:

$$E_{kp}^{(n)} = -\frac{4\pi i e^2}{mk} \frac{1}{D_{kp}} \int \frac{S_{kp}^{(n)}(v)dv}{p+ikv}, \tag{8}$$

$$f_{kp}^{(n)} = \frac{e/m}{p+ikv} \{ f_0'(v) E_{kp}^{(n)} + S_{kp}^{(n)} \}, \tag{9}$$

$$D_{kp} \equiv 1 - \frac{4\pi i e^2}{mk} \int \frac{f_0'(v)dv}{p+ikv}. \tag{10}$$

For $n=1$, the $t \rightarrow \infty$ behavior of $E_k^{(1)}(t)$ is determined by the singularities of $E_{kp}^{(1)}$. For $f_0(v)$ and $f_k^{(1)}(v,0)$ entire, and absolutely integrable for v real, these can be shown[2] to lie at the zeros of D_{kp}. If these singularities all lie to the left of the imaginary p axis, $E_k^{(1)}(t) \rightarrow 0$ as $t \rightarrow \infty$; this phenomenon is usually called "Landau damping."

In the present section we shall show that

$$\mathfrak{S}^{(n)}(k,p) \equiv \int_{-\infty}^{\infty} \frac{S_{kp}^{(n)}(v)dv}{p+ikv} \tag{11}$$

is analytic if $\mathrm{Re}p$ is greater than some number $\alpha_n(k)$, which is less than the real parts of the rightmost singularities of $E_{kp}^{(1)}$. This being the case, the rightmost singularities of $E_{kp}^{(n)}$ lie in the same half-plane as those of $E_{kp}^{(1)}$. So the behavior of $E_k^{(n)}(t)$ as $t \rightarrow$ % will be the same as $E_k^{(1)}(t)$, provided only that the $E_{kp}^{(n)}$ obey some simple integrability requirements in a strip containing the imaginary axis.

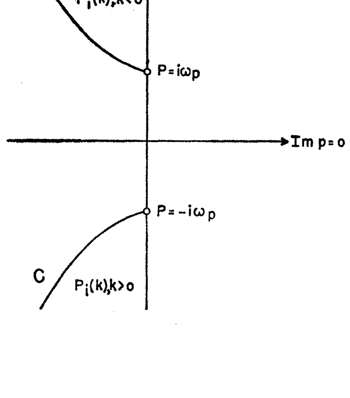

FIG. 1. The zeros of D_{kp}. For a right-traveling wave, the branch $\bar{C}$ is for $k<0$ and C is for $k>0$; for a left-traveling wave, just the reverse. In general, the curves C, $\bar{C}$ will be covered twice.

In fact, the complete statement of the condition of $f^{(1)}(x,v,0)$ in order that $E_{kp}^{(n)}$ be integrable is very involved, so we postpone the question of integrability until Secs. III–V, where, within the framework of normal modes, it becomes transparent. Now, we shall simply calculate the singularities, assuming that line integrals of non-singular functions always converge. This was done by Landau, also.

We assume that $f_0(v)$ and $f_k^{(1)}(v,0)$ are entire functions of v, and that their derivatives are absolutely integrable in v in $[-\infty, \infty]$. We also assume that D_{kp}^{-1} has only simple poles in the finite part of the p plane, and that these lie at $p=p_i(k)$.

Since $D_{-k,p}=(D_{k,p^*})^*$, it follows that $p_i(k)=p_i^*(-k)$. $E_{kp}^{(1)}=0$ for $k=0$, but $p_i(k) \rightarrow \pm i\omega_p$ as $k \rightarrow 0$, where $\omega_p \equiv$ the plasma frequency $=(4\pi N_0 e^2/m)^{\frac{1}{2}}$.

Figure 1 shows a typical plot of $p_i(k)$ in the complex p plane. The lower branch C corresponds to $\mathrm{Im}p_i<0$, the upper branch $\bar{C}$ to $\mathrm{Im}p_i>0$. Figure 1 shows a "stable" case; an "unstable" case would have C, $\bar{C}$ protruding to the right of the imaginary axis.

To discover the region of analyticity of the $\mathfrak{S}^{(n)}(k,p)$, we introduce a more general class of functions, defined by

$$\mathfrak{S}_{lm}^{(n)}(k_0p_0k_1p_1\cdots k_mp_m) \equiv \int_{-\infty}^{\infty} dv \frac{\partial^l S_{k_0p_0}^{(n)}(v)/\partial v^l}{(p_0+ik_0v)(p_1+ik_1v)\cdots(p_m+ik_mv)}. \tag{12}$$

We intend to show by induction that for any $l \geq 0$, $m \geq 0$, and $n \geq 1$, $\mathfrak{S}_{lm}^{(n)}$ is an analytic function of its p variables for the region defined by $\mathrm{Re}p_j > \alpha_n(k_j)$, and for all the k variables nonvanishing. This will show $\mathfrak{S}^{(n)}(k,p)$ is analytic for $\mathrm{Re}p > \alpha_n(k)$, by taking $l=m=0$.

$\mathfrak{S}_{lm}^{(1)} \equiv 0$ is clearly analytic for all l, m. Assume now that for $j=1, \cdots, n-1$, the $\mathfrak{S}_{lm}^{(j)}$ are analytic for the real parts of all the p variables greater than the corresponding $\alpha_j(k)$; recall that α is less than the real part of the rightmost zero of D_{kp}. [The rightmost zero of D_{kp} for the situation depicted in Fig. 1 is just $\lim_{\epsilon\rightarrow 0-}(\pm i\omega_p+\epsilon)$.]

For $\sigma >$ the real part of the rightmost singularity of $E_{kp}^{(j)}$, and $\mathrm{Re}(p_0-\sigma) >$ the same number, we have the following convolution:

$$\mathfrak{S}_{lm}^{(n)}(k_0p_0\cdots k_mp_m) = \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp' \int_{-\infty}^{\infty} dk' \, E_{k_0-k',\, p_0-p'}^{(n-j)} \times \int_{-\infty}^{\infty} \frac{dv \frac{\partial^{l+1}}{\partial v^{l+1}} f_{k'p'}^{(j)}(v)}{(p_0+ik_0v)\cdots(p_m+ik_mv)}. \tag{13}$$

Substituting from (9) for $f^{(j)}$,

$$\mathfrak{S}_{lm}{}^{(n)}=\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}dp'\int_{-\infty}^{\infty}dk' \times\frac{e}{m}E_{k_0-k',\,p_0-p'}{}^{(n-j)}E_{k'p'}{}^{(j)}\mathfrak{F}_{lm}{}^{(0)} +\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_{-i\infty}^{\sigma+i\infty}dp'\int_{-\infty}^{\infty}dk' \times\frac{e}{m}E_{k_0-k',\,p_0-p'}{}^{(n-j)}\mathcal{G}_{lm}{}^{(j)}, \quad (14)$$

where $\mathfrak{F}_{lm}{}^{(0)}$ and $\mathcal{G}_{lm}{}^{(j)}$ are given by

$$\mathfrak{F}_{lm}{}^{(0)}(k'p'|k_0p_0\cdots k_mp_m) \equiv\int_{-\infty}^{\infty}\frac{\dfrac{\partial^{l+1}}{\partial v^{l+1}}\left[\dfrac{f_0'(v)}{p'+ik'v}\right]}{(p_0+ik_0v)\cdots(p_m+ik_mv)}dv, \quad (15)$$

$$\mathcal{G}_{lm}{}^{(j)}(k'p'|k_0p_0\cdots k_mp_m) \equiv\int_{-\infty}^{\infty}\frac{\dfrac{\partial^{l+1}}{\partial v^{l+1}}\left[\dfrac{S_{k'p'}{}^{(j)}(v)}{p'+ik'v}\right]dv}{(p_0+ik_0v)\cdots(p_m+ik_mv)}. \quad (16)$$

$\mathfrak{F}_{lm}{}^{(0)}$ and $\mathcal{G}_{lm}{}^{(j)}$ are both analytic in the region $\mathrm{Re}p_l>\alpha_j(k_l)$: $\mathfrak{F}_{lm}{}^{(0)}$ by the entirety of $f_0(v)$; $\mathcal{G}_{lm}{}^{(j)}$ by our inductive hypothesis for $j=1,\cdots,n-1$. (The continuations of these functions are obvious generalizations of Landau's technique; just drop the contour of v-integration below the points $v=ip_0/k_0,\cdots,ip_m/k_m$, and ip'/k', as the p's pass into their left half-planes.)

To complete the proof, we need only show that each of the functions,

$$\int_{-i\infty}^{\sigma+i\infty}dp'\int_{-\infty}^{\infty}dk'E_{k_0-k',\,p_0-p'}{}^{(n-j)}E_{k'p'}{}^{(j)} \times\mathfrak{F}_{lm}{}^{(0)}(k'p'|k_0p_0\cdots k_mp_m), \quad (17)$$

and

$$\int_{\sigma-i\infty}^{\sigma+i\infty}dp'\int_{-\infty}^{\infty}dk'E_{k_0-k',\,p_0-p'}{}^{(n-j)} \times\mathcal{G}_{lm}{}^{(j)}(k'p'|k_0p_0\cdots k_mp_m), \quad (18)$$

is analytic for $\mathrm{Re}p_0>\alpha_n(k_0)$. Clearly they are analytic in $p_1,\cdots,p_m$, so we need only concentrate our attention to their behavior as a function of p_0.

Thus we have reduced the problem to showing that if there exists a number $\alpha_n(k)$, less than the rightmost singularity of $E_{kp}{}^{(1)}$, such that if $h(p,p',k')$ is analytic in the region $\mathrm{Re}p>\alpha_n(k)$, $\mathrm{Re}p'>\alpha_{n-1}(k')$, then

$$\int_{-\infty}^{\infty}dk'\int_{\sigma-i\infty}^{\sigma+i\infty}dp'\,E_{k-k',\,p-p'}{}^{(n-j)}E_{k'p'}{}^{(j)}h(p,p'k'), \quad (19)$$

and

$$\int_{-\infty}^{\infty}dk'\int_{\sigma-i\infty}^{\sigma+i\infty}dp'\,E_{k-k',\,p-p'}{}^{(n-j)}h(p,p',k'), \quad (20)$$

are analytic for $\mathrm{Re}p>\alpha_n(k)$.

Let us refer to Fig. 1, and study (19); (20) is handled similarly. Clearly (19) is analytic for $\mathrm{Re}p>0$ with $\sigma=0$. Now allow p to move into the left half-plane and observe how a singularity can arise. The integral (19) can be analytically continued for $\mathrm{Re}p<0$ so long as the singularities of $E_{k-k',p-p'}{}^{(n-j)}$ and $E_{k'p'}{}^{(j)}$ do not meet. When they meet, the contour of p' integration can no longer be deformed to avoid them, since it must run between.

The first time they do meet is when

$$p'=p_j(k'), \quad (21)$$

$$p-p'=p_{n-j}(k-k'), \quad (22)$$

for some k, k', p'. Eliminating p',

$$p=p_j(k')+p_{n-j}(k'). \quad (23)$$

$p_j(k')$ and $p_{n-j}(k-k')$ are the singularities of $E_{k'p'}{}^{(j)}$ and $E_{k-k',p-p'}{}^{(n-j)}$, which have negative real parts, by our inductive hypothesis. The maximum value of the real part of the right-hand side of (23) is a finite negative number for all $k\neq 0$ (see Fig. 1) and our theorem is proved.

III. NORMAL MODES

In this section we shall limit ourselves to the case which is "stable" in first order, and in particular to disturbances of an equilibrium distribution:

$$f_0(v)=N_0(m/2\pi KT)^{\frac{1}{2}}\exp(-mv^2/2KT). \quad (24)$$

As Van Kampen pointed out, not all solutions to the first-order equations are damped, and it is by requiring the initial disturbances to be non-singular and square-integrable in velocity that we limit consideration to those which are. In addition, we will impose the condition of infinite differentiability with respect to velocity, and the square-integrability of all these derivatives in $[-\infty,\infty]$.

The general normal-mode solution to (5) and (6) for $n=1$ is

$$f^{(1)}(x,v,t)=\iint_{-\infty}^{\infty}dkd\mu\,A(k,\mu)g_{k\mu}(v)e^{ik(x-\mu t)}, \quad (25)$$

where the normal modes $g_{k\mu}(v)$ are given by:

$$g_{k\mu}(v)=\frac{\omega_p{}^2}{k^2}F(v)\left[P\frac{1}{\mu-v}+\lambda(k,\mu)\delta(\mu-v)\right]. \quad (26)$$

$A(k,\mu)$ is arbitrary; $F(v)$ is given by $2\pi v f_0(v)/N_0$; the Cauchy principal value symbol "P" means that in integrating the quantity $(\mu-v)^{-1}$, the principal value is to be taken; $\delta(\mu-v)$ is a Dirac delta function; and $\lambda(k,\mu)$ is determined by the normalization condition

$$\int_{-\infty}^{\infty} g_{k\mu}(v)dv=1, \tag{27}$$

for all real k and μ.

The eigenfunctions $g_{k\mu}(v)$ are a complete set. That is, (27) may be solved for $A(k,\mu)$ at $t=0$ for arbitrary $f^{(1)}(x,v,0)$.

By the device of decomposing the various functions of v into their "positive and negative-frequency" parts,[4] for example, if $g_{0\alpha}$ is the Fourier transform of $g_0(v)$,

$$g_0(v)=g_{0+}+g_{0-}=\int_0^{\infty} g_{0\alpha}e^{i\alpha v}d\alpha+\mathrm{e}\int_{-\infty}^{0} g_{0\alpha}e^{i\alpha v}d\alpha, \tag{28}$$

Van Kampen is able to exhibit the solution for any plane density wave of the form

$$f^{(1)}(v,x,0)=g_0(v)e^{ik_0x}. \tag{29}$$

(In first order, of course, any sum of such solutions is also a solution.) For $t>0$, the solution is

$$f^{(1)}(x,v,t)=\frac{e^{ik_0x}}{2\pi}\int_{-\infty}^{\infty} d\mu\, e^{-ik_0\mu t} \times[Z(k_0,v)\delta_+(v-\mu)+Z^*(k_0,v)\delta_-(v-\mu)] \times\left[\frac{g_{0+}(\mu)}{Z(k_0,\mu)}+\frac{g_{0-}(\mu)}{Z^*(k_0,\mu)}\right], \tag{30}$$

where

$$Z(k_0,\mu)=1+2\pi i\frac{\omega_p^2}{k_0^2}F_+(\mu), \tag{31}$$

and $\delta_\pm(v)$ are defined in the standard way

$$\delta_+(v)+\delta_-(v)=\lim_{\epsilon\to 0+}\frac{1}{2\pi}\left\{\int_0^{\infty} d\alpha\, e^{-i\alpha v-\epsilon\alpha} +\int_{-\infty}^{0} d\alpha\, e^{-i\alpha v+\epsilon\alpha}\right\}. \tag{32}$$

Integrating (30) over v gives the density

$$n^{(1)}(x,t)=e^{ik_0x}\int_{-\infty}^{\infty}\frac{g_{0+}(v)e^{-ik_0vt}}{Z(k_0,v)}dv, \tag{33}$$

for $t>0$. Since the integrand is bounded and square-integrable, $n^{(1)}\to 0$ as $t\to\infty$ as a simple consequence of a standard theorem of Fourier analysis.[5] This is the proof of first-order damping as given in reference 3.

[4] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, England, 1937), Chap. V.

IV. NORMAL MODE SOLUTION FOR THE Nth ORDER

Fourier-analyze (5) and (6) in space and time, indicating transforms by subscripts k, ω (in Van Kampen's notation, $\omega\equiv k\mu$). The result is

$$i(-\omega+kv)f_{k\omega}{}^{(n)}-(e/m)E_{k\omega}{}^{(n)}f_0(v)=(e/m)S_{k\omega}{}^{(n)}, \tag{34}$$

$$ikE_{k\omega}{}^{(n)}=-4\pi e\int f_{k\omega}{}^{(n)}dv, \tag{35}$$

where $f^{(n)}(x,v,t)$ is given by

$$f^{(n)}(x,v,t)=\iint_{-\infty}^{\infty} e^{i(kx-\omega t)}f_{k\omega}{}^{(n)}dkd\omega, \tag{36}$$

and similarly for $E^{(n)}$ and $S^{(n)}$.

Equations (34) and (35) are a linear, inhomogeneous system. The general solution is *any* solution, plus the most general solution of the associated homogeneous system [i.e., (34) and (35) without the $S_{k\omega}{}^{(n)}$ term].

The homogeneous system has already been solved, since it is identical with the first-order system. It is easy to find an inhomogeneous solution for which $E_{k\omega}{}^{(n)}$ is identically zero. Just set

$$f_{k\omega}{}^{(n)}(\mathrm{inh})=\frac{e}{mi}P\frac{S_{k\omega}{}^{(n)}}{-\omega+kv}+\eta_{k\omega}{}^{(n)}\delta(-\omega+kv), \tag{37}$$

with $\eta_{k\omega}{}^{(n)}$ determined by

$$\int_{-\infty}^{\infty} f_{k\omega}{}^{(n)}(\mathrm{inh})dv=0. \tag{38}$$

We can write down the general $f^{(n)}$ at once, using (37) and the results of the first-order theory, for $n>1$:

$$f^{(n)}(x,v,t)=\iint_{-\infty}^{\infty} dkd\mu\, A^{(n)}(k,\mu)g_{k\mu}(v)e^{ik(x-\mu t)} +\iint_{-\infty}^{\infty} dkd\omega e^{i(kx-\omega t)} \times\left\{\frac{e}{mi}P\frac{S_{k\omega}{}^{(n)}}{-\omega+kv}+\eta_{k\omega}{}^{(n)}\delta(-\omega+kv)\right\}. \tag{39}$$

$A^{(n)}(k,\mu)$ is arbitrary, and $g_{k\mu}(v)$ is given by (26).

$A^{(n)}(k,\mu)$ is to be fixed by the initial value of $f^{(n)}$ as given by (4) for $n>1$. Equation (39) can be solved for $A^{(n)}(k,\mu)$ in exactly the same way as for the first order.

[5] See reference 4, Chap. III. We do not consider the unphysical case in which $g_0(v)$ does not approach a limit as $v\to\pm\infty$

Van Kampen's proof of completeness insures that this can be done. The "initial value" which determines $A^{(n)}$ will just be the second term on the right of (39), taken at $t=0$.

V. HIGHER-ORDER DAMPING BY NORMAL MODES

The quantity which plays mathematically the same role in higher order as $f_k^{(1)}(v, t=0)$ does for the first order is what we shall call "$f_k^{(n)}(v,0)$", defined by

$$\text{“}f_k^{(n)}(v,0)\text{”}=-\frac{e}{mi}P\int\frac{S_{k\omega}^{(n)}d\omega}{-\omega+kv}-\eta_{k,kv}^{(n)}. \quad (40)$$

Using (38), (40) may be written as

$$\text{“}f_k^{(n)}(v,0)\text{”}=\frac{e}{mi}P\int\frac{S_{k,kv}^{(n)}(v')+S_{k,kv'}^{(n)}(v)}{v-v'}dv'. \quad (41)$$

If this quantity can be shown to satisfy the same condition of nonsingularity and square-integrability that $f_k^{(1)}(v,0)$ required for damping of $n_k^{(1)}(t)$, then it is clear that $n_k^{(n)}(t)$ [and consequently $E_k^{(n)}(t)$] will also be damped.

In order to show "$f_k^{(2)}(v,0)$" is nonsingular and square-integrable, it is sufficient to show that $S_{k\omega}^{(2)}(v)$ is non-singular in ω and v and tends to zero as either $|\omega|$ or $|v|$ tends to infinity. But $S_{k\omega}^{(2)}(v)$ is just the Fourier transform of the quantity $E^{(1)}\partial f^{(1)}/\partial v$, which is nonsingular and square-integrable with respect to v (and t) if $f_k^{(1)}(v,0)$ is. This implies that $S_{k\omega}^{(2)}(v)$ is square-integrable and nonsingular in v and ω, and thus "$f_k^{(2)}(v,0)$" does in fact possess nonsingularity and square-integrability. It is clear that a proof by induction can be carried out for the nth order, since the result holds up to order $n-1$.

VI. DISCUSSION

Two important questions remain, neither of which we are now prepared to answer: (1) Is the series (3) convergent or only asymptotic? (2) What happens to the normal mode approach in an "unstable" case where, in first order, there exist solutions which $\sim e^{i(kx-\omega t)}$ with $\mathrm{Im}\omega>0$?

If the series (3) converges uniformly over the whole range of t, then we are rigorously justified in asserting that damping to all orders implies damping of the exact nonlinear solution for $E(x,t)$. On the other hand, if the series converges, but nonuniformly in time, then it is entirely possible that even though there is damping to all orders, the exact nonlinear field might not be damped. Probably, uniform convergence is too much to hope for and our result is only an asymptotic series.

As to the second question, we can assert that the Laplace transform treatment of Sec. II indicates the presence of exponentially growing terms in first order imply *more rapidly* growing exponential terms in the higher orders. How their coefficients add is not known. The normal mode approach as we have given it cannot readily be modified to include growing solutions since $S_{k\omega}^{(n)}(v)$ does not in general exist for this case. (Parenthetically it seems to us that there may be a question as to whether the unstable situation has any physical meaning in the unbounded case, anyway, and that the inclusion of boundary conditions may substantially modify our idea of what is to be called "instability" for a rarified plasma. It may not have such an unequivocal connection with exponentially growing normal modes as it does in magnetohydrodynamics.)

Very little exact work has been done on nonlinear plasma oscillations. Iordanskii[6] has put the problem on a solid mathematical basis by proving the existence of a unique solution by successive approximations. Computationally, his method is involved, but it seems clear that if any order of his approximations is damped, all higher orders will be, too. Even though his approximations converge, possible nonuniformity in this convergence precludes the rigorous conclusion that the exact nonlinear E is damped.

Bernstein, Green, and Kruskal[7] have shown how to construct a special class of time-independent space-charge waves. These are, at least in some cases, asymptotically stable.[8] These are *not*, in general, analytic functions of the amplitude, and could not be expected to exhibit damping of the sort we have discussed. The physical significance of analyticity of distributions is unclear.

Dawson[9] has studied large-amplitude waves in a nearly cold plasma, and has found that for amplitudes above a critical amount, "breaking" occurred, with very rapid dissipation of ordered motion into chaotic motion. Since near $T=0$, all wave numbers oscillate with the same frequency, we believe part of the dissipation in Dawson's case is due to mixing of various Fourier components, rather than being pure Landau damping, which requires a dispersion in velocity.

[6] S. Iordanskii, Doklady Akad. Nauk. U.S.S.R. **127**, 509 (1959).
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