PHYSICAL REVIEW VOLUME 124, NUMBER 5 DECEMBER 1, 1961

Collective Behavior in Solid-State Plasmas*

DAVID PINES† AND J. ROBERT SCHRIEFFER†
John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic, Division of General Dynamics Corporation, San Diego, California

(Received July 27, 1961)

The conditions for the existence of plasma wave instabilities in the plasma formed by the electrons and holes in semiconductors are discussed. The dispersion relations for both the high-frequency optical mode in which electrons and holes move out of phase, and the low-frequency acoustic mode in which electrons and holes move in phase are calculated. Growing acoustic waves are shown to occur for a sufficiently large relative drift velocity of the electrons and holes, and the boundary between growing and damped waves is determined for various electron-hole temperature ratios. Growth rates are calculated for several cases of interest; when the influence of impurity and phonon scattering on the electron-hole behavior is taken into account it is concluded that InSb is perhaps the most promising semiconductor in which to observe such instabilities. An investigation of the hole and electron temperatures and the relative electron-hole drift velocity as a function of field strength is carried out for InSb. It is shown that moderate field strengths (~100 v/cm) suffice to produce electron-hole drifts of the required order of magnitude for the observation of plasma wave instability; however, the scattering mechanisms present are sufficiently effective that it appears marginal whether the other condition (long hole relaxation times) necessary for the observation of the plasma wave instability is achievable in practice. In an Appendix the conditions for the occurrence of similar plasma wave instabilities in semimetals are analyzed briefly.

I

THIS paper is devoted mainly to the theoretical investigation of certain aspects of collective behavior in the "classical" plasma formed by electrons and holes in semiconductors at not too high carrier densities and not too low temperatures. The extension of the ideas developed herein to the quantum electron-hole plasma found in semimetals or certain semiconductors is discussed in the Appendix.

In general, there may exist two modes of collective oscillation for a two-component plasma, such as the classical plasma of electrons and ions or the electron-hole plasma in a semiconductor. One mode consists of a high-frequency oscillation in which the electrons and holes oscillate out of phase with one another; the frequency ω_1 of a long-wavelength oscillation is

$$\omega_1^2=\omega_+^2+\omega_-^2,$$

where ω_+ and ω_- are the electron and hole plasma frequencies, respectively. The other mode corresponds to a low-frequency oscillation in which the holes (assumed to be heavy) and electrons (assumed to be light) move in phase with one another. It is thus a plasma oscillation appropriate to the holes plus their associated screening cloud of electrons; for equal densities of holes and electrons the frequency of a long-wavelength oscillation of wave number k is

$$\omega_2\cong(T_-/2T_+)^{\frac{1}{2}}kv_+=(m_-/2m_+)^{\frac{1}{2}}kv_-,$$

provided the ratio of electron temperature to hole temperature, T_-/T_+, is large enough, and the electron-hole mass ratio, m_-/m_+, is small enough. These are just the requirements that the low-frequency mode possess a frequency which is distinct from typical individual hole and electron excitation frequencies, kv_+ and kv_-. Here v_+ and v_- are defined by $(m_-v_-^2/2)=\kappa T_-$; $(m_+v_+^2/2)=\kappa T_+$, and the foregoing requirements are necessary in order that the collective mode not be too strongly damped by the individual particle excitations. By analogy to the vibration spectrum of polar crystals, we may call the high-frequency mode an optical mode of plasma oscillation, the low-frequency mode an acoustic plasma mode.

We shall be particularly concerned with the possible existence of a two-stream instability in the semiconductor associated with the electron drift under the application of an electric field. Our motivation for this study is the obvious need for an understanding, both theoretical and experimental, of the high-frequency instabilities in fully ionized plasmas. From a theoretical point of view, the conditions for the existence of certain classes of such instabilities (and in particular for the two-stream instability in a homogeneous plasma) are well understood; the calculations of the growth rate of the instability for short times (where the linear approximation is valid) are also reliable. However, once the amplitude of the growing plasma oscillation becomes sufficiently large that nonlinear effects (such as the coupling between plasma modes of different wavelengths) begin to play a role, comparatively little is known of the resulting behavior of the plasma. The experimental situation is even less satisfactory. The unambiguous observation of the two-stream instability has proved an extremely challenging problem for the plasma experimentalist, and only recently has a measure of success been achieved in its detection.[1] It seems not unlikely that quantitative experiments which bear on the above nonlinear aspects will prove equally difficult to carry out.

The electron-hole plasma in a semiconductor would seem to offer a promising tool for the investigation of such instabilities. It possesses the obvious advantage that the relative concentrations and temperatures of

* This work was supported by a joint General Atomic-Texas Atomic Energy Research Foundation program on controlled thermonuclear reactions.

† Present address: Department of Physics, University of Illinois, Urbana, Illinois.

[1] G. D. Boyd, L. M. Field, and R. Gould, Phys. Rev. **109**, 1393 (1958).

the electrons and holes can be measured and varied over quite a substantial range of interest. The principal disadvantage is that the electrons and holes are scattered by phonons, impurities, and, in some cases, one another; hence one is hampered by the need to find temperatures and concentrations such that $\omega\tau_{\pm}\gg 1$, where ω is the characteristic frequency one wishes to study and τ_+ and τ_- represent the hole and electron lifetimes against the extraneous scattering mechanisms. As we shall see, therefore, the possibility of achieving conditions for observing the two-stream instability appears somewhat touch and go for semiconductors, where one has to contend with lifetimes $\cong 10^{-12}$ sec. The prospect may be brighter in semimetals, where the lifetimes are $\cong 10^{-10}$ sec or perhaps an order of magnitude longer for low field strengths; however, interband transitions may act to reduce these lifetimes for a large applied field.

In a two-component plasma one finds that for a sufficiently large drift velocity of electrons vs ions (or holes) the plasma becomes unstable against a growing wave of plasma oscillation, corresponding to a coherent excitation of the oscillations by the electron beam. The conditions for the existence of this instability and its growth rate have been studied by Rosenbluth,[2] Buneman,[3] and Jackson[4] for a plasma of electrons and ions at equal temperatures. They find that the instability comes into play for an electron drift velocity $v_d \geqslant 1.32 v_-$. When the electron-ion temperature ratio, T_-/T_+, is sufficiently large the critical drift velocity required to produce an instability is reduced, being of order $(m_-/m_+)^{\frac{1}{2}} v_-$ or smaller.[5] Drift velocities of this latter order seem definitely achievable in high-mobility semiconductors; the first problem, then, in producing a two-stream instability in a semiconductor is that of achieving a sufficiently large electron-hole temperature ratio.

InSb appears a likely material, because the electron-hole mass ratio is large, ($\sim$14) and the electron mobility is quite high ($\sim 10^5$ cm^2/v). The high electron mobility means that the application of modest electric field strengths will act to produce substantial deviations from Ohm's law, with appreciable heating of the electrons. On the other hand, the large value of m_+/m_- means that the hole mobility is an order of magnitude smaller than the electron mobility; as a result one gets much less hole heating than electron heating for a given field strength, so that a large value of m_+/m_- tends to favor a large value of T_-/T_+. This aspect of semiconductor behavior favors the existence of acoustic plasma oscillations and the existence of a lower threshold for the observation of the two-stream instability.

As we have mentioned, the principle obstacle to carrying out such observations is the incoherent scattering of the electrons and holes. The instability will occur only if the product of its growth rate (neglecting particle relaxation times), Ω_g, and the relevant electron- or hole-scattering lifetime, $\tau_{\pm}$, is greater than unity. The growth rates we calculate for InSb under favorable circumstances are of the order of $\omega_+/10$; the reciprocal of the hole lifetime (which is the relevant one here) is of this same order, so that the production and observation of the growth of acoustic plasma oscillations may or may not be feasible for this material.

In Sec. II we discuss the dispersion relation for both high- and low-frequency modes in a system of two coupled plasmas at rest having different masses, densities, and temperatures. The treatment is extended to include the effect of directed particle drift velocities and the scattering of the carriers of the crystal lattice. The effect of an external electric field on the dispersion relation is also included. The conditions under which the electron drift may be just sufficient to excite a growing acoustic wave instability are treated in Sec. III. The growth rate of the oscillations is discussed in Sec. IV. In Sec. V we discuss the semiconductor aspects of the problem with particular reference to the magnitude of the particle drift velocities and temperatures one can expect in moderate electric fields. In Sec. VI a brief discussion of the possibility of observing the two-stream instability by pulsed conductivity measurements is given.

II

We consider an idealized situation in which there are n_+ holes and n_- electrons per unit volume distributed uniformly over a large sample. The holes are assumed to be free particles of effective mass m_+; the electrons likewise possess an effective mass m_-. For the low frequencies of interest to us here the interaction between the charged particles is well described by $e^2/\epsilon_0|\mathbf{r}_i-\mathbf{r}_j|$, where ϵ_0 is the static dielectric constant of the semiconductor. The coupled electron-hole plasma oscillations of this system have been studied using the collective variables approach and the random phase approximation by Nozières and Pines.[6] In this report we shall use an alternative approach; we work with the Boltzmann equation for the one-particle distribution functions, $f_{\pm}(\mathbf{r},\mathbf{v},t)$, and take into account the influence of charged particle interaction by means of a self-consistent field.[7] The equivalence of the collective-variables random phase approximation approach with use of the collisionless Boltzmann equation plus the self-consistent field is by now well understood.[8]

[2] M. Rosenbluth (private communication).
[3] O. Buneman, Phys. Rev. Letters **1**, 8 (1958); Phys. Rev. **115**, 503 (1959).
[4] J. D. Jackson, J. Nuclear Energy **Pt. C**: **1**, 171 (1960).
[5] M. Rosenbluth (private communication); I. Berstein, E. A. Frieman, R. M. Kulsrud, and M. N. Rosenbluth, Phys. Fluids **3**, 136 (1960).
[6] D. Pines, Can. J. Phys. **34**, 1379 (1956); P. Nozières and D. Pines, Phys. Rev. **109**, 1062 (1958); P. Nozieres, Ann. phys. **4**, 865 (1959).
[7] A. Vlasov, J. Phys. U.S.S.R. **9**, 25, 130 (1945); L. D. Landau, *ibid.* **10**, 25 (1946).
[8] J. Goldstone and K. Gottfried, Nuovo cimento **13**, 849 (1959); M. Cohen and H. Ehrenreich, Phys. Rev. **115**, 786 (1959); D. Pines, J. Nuclear Energy **Pt. C**: **2**, 5 (1960).

The random phase approximation is valid when the average potential energy per electron is small compared to the average kinetic energy. Thus our treatment will be valid when

$$\frac{e^2k_\pm}{\epsilon_0\kappa T_\pm}\lesssim 1 \quad\text{or}\quad \left(\frac{4\pi n_\pm e^2}{\epsilon_0\kappa T_\pm}\right)^{\frac{3}{2}}\frac{1}{4\pi n_\pm}\lesssim 1, \tag{2.1}$$

where $k_\pm$ are the hole and electron Debye screening wave vectors defined by

$$k_\pm{}^2=4\pi n_\pm e^2/\epsilon_0\kappa T_\pm. \tag{2.2}$$

For electron and hole densities of the order of 10^{15}, the weak coupling condition (2.1) is satisfied for temperatures in excess of 20°K for InSb ($\epsilon_0\cong 16$). Therefore, we shall see the weak coupling condition is satisfied for the experimental condition of interest to us in what follows.

The time evolution of the distribution function of a two-component plasma in the presence of an external electric field and an external scattering mechanism is extremely complicated. No general solution of this problem has thus far been obtained. Our treatment is based on the assumption that there are two reasonably distinct time scales which characterize the change in the distribution function. The first time scale is associated with the macroscopic drift of the particles induced by the external field. The second time scale, which we assume to be short compared to the first, is that which characterizes the coherent behavior of the system; that is, the collective oscillations and their growth for sufficiently large relative hole-electron drift velocity. Thus we regard the drift velocity of the particles as changing adiabatically with regard to the times characteristic of plasma effects.

Consider, for the moment, the collisionless Boltzmann equation for the distribution functions $f_\pm(\mathbf{r},\mathbf{v},t)$ in the absence of an external electric field:

$$\partial f_\pm/\partial t+\mathbf{v}\cdot\boldsymbol{\nabla} f_\pm\pm(e/m_\pm)\mathbf{E}\cdot\boldsymbol{\nabla}_v f_\pm=0. \tag{2.3}$$

In (2.3), the electric field $\mathbf{E}$ is that arising from the averaged field of the charged particles, and is determined by Poisson's equation:

$$\epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi e\int d\mathbf{v}(f_+-f_-). \tag{2.4}$$

The dispersion relation for the collective modes of the system is determined by solving (2.3) and (2.4) simultaneously; this may be done easily if one assumes the departures of $f_\pm$ from their respective Maxwellian values, $f_{0\pm}$, are small. Thus one writes

$$f_\pm=f_{0\pm}+f_{1\pm} \tag{2.5}$$

and linearizes the resulting equations by neglecting the cross-terms $\mathbf{E}\cdot\boldsymbol{\nabla}_v f_{1\pm}$ as being of higher order. The solution is perhaps most easily obtained by taking the Fourier transform in space and time of (2.3) and (2.4); one then arrives at the dispersion relation for plasma oscillations:

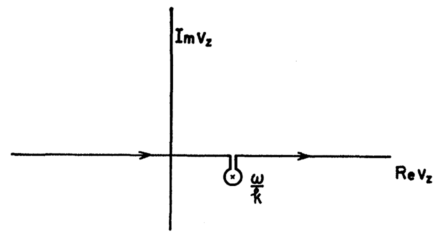

FIG. 1. Contour of integration for evaluating integrals appearing in (2.6); v_Z is $\mathbf{k}\cdot\mathbf{v}/k$.

$$1+\frac{4\pi e^2}{\epsilon_0k^2m_+}\int d\mathbf{v}\,\frac{\mathbf{k}\cdot\boldsymbol{\nabla}_v f_{0+}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\delta}+\frac{4\pi e^2}{\epsilon_0k^2m_-}\int d\mathbf{v}\,\frac{\mathbf{k}\cdot\boldsymbol{\nabla}_v f_{0-}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\delta}=0. \tag{2.6}$$

The small imaginary part $i\delta$ arises from the choice of a boundary condition corresponding to a retarded solution of (2.3) and (2.4). Such a solution is valid only for $\mathrm{Im}\omega\geqslant 0$. For $\mathrm{Im}\omega<0$, it is necessary to analytically continue the functions of ω appearing here; this may be accomplished by the choice of the contour of integration indicated in Fig. 1. This prescription is equivalent to the result obtained by Landau using Laplace transforms and treating the problem as an initial value one; it leads to damping of the plasma oscillations by the individual electron excitations in the absence of directed motion of the electrons.

It is convenient to write the dispersion relation (2.6) in the following form:

$$1+\frac{k_+{}^2}{k^2}W\left(\frac{\omega}{kv_+}\right)+\frac{k_-{}^2}{k^2}W\left(\frac{\omega}{kv_-}\right)=0, \tag{2.7}$$

where the response function, W, is given by

$$\begin{aligned}W(Z)&=\lim_{\delta\to 0}-\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}dq\,\frac{q\exp(-q^2/2)}{Z-q+i\delta}\\&=+i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}Z\exp(-Z^2/2)+1\\&\quad-Z\exp(-Z^2/2)\int_0^Z dq\exp(q^2/2).\end{aligned} \tag{2.8}$$

The result (2.8) follows after substitution of the Maxwellian values, $f_{0\pm}=a_\pm\exp(-mv^2/2\kappa T_\pm)$ in (2.6) and a certain amount of straightforward algebra.

As we shall see, the W function plays a central role in the discussion of dynamics of plasma oscillations. It has the following expansions for large and small

argument:

$Z \gg 1$:

$$W(Z)=i\pi^{\frac{1}{2}}Z\exp(-Z^2)-\frac{1}{2Z^2}-\frac{3}{4Z^4}-\frac{15}{8Z^6}-\cdots-\frac{(2n+1)!!}{2^nZ^{2n}};\quad(2.9)$$

$Z \ll 1$:

$$W(Z)=i\pi^{\frac{1}{2}}Z\exp(-Z^2)+1-2Z^2\left[1-\frac{2}{3}Z^2+\frac{4}{15}Z^4+\cdots\frac{(-2)^nZ^{2n}}{(2n+1)!!}\right],\quad(2.10)$$

where $(2n+1)!!\equiv 1\times3\times5\cdots\times(2n+1)$. The high-frequency plasma oscillation solution of (2.7), ω_1, is obtained by replacing the W functions by their high-frequency expansions; one obtains

$$1=\frac{\omega_+^2}{\omega^2}+\frac{\omega_-^2}{\omega^2}-i\pi^{\frac{1}{2}}\left[\frac{k_-^2}{k^2}\frac{\omega}{kv_-}\exp\left(-\frac{\omega^2}{k^2v_-^2}\right)+\frac{k_+^2}{k^2}\frac{\omega}{kv_+}\exp\left(-\frac{\omega^2}{k^2v_+^2}\right)\right],\quad(2.11)$$

where the plasma frequency for each type of carrier is defined by

$$\omega_\pm^2=4\pi n_\pm e^2/m_\pm\epsilon_0.\quad(2.12)$$

For long wavelengths ($kv_- \ll \omega$) the solution of (2.11) may be written as

$$\omega=\omega_1\left\{1-i\frac{\pi^{\frac{1}{2}}}{2}\left[\frac{\omega_1}{kv_-}\frac{k_-^2}{k^2}\exp\left(-\frac{\omega_1^2}{k^2v^2}\right)+\frac{\omega_1}{kv_+}\frac{k_+^2}{k^2}\exp\left(-\frac{\omega_1^2}{k^2v_+^2}\right)\right]\right\},\quad(2.13)$$

since in this limit the damping of the wave due to individual electron and hole excitation is slight. The real part of the frequency is, as we have indicated,

$$\omega_1=\left[\frac{4\pi e^2}{\epsilon_0}\left(\frac{n_+}{m_+}+\frac{n_-}{m_-}\right)\right]^{\frac{1}{2}}.\quad(2.14)$$

Thus if $n_+=n_-$, the plasma frequency of the coupled system is given by the plasma frequency appropriate to the reduced mass,

$$\left(\frac{1}{\mu}\right)=\frac{1}{m_+}+\frac{1}{m_-},$$

which is to be expected since the holes and electrons are moving completely out of phase with respect to each other.

The physical origin of the damping is simply understood by considering particles whose velocity is approximately equal to the phase velocity of the wave, $v_p=\omega/k$. If the wave has sufficiently large amplitude, particles with velocities slightly greater than v_p will be trapped in a potential trough and decrease their average velocity to v_p transferring their extra kinetic energy to the wave. Particles with velocities slightly less than v_p will be accelerated by this trapping mechanism and absorb energy from the wave. Therefore if the distribution function decreases with increasing velocity, damping takes place since there will be more slow particles to absorb energy than fast particles to transfer energy to the wave. If, on the other hand, the distribution function increases with velocity near v_p it is possible that the wave will grow in amplitude.

The low-frequency mode of the electron-hole system takes on a simple form if the phase velocity of the wave is large compared to the hole thermal velocity and small compared to the electron thermal velocity. In this case the small- and large-Z approximations can be made in the dispersion relation for the electrons and holes, respectively. The dispersion relation then becomes

$$1=\frac{\omega_+^2}{\omega^2}\left(1+\frac{3}{2}\frac{k^2v_+^2}{\omega^2}+\cdots\right)-\frac{k_-^2}{k^2}-i\pi^{\frac{1}{2}}\left\{\frac{\omega}{kv_-}\frac{k_-^2}{k^2}+\frac{\omega}{kv_+}\frac{k_+^2}{k^2}\exp\left(-\frac{\omega^2}{k^2v_+^2}\right)\right\},\quad(2.15)$$

and possesses the long-wavelength solution

$$\omega=\omega_2\left[1-i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{\left(\frac{m_-}{m_+}\frac{n_+}{n_-}\right)^{\frac{1}{2}}+\left(\frac{T_-}{T_+}\frac{n_+}{n_-}\right)^{\frac{3}{2}}\times\exp\left(-\frac{1}{2}\frac{T_-}{T_+}\frac{n_+}{n_-}\right)\right\}\right],\quad(2.16)$$

$$\omega_2=\left(\frac{n_+}{n_-}\frac{\kappa T_-}{m_+}+\frac{3\kappa T_+}{m_+}\right)^{\frac{1}{2}}k.\quad(2.17)$$

We see directly from (2.16) that the damping of the long-wavelength acoustic wave will be small only if the following conditions are satisfied:

$$\frac{m_-}{m_+}\frac{n_+}{n_-}\ll 1,\quad \frac{T_-}{T_+}\frac{n_+}{n_-}\gg 1.\quad(2.18)$$

These are just the conditions, $\omega^2\ll k^2v_-^2$ and $\omega^2\gg k^2v_+^2$, which are necessary in order that we be justified in our use of the low-frequency and high-frequency expansions, respectively, of the electron and hole response functions, W, in the long-wavelength limit. Thus where the frequency of the acoustic wave satisfies the condition

$$k^2v_+^2\ll\omega_2^2\ll k^2v_-^2,\quad(2.19)$$

the damping of the wave will be small, and one finds a simple analytic form (2.17) for ω_2; where (2.19) is not satisfied, one must use the exact values of the W functions, and the damping is appreciable.

We now consider the behavior of the system if the holes and electrons possess net drift velocities v_{d+} and v_{d-}, respectively. If collisions between the particles are more efficient in relaxing the momentum and energy distributions of the holes and electrons than their interactions with the lattice, then the distribution functions in the presence of an external electric field, E_0, will take the form of displaced Maxwellians:

$$f_\pm(E_0)=a_\pm \exp[-m_\pm(v-v_{d\pm})^2/2\kappa T_\pm]. \quad (2.20)$$

The temperatures of the holes and electrons may differ from the lattice temperature as a result of the energy absorbed from the external field. With the aid of (2.20) it is easy to show that the dispersion relation for plasma oscillations is given by a modified form of (2.7), in which

$$Z_\pm=(\omega-\mathbf{k}\cdot\mathbf{v}_{d\pm})/kv_\pm$$

replaces the arguments $\omega/kv_\pm$ appearing there. This result follows simply by noting that the integration over velocity in (2.6) can be reduced to that for $v_d=0$ by the change of variables $\mathbf{v}\rightarrow\mathbf{v}+\mathbf{v}_{d\pm}$, the only change being that ω is replaced by the Doppler shifted value $\omega-\mathbf{k}\cdot\mathbf{v}_d$. As we discussed above, the Landau damping term depends upon the sign of $\omega-\mathbf{k}\cdot\mathbf{v}_d$. If $\omega-\mathbf{k}\cdot\mathbf{v}_d<0$ the Landau prescription for treating the pole in the integrand tends to make the wave grow rather than decay. This result follows simply from the fact that for $v_p\equiv\omega/k<v_d$, the distribution function is increasing with increasing v while for $\omega/k>v_d$ the maximum of the Maxwellian has been passed and damping results, in agreement with the physical description of the effect given earlier. The growth of oscillations due to the relative drift of the particles is known as the two-stream instability; a study of this instability in semiconductors will form the major part of this paper.

The particle-lattice interactions can be included in the Boltzmann equation by a relaxation time approximation in the case of impurity scattering while, in general, the phonon scattering requires a more detailed treatment. If one uses a collision time approximation,

$$(\partial f/\partial t)_{\rm coll}=-(f-f_0)/\tau(v), \quad (2.21)$$

then the only effect of the collisions on the dispersion relation is to replace $\omega/kv_\pm$ by $[\omega+i/\tau_\pm(v)]/kv_\pm$. It is clear that the collision term leads to damping of the oscillation since if ω is satisfied the dispersion relation with $\tau_+=\tau_-=\infty$, then $\omega-i/\tau$ satisfies the dispersion relation when $\tau_+=\tau_-=\tau$.

The external field $\mathbf{E}_0$ which produces the drift velocities $\mathbf{v}_{d\pm}$, can have a direct effect on the plasma oscillations. If the electric field $\mathbf{E}$ in (2.3) is extended to include $\mathbf{E}_0$, then the perturbed distribution functions $f_{1\pm}$ satisfy

$$i\left(\omega-\mathbf{k}\cdot\mathbf{v}\pm e\frac{E_0}{m_\pm}\frac{\partial}{\partial v_Z}\right)f_1=\pm e\frac{\mathbf{E}_s}{m_\pm}\cdot\nabla_v f_{0\pm}, \quad (2.22)$$

where $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_s$. It is straightforward to solve (2.22) for $f_\pm$ and combine the result with Poisson's equation,

$$i\mathbf{k}\cdot\mathbf{E}_s=4\pi e\int(f_{1+}-f_{1-})d^3v, \quad (2.23)$$

to obtain the modified dispersion relation. The result is equivalent to replacing the temperatures $T_\pm$ by complex temperatures,

$$T_\pm'=T_\pm\left[1\mp\frac{ie\mathbf{E}_0\cdot\mathbf{k}}{k^2\kappa T_\pm}\right], \quad (2.24)$$

in both W and $k_\pm^2$. The external electric field can therefore have a strong effect on the dispersion relation if a particle gains an energy from the external field which is large compared to κT while moving a distance of one wavelength. The sign of the effect depends upon both the charge of the particle and the angle between $\mathbf{E}_0$ and $\mathbf{k}$. This correction to the dispersion relation vanishes for $Z\gg1$ since in this limit thermal effects are negligible compared to potential energy effects.

III

In order to obtain a feeling for the conditions under which an instability corresponding to growing waves of frequency ω will occur it is convenient to calculate the wave number k for which $\text{Im}\omega=0$, as a function of the drift velocity of the electrons. This yields the boundary between the growing waves and the damped waves for a given drift velocity. We choose a coordinate system in which the holes have zero drift velocity and neglect the damping due to particle-lattice interactions and the external field E_0. The basic equation we wish to investigate is, therefore, from (2.7) and (2.20)

$$1+\frac{k_+^2}{k^2}W(Z_+)+\frac{k_-^2}{k^2}W(Z_-)=0, \quad (3.1)$$

where

$$Z_+=\omega/kv_+;\quad Z_-=(\omega-\mathbf{k}\cdot\mathbf{v}_d)/kv_-.$$

Since there are a large number of parameters to be specified, i.e., $m_\pm$, $T_\pm$, $n_\pm$, and v_d, we choose some typical cases of interest to study:

1. $n_+=n_-$, $T_+=T_-$;
2. $T_+=0$, $T_->0$;
3. $n_+=n_-$, $T_-=10T_+$;
4. $n_+=n_-$, $T_-=4T_+$;
5. $n_+=50n_-$, $T_-=4T_+$.

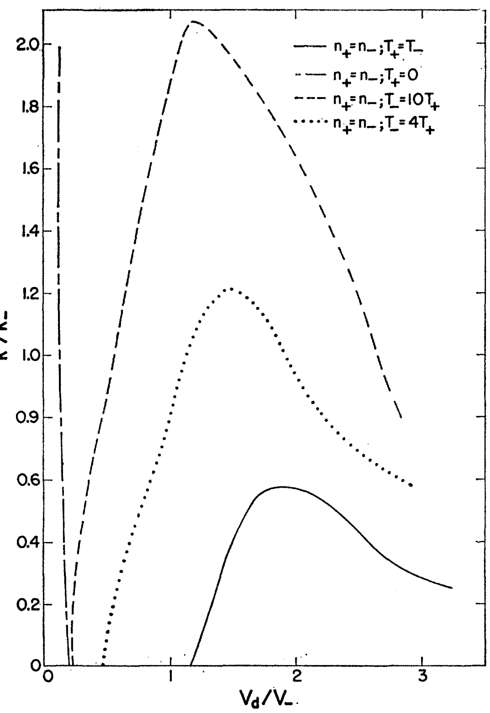

FIG. 2. The boundary between growing waves and damped waves for an electron-hole plasma with $n_+=n_-$, $m_+=14m_-$, and varying values of T_-/T_+.

In all of these cases we keep the mass ratio, m_+/m_-, fixed at 14. The mass ratio of 14 is appropriate to the heavy holes and low-energy electrons in InSb, which is likely to be one of the most favorable materials for observing this instability.

Case 1 has been investigated by Buneman[3] and Jackson.[4] Under the condition $\text{Im}\omega=0$, Z_+, and Z_- are real and therefore the imaginary part of (3.1) reduces to

$$Z_+ \exp(-Z_+^2)=-Z_- \exp(-Z_-^2), \tag{3.2}$$

which may be satisfied by $Z_+=-Z_-\equiv Z$. The real part of (3.1) becomes

$$1=2\frac{k^2}{k_-^2}\left(-1+2Z\exp(-Z^2)\int_0^Z \exp(t^2)dt\right). \tag{3.3}$$

The relation between k and v_d is plotted in Fig. 2. The critical drift velocity for which infinitely long wavelengths just begin to grow is given by

$$v_d=0.926v_-[1+(m_-/m_+)^{\frac{1}{2}}]. \tag{3.4}$$

As the drift velocity increases shorter wavelength oscillations will grow. This condition persists until one attains the velocity $1.52v_-[1+(m_-/m_+)^{\frac{1}{2}}]$; at this value of v_d the maximum range of wave numbers is unstable; the maximum wave number of a growing wave is given by $0.722k_-$. For very large velocities, the maximum wave number for which plasma oscillations can grow is given by

$$k=\sqrt{2}(\omega_-/v_d)[1+(m_-/m_+)^{\frac{1}{2}}]. \tag{3.5}$$

The relation $Z_+=-Z_-$ gives

$$\omega=\frac{\mathbf{k}\cdot\mathbf{v}_d}{(m_+/m_-)^{\frac{1}{2}}+1}. \tag{3.6}$$

Thus for long wavelengths and low velocities, the waves have a dispersion law corresponding to phonons, that is, ω is proportional to k. For high velocities, (3.5) and (3.6) lead to

$$\omega=\sqrt{2}\omega_+. \tag{3.7}$$

Thus, we see that if $m_+\gg m_-$ the unstable oscillations behave much like the low-frequency acoustic oscillations in which the holes and electrons move essentially in phase with each other.

In case 2, because $T_+=0$ the hole response function may be approximated by its high-frequency limit while the electrons may be treated in the low-frequency approximation at long wavelengths. The real part of the dispersion relation (3.1) becomes

$$1=(\omega_+^2/\omega^2)-(k_-^2/k^2), \tag{3.8}$$

or

$$\omega=\frac{k}{(k^2+k_-^2)^{\frac{1}{2}}}\omega_+\equiv s(k)k; \tag{3.9}$$

the imaginary part gives $Z_-=0$ or

$$\omega=\mathbf{k}\cdot\mathbf{v}_d. \tag{3.10}$$

On combining (3.9) and (3.10) we see that the relation between k and v_{d-} for which oscillations are just stable is given by

$$v_d=\frac{\omega_+}{(k^2+k_-^2)^{\frac{1}{2}}}. \tag{3.11}$$

Thus, for sufficiently large k, unstable acoustic plasma oscillations can be excited by electrons with a vanishing small drift velocity. For $v_d>\omega_+/k_-$ the spectrum of growing waves extends down to $k=0$. The boundary given by (3.11) is likewise plotted in Fig. 2. The distinctly different behavior between cases 1 and 2 should be noted. The essential reason for the great difference in the boundaries is the lack of Landau damping for the holes in case 2. This has the effect of allowing growth of oscillations whenever the drift velocity of the electrons is greater than the sound velocity $s(k)$ for these oscillations. Since $s(k)$ decreases as $1/k$ for large k, the condition for growing waves can always be satisfied for sufficiently large k. We conclude that if the temperature of the holes can be made very small, the critical velocity for excitation of plasma oscillations can be appreciably lowered. Thus, it is desirable to obtain a material with a high rate of energy loss for one type of carrier, e.g., the holes, so that their temperature can be maintained at a low value. Also these carriers should have a large effective mass since from (3.11) the minimum drift velocity for a given wave to grow is proportional to $1/\sqrt{m_+}$. However, as we shall

see in the next section, the growth rate ω_+ is proportional to $1/\sqrt{m_+}$ so that there will be an optimum value of m_+ for a given drift velocity to maximize the growth rate.

The boundary of the region of growing waves for case 3 is also shown in Fig. 2. For the temperature ratio $T_-/T_+=10$ the critical drift velocity is $v_d=0.22v_-$. As one might expect, this value is intermediate between the critical velocities for cases 1 and 2. The boundary does not exhibit the characteristic folding back into the k axis which occurs for somewhat larger temperature ratios. By using the high-frequency expansion for the hole polarizability and the low-frequency expansion for the electron contribution, one easily obtains the following expression for the boundary curve:

$$\frac{v_d}{v_-}=\frac{1}{(2+2x^2)^{\frac{1}{2}}}\left\{\left(\frac{T_-}{T_+}\right)^{\frac{3}{2}}\exp\left(-\frac{T_-}{2T_+(1-x^2)}\right)+\left(\frac{m_-}{m_+}\right)^{\frac{1}{2}}\right\}, \quad (3.12)$$

where $k/k_-=x$. In order for the curve to fold in toward the k axis, it follows from (3.12) that

$$\ln\left\{\left[\left(\frac{T_-}{T_+}\right)^{\frac{5}{2}}-\left(\frac{T_-}{T_+}\right)^{\frac{3}{2}}\right]\left(\frac{m_+}{m_-}\right)^{\frac{1}{2}}\right\}\leqslant\frac{T_-}{2T_+}. \quad (3.13)$$

For a mass ratio of 14 the criterion is satisfied for $T_-/T_+\geqslant 17$. Thus, the temperature ratio must be quite large in order for the high wave numbers to become unstable before the low wave numbers do so. We remark that

$$Z_1^2\cong\frac{T_-}{2T_+}\frac{1}{1+x^2}; \quad Z_2\cong-\frac{T_-}{T_+}Z_1\exp(-Z_1^2),$$

so that for $T_-/T_+>10$, the parameters satisfy $Z_1\gg1$ and $|Z_2|\ll1$. Therefore the expansions used in deriving (3.12) and (3.13) are valid.

Case 4 is, as one would expect, intermediate between case 3 and case 1. The influence of having unequal hole and electron densities (always a possibility since the lack of charge balance may be supplied by ionized impurity atoms) is considered in case 5, and the boundary between growing waves and damped waves is shown in Fig. 3. The increase of hole density by a factor of 50 has the effect of increasing the threshold for growth of long-wavelength oscillations relative to the other cases considered; however, very short-wavelength oscillations continue to grow at lower drift velocities. For a given drift velocity, a considerably larger number of wave numbers can grow than for case 3 which has the same temperature ratio but equal densities.

It is clear from the foregoing considerations that an appreciable lowering of the threshold drift velocity for creating unstable waves can be attained by maintaining one of the plasmas at a low temperature. Of course it would be best if both plasmas could be maintained at a low temperature with a large relative drift velocity; this is made difficult due to the scattering of the particles with the lattice and with each other. For the best situations in semiconductors we will see that the ratio of electron to hole temperatures is between 5 and 10; thus, neglecting other relaxation processes, the electrons would be required to have a drift velocity between 0.22 and 0.40 times the electron thermal velocity in order that a growing wave instability be excited. In general, somewhat higher drift velocities are required due to particle-lattice collisions. Clearly any drift that the holes attain from the external field works in favor of instabilities since the holes have a drift velocity which is opposite to that of the electrons and only the relative drift velocity enters the criterion for growth. However, if the holes have a high mobility their temperature will increase and therefore a balance is required between the hole drift velocity, temperature increase, and relaxation time.

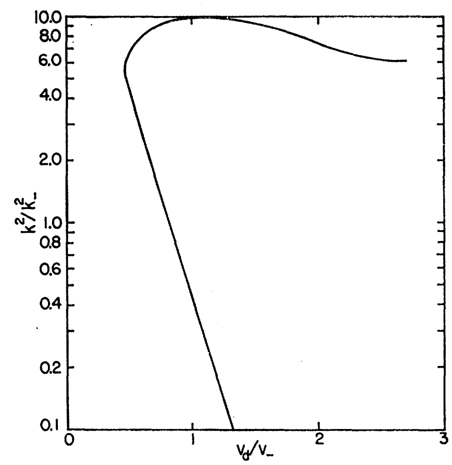

FIG. 3. The boundary between growing waves and damped waves for an electron-hole plasma such that $m_+=14m_-$, $n_+=50n_-$, and $T_-=4T_+$.

IV

Growth rates for the instabilities we have been considering may be calculated in a straightforward fashion as long as we stay within the linear region for which the dispersion relations of the preceding sections apply. In the calculations which follow we take into account the influence of hole lattice collisions by introduction of the relaxation time τ_+; the electron-lattice collision time τ_- will be an order of magnitude larger than τ_+ for the cases which interest us, and its influence on the electronic polarizability may be safely be neglected. The introduction of τ_+ has the effect of simply reducing the growth rates by an amount $1/\tau_+$.

Approximate expressions can be derived for the growth rates for cases 2 and 5 in the region of interest, i.e., $v_d\gtrsim v_-$, since the high-frequency and low-frequency

expansions of the polarizabilities may be used for holes and electrons respectively. The real part of the dispersion relation is given by

$$1\cong\frac{\omega_+^2}{[\omega_r+(\omega_i+1/\tau_+)]^2}\left[\omega_r-\left(\omega_i+\frac{1}{\tau_+}\right)\right]^2 -\left(1-\frac{2(\omega_r-kv_d)^2}{(kv_-)^2}\right)\frac{k_-^2}{k^2}; \quad (4.1)$$

here $\omega=\omega_r+i\omega_i$, and we have assumed that $kv_-\tau_-\gg1$. Typical growth rates for $\tau_+=\infty$ are of the order of $\omega_+/20$ to $\omega_+/10$ so that it is necessary for $\omega_+\tau_+$ to be of the order of 20 to obtain growth. Since the waves with maximum growth rate occur for $k\simeq k_-$, it follows that $\omega\sim\omega_p$ and it is legitimate to neglect $1/\tau_+^2$ in the real part of the dispersion relation compared to ω^2. Thus,

$$\omega\cong s(k)k, \quad (4.2)$$

where

$$s^2(k)=\frac{\omega_+^2}{k^2+\{1-2[s(k)-v_d]^2/v_-^2\}k_-^2}\simeq\frac{\omega_+^2}{k^2+k_-^2}. \quad (4.3)$$

The imaginary part of the dispersion relation may be expressed as

$$\omega_i=-\frac{1}{\tau_+}-\frac{\pi^{\frac12}}{2}\frac{\omega_r^3}{\omega_+^2}\frac{k_{D-}^2}{k^2}\left(\frac{\omega_r-\mathbf{k}\cdot\mathbf{v}_{d-}}{kv_-}\right)\times\exp\left[-\left(\frac{\omega_r-\mathbf{k}\cdot\mathbf{v}_d}{kv_-}\right)^2\right]; \quad (4.4)$$

on inserting the expression for ω we obtain the growth rate

$$\omega_i\cong-\frac{1}{\tau_+}-\frac{\pi^{\frac12}}{2}\frac{\omega_+x}{(1+x^2)^{\frac32}}Z_{1-}\exp(-Z_{1-}^2), \quad (4.5)$$

where

$$Z_{1-}=\frac{(\omega_+/\omega_-)}{[2(1+x^2)]^{\frac12}}-\frac{\mathbf{k}\cdot\mathbf{v}_d}{kv_-}.$$

There are a number of conclusions one can draw from this expression. For growth of oscillations k must be essentially in the direction of the drift velocity. In the absence of $1/\tau_+$, there always exists a value of k above which growth can occur for arbitrarily small drift velocity; however, the growth rate decreases as $1/x^3$ for $x\gg1$. If we had chosen a low but finite temperature for the holes, Landau damping of the holes would become important for large k and eventually lead to damped oscillations in accordance with the boundary curves discussed above. We note that the growth rate increases linearly with ω_+ for fixed ω_+/ω_- so that it is desirable to have a high density of carriers. Also, the growth rate is increased if ω_+/ω_- is decreased; this can be attained either by increasing the number of electrons or by choosing a material with small electron mass. It is of course desirable to have a large ratio of v_d/v_- and a large relaxation time τ_+. A plot of the growth rate based on (4.5) for $T_+=0$, $m_+/m_-=14$, $n_+=n_-$, $v_d=\frac12v_-$, and $\tau_+=\infty$ is given in Fig. 4. The maximum growth rate is approximately $0.107\omega_+$ and therefore it is necessary to attain a relaxation time τ_+, such that $\omega_+\tau_+>10$ for unstable oscillations to occur.

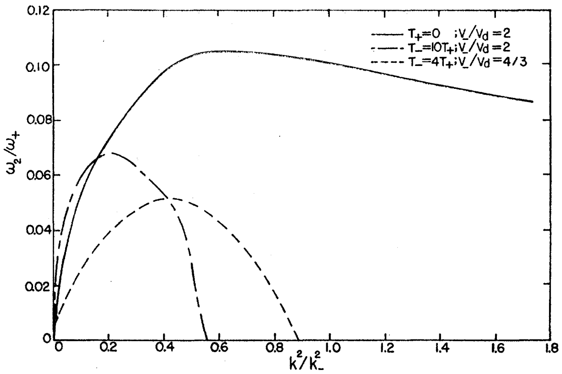

FIG. 4. Growth rate curves for several cases of interest.

Thus far the direct influence of the external field on the plasma oscillations has been neglected. This approximation is valid if the high frequency expansion of the hole polarizability is appropriate; however the correction to the imaginary part of the low-frequency electron polarizability leads to an extra contribution to the growth rate of the form

$$\pi^{\frac12}Z_1\exp(-Z_1^2)\to\pi^{\frac12}Z_1\exp(-Z_1^2)+\eta/(1+\eta^2),$$

where $\eta=e\mathbf{E}_0\cdot\mathbf{k}/k^2\kappa T_-$. For k in the direction of the drift velocity, η is negative and hence the external field diminishes the instability. For typical values of parameters, e.g., $E_0=100$ v/cm, $k=k_D$, $n_-=3\times10^{15}$, and $T_-=150$°K, $\eta=0.05$ which is to be compared with $\pi^{\frac12}Z_1\exp(-Z_1^2)=0.55$ at the fastest growing k. Thus, the influence of the external field is relatively weak in this case.

Growth curves for cases 3 and 4 were calculated by expanding the W functions about their values for real arguments and keeping first order terms in the imaginary parts. The results are also plotted in Fig. 4. Case 3 was calculated for a drift velocity $v_d=v_-/2$ and exhibits a maximum growth rate which is approximately half of that given by case 2, that is, $T_+=0$. Since case 4 would give a very small growth rate for $v_d/v_-=\frac12$, we have chosen the value of $\frac34$ in drawing our plot. The maximum growth rate is approximately $0.067\omega_{p+}$ for this drift velocity. Case 5 was treated by using the expression (4.5) and including the Landau damping of the holes. The results are shown in Fig. 5 for $v_d/v_-=\frac34$. The small apparent value of the maximum growth rate is somewhat misleading, since the quantity plotted is ω_2/ω_+. Due to the fact that n_+ is increased by a factor of 50 over the value in case 4, the maximum growth rate is actually larger by a factor of 4.5 than that in case 4.

We conclude from this brief analysis that in order to obtain growth of unstable oscillations it is desirable to work with a material with high densities of carriers and long relaxation times for particle-lattice scattering. For very high carrier densities hole-electron scattering via the screened Coulomb potential must be taken into account. The plasmas should have a relatively large temperature ratio and the relative drift velocity should be of the order of the thermal velocity of the hot plasma. The possibility of attaining these conditions is discussed in the next section.

V

In order to decide upon the feasibility of producing and observing the two-stream instability in solid-state plasmas, it is necessary that we investigate in some detail the extent to which an applied electric field $\mathbf{E}_0$ may give rise to an appreciable relative electron-hole drift velocity in the solid. Such electric fields not only shift the average electron and hole velocities but also alter the effective electron and hole temperatures. An added complication is the presence of a variety of temperature-dependent, lattice-scattering mechanisms for both the holes and electrons: we consider acoustic phonon scattering, optical phonon scattering, and ionized impurity scattering here.

We carry out our calculations under the assumption that inter-electron or inter-hole collisions dominate in determining the form of the distribution function; the particle distribution function in the presence of the applied field may then be taken as Maxwellian with respect to an average drift velocity $v_{d\pm}$, and an effective temperature (greater than that of the lattice) $T_\pm$. This assumption, which was introduced by Fröhlich[9] for the hot-electron problem, obviously requires that the exchange of energy and momentum among the electrons (or holes) take place at a rate which is fast compared to the rate $\tau_\pm$, which characterizes the particle-lattice interaction. We have, in fact, made a similar assumption in our treatment of the two-stream instability,

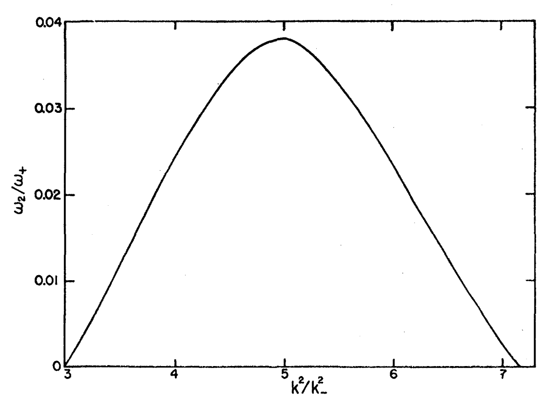

FIG. 5. Growth rate curve for $n_+=50n_-$; $T_-=4T_+$.

[9] H. Frölich, Proc. Roy. Soc. (London) **A188**, 521 (1947).

when we assumed that the influence of the external field could be characterized by a drift velocity, and that the particle-lattice interaction could be characterized by a relaxation time. We postpone until the latter part of this section a discussion of the validity of this assumption for the case under consideration.

We assume, therefore, that the hole and electron distribution functions in the presence of external field, $\mathbf{E}_0$, $f_\pm(\mathbf{E}_0)$, may be written as

$$f_\pm(E_0)=a_\pm \exp\left(-\frac{m_\pm(v-v_{d\pm})^2}{2\kappa T_\pm}\right). \tag{5.1}$$

For definiteness, we consider the electron distribution function which satisfies the Boltzmann equation

$$\frac{e\mathbf{E}_0}{m_-}\cdot\nabla_v f_-=\left(\frac{\partial f_-}{\partial t}\right)_{\text{coll}}. \tag{5.2}$$

To determine the two parameters v_{d-} and T_-, it is convenient to take the momentum and energy moments of (3.2). In this manner one obtains the relations

$$\frac{eE_0}{m}\int p_z\frac{\partial}{\partial v_z}fd^3v=-neE_0$$
$$=\int p_z\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}d^3v\equiv F(v_{d-},T), \tag{5.3}$$
$$\frac{eE_0}{m}\int\frac{mv^2}{2}\frac{\partial}{\partial v_z}fd^3v=-nev_{d-}E_0$$
$$=\int\frac{mv^2}{2}\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}d^3v\equiv G(v_{d-},T_-). \tag{5.4}$$

The electron-electron collisions conserve energy and momentum; therefore their influence may be neglected in computing the right-hand sides of (5.3) and (5.4). The simultaneous solution of this pair of equations determines v_{d-} and T_- as a function of E_0.

Before proceeding to a detailed consideration of these equations, we make the following qualitative remarks concerning the role of the different scattering mechanisms. First, a given scattering mechanism may play a quite different role for energy transfer than for momentum transfer, that is, the dominant contributions to F and G may be from different mechanisms. Thus the scattering by ionized impurities is the dominant scattering mechanism at low temperatures, but is ineffective as an energy transfer mechanism because of the large impurity atom mass. On the other hand, the scattering by optical phonons (which possess an approximately constant energy $\hbar\omega\cong\kappa\theta$, where θ is the Debye temperature) provides the most effective energy transfer mechanism over a wide range of temperatures for which the acoustic scattering provides the dominant momentum transfer mechanism.

Indium antimonide seems in many respects one of the

most promising semiconductors for the possible observation of the two-stream instability. As we have mentioned, the electrons and holes possess quite different masses, so that their mobilities are correspondingly different. The low-field mobilities of both holes and electrons in InSb have been measured as a function of temperature by Putley.[10] He finds that for a concentration of 2×10^{14}, electrons/cc the electron mobility at 20°K is $\cong10^5$ cm²/volt sec. As the temperature increases, the impurity scattering decreases in effectiveness, so that the electron mobility increases until it reaches a maximum of $\cong7\times10^5$ cm²/volt sec at 60°K after which it falls off. For holes, the dominant scattering mechanism for the specimen used by Putley from 30°K upwards appears to be acoustic scattering; Putley finds a fit for his measured values of mobility of the form

$$\mu=5.4\times10^6T^{-1.45}\ \mathrm{cm^2/volt\ sec},$$

where T is in °K. Thus at 20°K, for a sufficiently low impurity atom concentration, the hole mobility would be 7×10^4 cm²/volt sec. If one uses the Brooks-Herring formula for the scattering of the holes by ionized impurities,

$$\mu=\frac{2^{7/2}}{\pi^{\frac{3}{2}}}\frac{\epsilon_0{}^2(\kappa T)^{\frac{3}{2}}}{m_+{}^{\frac{1}{2}}e^3n_\pm}\frac{1}{\ln(1+b)-b/(1+b)},\tag{5.5}$$

where $n_\pm$ is the impurity density, and

$$b=\frac{2}{\pi}\frac{\epsilon_0 m(\kappa T)^2}{n_+\hbar^2e^2},\tag{5.6a}$$

one finds that for a density of 3×10^{15} holes/cc, the mobility due to impurities is $\sim10^4$ cm²/volt sec, assuming $m_+=0.18m$.

In general, as one goes to larger field strengths, the mobility will become field-dependent, since the particle temperature increases with increasing electric field. Indeed, one may pass from a region in which impurity scattering is dominant in determining the relaxation time through a region in which acoustic scattering is dominant up to a region in which optical scattering determines the relaxation time, by simply increasing the field strength. The detailed calculations of the variation on the drift velocity and particle temperature with field strength may be carried out using (5.3) and (5.4).

The calculation of the contributions to F and G due to acoustic phonon and optical phonon scattering has been carried out by Stratton.[11] He evaluates F and G by expanding $f(E_0)$ to first order in v_{d-} so that

$$f(E_0)\cong\exp\left(-\frac{mv^2}{KT_-}\right)\left[1+\frac{m_-\mathbf{v}\cdot\mathbf{v}_{d-}}{\kappa T_-}\right],\tag{5.6}$$

an approximation which is valid if $mv_{d-}{}^2/2\kappa T_-\ll1$. We are interested in the case $v_{d-}\sim\frac{1}{2}v_-$ or $mv_{d-}{}^2/2\kappa T_-\simeq\frac{1}{4}$ and therefore we might expect to obtain a drift velocity which is accurate to within 25% by following this procedure. Stratton gives the following expressions for F due to both acoustic and polar optical mode scattering:

$$F_{\mathrm{ac}}=\left(\frac{T_-}{T}\right)^{\frac{1}{2}}\frac{nev_{d-}}{\mu_{\mathrm{ac}}},\qquad \kappa T\ll(\kappa T_-m_-u_L{}^2)^{\frac{1}{2}}\tag{5.7}$$

$$=\left(\frac{T_-}{T}\right)\frac{256}{27}\frac{\sqrt{2}}{\pi^{\frac{3}{2}}}\left(\frac{m_-u_L{}^2}{\kappa T}\right)^{\frac{1}{2}}\frac{nev_{d-}}{\mu_{\mathrm{ac}}(T)},\qquad \kappa T\gg(\kappa T_-m_-u_L{}^2)^2,\tag{5.8}$$

where u_L is the longitudinal sound velocity and μ_{ac} is the zero-field mobility due to acoustic phonon scattering;

$$F_{\mathrm{op}}=\frac{1}{3\sqrt{\pi}}N_0F_0\frac{\gamma_-{}^{\frac{3}{2}}e^{\gamma_-/2}}{(2m_-\kappa\theta)^{\frac{1}{2}}}\times[\{e^{\gamma-\gamma_-}-1\}K_0(\gamma_-/2)+\{e^{\gamma-\gamma_-}+1\}K_1(\gamma_-/2)nemv_{d-}],$$

where $N_0=[e^\gamma-1]^{-1}$, $\gamma=\theta/T$, $\gamma_-=\theta/T_-$, and K_0 and K_1 are modified Bessel functions of the second kind. Also, $\kappa\theta$ is the energy of optical phonons, taken to be a constant and the parameter F_0, having the dimensions of an electric field strength, is given by

$$F_{0\pm}=(\epsilon_\infty{}^{-1}-\epsilon_0{}^{-1})em_\pm\kappa\theta/\hbar^2,$$

where ϵ_0 and ϵ_∞ are the static and optical dielectric constants of the material.

The Brooks-Herring formula for ionized impurity scattering leads to the expression

$$F_i=\frac{8\pi^2}{3}\frac{e^4N_im_-{}^2n_-v_{d-}\phi(b)}{\epsilon_0{}^2(2m_-\kappa T_-)^{\frac{3}{2}}}=\frac{nev_{d-}}{\mu_i(T)}\left(\frac{T}{T_-}\right)^{\frac{3}{2}},\tag{5.10}$$

where $\phi(b)=\ln(1+b)-b/(1+b)$ and b is given by (5.6a). μ_i is the zero-field mobility due to impurity scattering and N_i is the ionized impurity concentration. The requirement of a long relaxation time demands that the experiments be carried out at a low lattice temperature so that the phonon scattering is small. As pointed out above, the effect of impurity scattering will decrease due to heating of the carriers by the electric field and for this reason it is desirable to carry out experiments at temperatures below the zero-field mobility maximum.

The G functions are given by

$$G_{\mathrm{ac}}=\frac{3neu_L{}^2}{\mu_{\mathrm{ac}}(T)}(1-T/T_-)(T_-/T)^{\frac{1}{2}},\tag{5.11}$$

[10] E. Putley, Proc. Phys. Soc. (London) **73**, 128 (1959); **73**, 280 (1959).

[11] R. Stratton, Proc. Roy. Soc. (London) **A246**, 406 (1958).

$$G_{\mathrm{op}}=neN_0F_0\left(\frac{2K\theta}{\pi m_-}\right)^{\frac{1}{2}}\times[e^{\gamma-\gamma_-}-1]\gamma_-^{\frac{1}{2}}e^{\gamma_-/2}K_0(\gamma_-/2), \quad (5.12)$$

and

$$G_i=\frac{\sqrt{2}neKT}{M\mu_i(T)}\left(\frac{T}{T_-}\right)^{\frac{1}{2}}\left(1-\frac{T}{T_-}\right), \quad (5.13)$$

where M is the mass of an impurity atom.

Calculations of the hole and electron temperatures and the ratio of the electron drift velocity to the electron thermal velocity for p-type InSb at $T=20°\mathrm{K}$ have been carried out for $N_i=10^{15}/\mathrm{cm}^3$ on the basis of the above formulas. The results of these calculations are plotted in Fig. 5. In carrying through the analysis, we have attempted to take into account the variation of the electron effective mass in InSb with temperature (the variation arising from the nonparabolic form of the bands due to spin orbit coupling) by using an effective mass appropriate to hot electrons, $m_-=0.03m$. The calculations for electrons show that the optical phonons dominate both the momentum loss and energy loss for electron temperatures greater than 100°K. For $T_-=60°\mathrm{K}$, the impurity and optical phonon scattering contribute essentially equally to the momentum loss of the electrons while the acoustic phonon contribution is smaller by a factor of 5; the optical phonons however, still dominate the energy loss by a factor of 20 over the acoustic phonons at this temperature. The situation is somewhat more complicated for the holes where the momentum loss is primarily due to acoustic phonons up to $T_+=80°\mathrm{K}$, at which point the optical phonons take over and become the dominant scattering mechanism; the impurity scattering plays essentially no roll for $T_+>40°\mathrm{K}$, due to the decrease in the Coulomb cross-section for high velocities. The energy loss for the holes is primarily due to optical phonons down to the hole temperature $T_+=30°\mathrm{K}$, where the acoustic phonons contribute approximately 10% of the energy loss. The impurity scattering is ineffective in the energy loss mechanism over the entire range of temperatures for both holes and electrons.

It should be emphasized that these calculations have been based on a value of $\epsilon_0=17.5$ taken from the infrared measurements of Spitzer and Fan.[12] Ehrenreich[13] finds that $\epsilon_0=18.9$ leads to better agreement between his calculation of the mobility associated with mode scattering and the experimental value. If this larger value of ϵ_0 is chosen, F_0 should be increased by a factor 1.7 and a somewhat smaller temperature ratio will obtain for a given value of E_0. Since the optical modes give by far the largest contribution to both the F and G functions for fields greater than 100 v/cm, the increase in F_0 will simply scale the E_0 axis by a factor of 1.7. The curves for the holes may be approximately

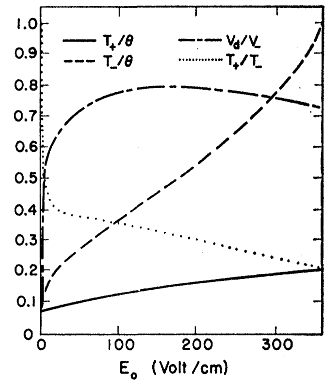

FIG. 6. Hole and electron temperatures and electron drift velocity as a function of field strength in InSb, assuming the following initial conditions: $T_{\mathrm{lattice}}=20°\mathrm{K}$; $m_-=0.03m$; $m_+=0.18m$; $N_i=10^{15}/\mathrm{cc}$.

adjusted by scaling the E_0 axis by $\sqrt{1.7}$ for $T_+<80°\mathrm{K}$ and 1.7 for $T_+>80°\mathrm{K}$.

The foregoing analysis of the hot-electron problem shows that substantial temperature ratios T_-/T_+ and drift velocities v_d/v_- may be achieved at moderate electric fields. Although we have not carried out detailed calculations of the growth rate for the conditions obtaining for the curves of Fig. 6, we can draw some tentative conclusions as to whether the conditions for the existence of the two-stream instability can be met in InSb. We first remark that on comparing Fig. 5 with Figs. 3 and 4 we may conclude that growth rates of the order of $\omega_+/15$ appear definitely achievable, provided hole-lattice scattering effects are negligible. Thus one requires, at the very least, a hole-lattice relaxation time τ_+, which is sufficiently long that

$$\omega_+\tau_+\gtrsim 15.$$

ω_+ is dependent on the hole density, and τ_+ is dependent on both the impurity atom density and on the temperature. The scattering due to acoustic and optical phonons can presumably be eliminated by going to a sufficiently low initial lattice temperature (and correspondingly lower particle temperatures) so that it is the hole-impurity scattering mechanism which will cause the greatest trouble. For example, for $n_+=3\times10^{15}$, one finds $\omega_+=1.8\times10^{12}\ \mathrm{sec}^{-1}$; on the other hand, for this concentration of ionized impurity atoms at $T_+=20°\mathrm{K}$ one finds $\tau_+=2\times10^{-12}$ sec, so that $\omega_+\tau_+\cong3.6$ in this case.

Matters improve slightly as one reduces the ionized impurity concentrations; at $T=20°\mathrm{K}$, the best one can do is to reduce it so much that acoustic scattering is dominant; in this case, one would have $\tau_+=7.25\times10^{-12}$ sec. What is wanted then is a higher density of holes than of impurity centers, a condition which can be achieved through ionization across the gap by the strong electric fields, according to Glicksman and Steele.[14] Thus one might start with a p-type sample, containing $\sim10^{14}$ impurity centers and holes/cc; on applying a strong E_0, one could produce perhaps several times 10^{15} electrons and holes/cc; an $\omega_+\tau_+$ of the order

[12] R. Spitzer and H. Y. Fan, Phys. Rev. **106**, 882 (1958).
[13] H. Ehrenreich, J. Phys. Chem. Solids **2**, 131 (1957).
[14] M. Glicksman and M. C. Steele, J. Phys. Chem. Solids **8**, 242 (1959).

of 15 would then seem not completely out of the question provided one further worked at values of initial lattice temperature such that the hole temperature T_+, for the value of E_0 in question, was somewhat lower than 20°K. It should be added that under these circumstances it is probably necessary to consider the relaxation time associated with the impact ionization. Detailed considerations are required to decide what would be the best experimental setup, though it does seem likely that unless the ionized impurity concentration is substantially below 3×10^{15}/cc, and the initial lattice temperature below 20°K, the conditions for the existence of the two-stream instability cannot be met in InSb.

We conclude this section with a brief discussion of the validity of our assumption of a displaced Maxwellian distribution function for the holes and electrons. Following Fröhlich and Paranjape[15] we may regard the electron-electron collisions as being more effective provided the energy loss per unit time by an electron (moving at a velocity greater than v_-) in electron-electron collisions is greater than that in electron-phonon collisions. Fröhlich and Paranjape find that this will be true for acoustic phonon collisions if

$$n_->n_a\equiv\frac{1}{4\pi}\frac{E^{\frac{3}{2}}m_-^{\frac{1}{2}}u_l^2\epsilon_0^2}{\kappa Te^4\tau_{\mathrm{ac}}(E)},\qquad(5.14)$$

where E is the average electron energy, and τ_{ac} is the relaxation time for collisions between electrons of energy E and acoustic phonons of lattice temperature T. This condition may be expressed in the form

$$n_->10^{13}\left(\frac{T}{293}\right)^{\frac{1}{2}}\left(\frac{E}{\kappa T}\right)^2\left(\frac{\epsilon_0}{10}\right)^2\times\left(\frac{u_l}{10^5}\right)^2\left(\frac{10^3}{\mu_{\mathrm{ac}}}\right)^2\left(\frac{m_-}{m_0}\right)^{\frac{1}{2}},\qquad(5.15)$$

where μ_{ac} is the zero-field mobility due to acoustic phonon scattering at the temperature T. For electrons in InSb with $T=20$°K and choosing $E=0.01$ ev, $m_-/m_0=0.03$, $u_l=3.6\times10^5$ cm/sec, $\epsilon_0=17.6$, and $\mu_{\mathrm{ac}}(20°\mathrm{K})=4.3\times10^7$ cm²/v sec corresponding to a deformation potential constant $E_1=-7.2$ ev, the criterion becomes

$$n_->2.4\times10^5\ \mathrm{cm}^{-3},$$

which is of course always met. As pointed out earlier, Putley finds the hole mobility in InSb can be well represented by a combination of impurity scattering and a mechanism giving rise to a mobility which varies as $5.4\times10^6T^{-1.45}$. We have assumed in our discussion that this mechanism is in fact acoustic mode scattering. The criterion (5.15) for holes then becomes

$$n_+>7.8\times10^{11}\ \mathrm{cm}^{-3},$$

and therefore the interparticle scattering dominates the particle-acoustic phonon interaction in determining the distribution function for all concentrations of practical interest at $T=20$°K in InSb.

The corresponding criterion for interparticle collisions to dominate the optical mode scattering in determining the distribution function is

$$n>\bar{n}=\frac{\epsilon_0^2F_0\kappa\theta}{2\pi e^3}\left(\frac{T_-}{\theta}\right)^{\frac{1}{2}}e^{-\theta/T_-}\quad\text{for}\quad T_-\lesssim\theta.\qquad(5.16)$$

For InSb, taking $T=20$°K, $\theta=290$°K, $T_-=145$°K, $T_+=50$°K, $\epsilon_0=17.5$, $\epsilon_\infty=16$, $m_+/m_0=0.18$, and $m_-/m_0=0.03$, we have

$$n_+>2.8\times10^{14}\ \mathrm{cm}^{-3}$$

and

$$n_->3.7\times10^{15}\ \mathrm{cm}^{-3}.$$

It is impossible to satisfy both of these criteria for steady state conditions in weak external fields at $T=20$°K. However, due to impact ionization of electrons across the energy gap by hot carriers, it may be possible to increase the minority carrier concentration sufficiently to satisfy these conditions for both holes and electrons.

It would appear from the preceding discussion that the growth rate could be increased by increasing the concentration of carriers. This could be accomplished by using a pulsed beam of high-energy electrons. Several difficulties arise however, if one goes to high carrier concentrations. The use of classical statistics is not longer valid if $T_\pm\lesssim T_{d\pm}$, where the degeneracy temperature is determined from the relation

$$\frac{1}{3\pi^2}\left(\frac{2m_\pm\kappa T_{d\pm}}{\hbar^2}\right)=n_\pm.$$

The concentrations appropriate to InSb for degeneracy temperatures $T_{d\pm}=100$°K are $n_+=5.7\times10^{16}$ cm^{-3} and $n_-=1.6\times10^{16}$ cm^{-3}. There is also the problem of a decrease in the relaxation time due to electron-hole scattering as the concentration is increased. The relaxation time for electrons may be estimated from an extension of the Brooks-Herring formula to include screening due to both holes and electrons. By treating the holes as being fixed scattering centers, we have the relation

$$\frac{1}{\tau(k)}=\frac{(4\pi e^2)^2}{8\pi\hbar^3k^3\epsilon_0}m_-n_+\left[\ln(1+b)-\frac{b}{1+b}\right],$$

where $b=4k^2/(k_{D+}^2+k_{D-}^2)$. For $n_\pm=10^{16}$ cm^{-3}, $T_+=50$°K, $T_-=200$°K, and $\hbar^2k^2/2m_-=0.01$ ev, the screening parameter b is equal to 7.65 and the electron relaxation time is 1.16×10^{-13} sec. The relaxation time for the holes is more difficult to estimate due to the large mass ratio of holes and electrons. The electron relaxation time in any event would appear to be sufficiently short

[15] H. Fröhlich and B. Paranjape, Proc. Phys. Soc. (London) B69, 21 (1956).

to make the approach of dubious merit in generating plasma instabilities.

VI

The behavior of the system once the threshold for growth of instabilities is reached is quite complicated; we can here only make some qualitative remarks, and hope to draw attention to some of the interesting features of the problem. In thermal equilibrium, each relatively undamped plasma mode will possess a small amplitude of vibration due to thermal excitation. As the particle drift velocity increases just beyond the threshold for growth of oscillations, a small number of modes will have their thermal level of oscillation amplified by the two-stream mechanism. These growing waves will absorb energy from the directed particle drift motion and therefore decrease the drift velocity slightly. The decreased drift velocity will then support fewer growing modes. This process will continue until the energy supplied by the external field is appropriately distributed amongst the lattice waves and the growing acoustic plasma modes

In order to calculate this distribution, it is necessary to know the level of plasma oscillation which exists, and a knowledge of this in turn requires an understanding of the nonlinear coupling between the plasma modes of different wavelength. This coupling causes a long wave-length mode to decay into higher wave-number modes. The higher wave-number modes decay in turn into still higher wave-number oscillations until presumably a fine grained random or "thermal" motion will be set up. Since the particles are coupled to the lattice by the various interactions discussed in the last section (i.e., impurity and phonon scattering, etc.) a large part of this thermal energy will be given to the lattice. It would appear that a quasi-steady state would eventually be set up in which the growth of the oscillation due to the two-stream mechanism is just balanced by a decay due to nonlinear interactions. The situation would not be a true steady state since the temperatures of the particles and lattice will increase slowly with time and as a result the threshold for excitation of unstable oscillations and the various scattering mechanisms will be time dependent. For weak nonlinear effects the modes will attain a large amplitude before this quasi-steady state sets in and the rate at which each mode absorbs directed drift energy will be correspondingly large. Now the external field delivers energy to the system at the rate of $\mathbf{j}\cdot\mathbf{E}_0$, part of which goes directly into thermal motion of the lattice by the usual Joule heating process and the remainder goes into excitation of plasma oscillations. It follows that for a given value of E_0 the weaker are the nonlinear effects, the larger is the portion of the available drift energy going into each plasma oscillation and therefore the fewer the modes that will be excited.

These arguments lead to the conclusion that the relative drift velocity will tend to saturate near the threshold for creation of unstable oscillations. Thus the effectiveness of the mechanism in producing a saturation drift velocity (or current) depends upon the effectiveness of the energy transfer from the growing oscillations to fine grained thermal motions. Due to the lack of theoretical and experimental understanding of this nonlinear decay process it is impossible to make predictions in this regard and indeed it is for this reason that experiments on the two-stream instability would be most interesting to carry out.

In our analysis we have assumed that the major portion of the distribution function in the quasi-steady state can be described by a displaced Maxwellian. The validity of this approximation rests upon the level of plasma oscillations which are excited and in turn depends upon the nonlinear effects about which little can be said at this time. It would appear, however, that the qualitative arguments in favor of a saturation current are likely to be correct. It should be noted that a saturation drift velocity may result directly from the optical phonon scattering, a fact which should be taken into account in the analysis of experimental data relating to the generation of the instability in polar crystals.

We mention that another type of electrostatic instability may occur under special circumstances. If the distribution of one of the carriers exhibits a hump other than the main hump of the Maxwellian, plasma oscillations may be exicted by particles whose velocities are in the region where the distribution function is increasing with velocity. Such a situation might exist when one is dealing with a low-density plasma in a polar crystal. In this case the optical phonons are more effective in the energy loss of high-energy particles than the particle-particle collisions which tend to restore the distribution to a Maxwellian form. Thus one might expect a hump to develop in the particle distribution just above the threshold for optical phonon emission. The hump would then lead to coherent excitation of plasma oscillations analogous to that expected for runaway electrons in a hot plasma.

In conclusion, the considerations presented above indicate that the production of a two-stream instability in the coupled hole-electron plasmas of a semiconductor is marginal due to the relatively short relaxation times which are attainable in practice. There is a good possibility that the situation would be brighter for producing the instability in semimetals such as bismuth where relaxation times for both holes and electrons of the order of 10^{-10} second or longer are observed. This case is discussed briefly in the appendix.

ACKNOWLEDGMENT

It is a pleasure to thank Dr. Marshall Rosenbluth for many stimulating discussions on these and related topics.

APPENDIX

Two-Stream Instability in Quantum Plasmas

The discussion of the two-stream instability given above is easily extended to the case in which one must take account of quantum effects. The random-phase approximation for the polarizability, which holds for both classical and quantum plasmas for long wavelengths, is

$$4\pi\alpha(q,\Omega)=\frac{-4\pi e^2}{q^2}\sum_k \{f(k)-f(k+q)\} \times\frac{1}{\hbar\Omega-E_{k+q}+E_k+i\delta}, \quad (A.1)$$

where for Fermi-Dirac statistics $f(k)$ is

$$f(k)=1/[e^{(E_k-\mu)/\kappa T}+1]. \quad (A.2)$$

Here μ is the Fermi energy and $E_k\equiv\hbar^2k^2/2m$. We consider a two-component plasma formed by particles of mass m_- drifting with an average velocity $\mathbf{v}_d$ with respect to a set of particles of mass m_+. The frequency Ω_q of the acoustic plasmon is given by

$$1+4\pi\alpha_+(q,\Omega_q)+4\pi\alpha_-(q,\Omega_q-\mathbf{q}\cdot\mathbf{v}_d)=0. \quad (A.3)$$

If the effective temperature T_+ of the particles of mass m_+ is sufficiently small such that

$$q^2\kappa T_+/m_+\ll\Omega_q^2, \quad (A.4)$$

the polarizability $4\pi\alpha_+$ may be replaced by its high-frequency limit $-\omega_{p+}^2/\Omega_q^2$ where $\omega_{p+}^2\equiv 4\pi ne^2/m_+$; this result is identical to that obtained for high frequencies in the classical plasma. If in addition the particles of mass m_- have a sufficiently large Fermi energy μ_- so that

$$q^2\mu_-/m_-\gg\Omega_q^2. \quad (A.5)$$

the solution of Eq. (A.3) for $\Omega_q=\Omega_{1q}+i\Omega_{2q}$ is

$$\Omega_{1q}=s(q)q,$$

$$\Omega_{2q}=-\frac{3\pi}{4}\frac{m_+}{m_-}\left[\frac{s(q)}{v_{F-}}\right]^3[s(q)q-\mathbf{v}_d\cdot\mathbf{q}], \quad (A.6)$$

where

$$s(q)^2=\frac{4\pi ne^2}{m_+[1+4\pi\alpha_-(q,\Omega_q)]}\cong\frac{1}{3}\frac{m_-}{m_+}v_{F-}^2$$

for $q^2v_{F-}^2\ll 4\pi ne^2/m_-$. Here v_{F-} is the Fermi velocity of the m_- particles. In order to have growing wave solutions ($\Omega_{2q}>0$) it is required that

$$v_q>s(q)\cong(m_-/3m_+)^{\frac{1}{2}}v^F_-. \quad (A.7)$$

One might hope to observe this instability in bismuth where the hole mass m_+ is roughly thirty times that of an electron m_-. Also, one has conductivity relaxation times for both holes and electrons longer than 10^{-10} second at liquid helium temperatures in this case.

If we use $v_{F-}=10^7$ cm/sec, $u_-=0.017$ ev and $\tau_+=\tau_-=10^{-10}$ sec, the growth rate for wave numbers q less than the screening wave number $k_s\cong10^7$ cm^{-1} is

$$\Omega_{2q}\cong10^{10}\{10^{-5}q(v_d/s-1)-1\}.$$

where $s\cong10^6$ cm/sec. Thus with drift velocities of the order of 2×10^6 cm/sec ($\cong v_{F-}/10$), a two-stream instability might be observed. In practice, this high drift velocity is likely to be difficult to attain due to excitation of electrons from the heavy hole band into the conduction band by high-energy conduction electrons.