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Dispersion Relations for Production Amplitudes. II*

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The question of single-variable dispersion relation is discussed in the framework of perturbation theory, for the production reactions: $\gamma+N\rightarrow\gamma+\pi+N$, $\pi+N\rightarrow\gamma+\gamma+N$, and $\pi+N\rightarrow\gamma+\pi+N$. It is shown that the contributions from single-loop diagrams satisfy the dispersion relation which was conjectured by Logunov *et al.* The discussion is extended to higher orders. The amplitude associated with a certain class of diagrams are shown to satisfy the cut plane representation. For the rest of the diagrams, a plausibility argument is given.

I. INTRODUCTION

Due to the large number of independent kinematical variables involved, questions of analyticity for production amplitudes are intrinsically complicated. A simplified and covariant version of the Polkinghorne-Kibble-Logunov kinematics[1] for production reactions has been given by the present author for the class of reactions when a nucleon collides with a boson, resulting in a single nucleon and two bosons, the bosons being photons and/or pions.[2] It has been shown in paper I that the transition amplitude for the double Compton effect $(\gamma+N\rightarrow\gamma+\gamma+N)$ satisfies three different single-variable dispersion relations. For the reactions having a two-pion final state, the conjectured dispersion relations of Kibble and Logunov have been shown to be invalid.[3] For the remaining processes,

$$\begin{aligned}&\gamma+N\rightarrow\gamma+\pi+N,\\&\pi+N\rightarrow\gamma+\gamma+N,\\&\pi+N\rightarrow\gamma+\pi+N,\end{aligned}\tag{1}$$

it is shown in this note that the perturbations analysis gives a hope for the cut-plane representation in the same variable as in the early work. For completeness we include also the double Compton effect in the present discussion.

Our perturbation theoretic discussion is based on a wide class of Feynman diagrams. We restrict the electromagnetic interaction to lowest order, that is, ignore all internal photons. Throughout the entire work, all particles are treated as neutral spinless bosons, though actual selection rules are taken properly into account.

In Sec. II, the kinematics of paper I is reintroduced with some algebraic detail. In Sec. II, we discuss the general analytic structure of the Feynman amplitude for the processes under consideration. A sufficient condition is formulated for the validity of the cut-plane representation. The discussion is so general that one may apply the formalism to every order of the perturbation series. In Sec. IV, the formalism of Sec. III is applied to single-loop diagrams, with the conclusion that the amplitude associated with every single-loop diagram satisfies the conjectured single-variable dispersion relation. In Sec. V, the method of majorization is introduced for purposes of dealing with more complicated diagrams. In Sec. VI, the cut-plane representation is shown, in fact, to be valid for a certain class of diagrams in every order. In Sec. VII, the remaining diagrams are discussed. For these, only a plausibility argument can be given.

II. KINEMATICS

Kinematical details have been fully discussed in Sec. II of paper I. For completeness, however, the five independent variables are introduced again in this section. Let p and p', respectively, denote the four-momenta of the initial and final nucleons; k, that of the initial bosons; and k' and k'', those of the final bosons; that is, we treat the process

$$k+p\rightarrow k'+k''+p',\tag{2}$$

where $k^2=-\mu_0^2$, $k'^2=-\mu_1^2$, $k''^2=-\mu_2^2$, and $p^2=p'^2=-m^2$; $\mu_i=0$ or μ, m and μ, respectively, being the nucleonic and pionic masses.

We investigate the analytic property of the transition amplitude in the variable

$$E=-(k'+k'')\cdot(p+p'),\tag{3}$$

fixing the following variables at their physical values.

$$\begin{aligned}&\eta=(k'-k'')\cdot(p+p')/(k'+k'')\cdot(p+p'),\\&x_1=-k'\cdot(p-p'),\ x_2=-k''\cdot(p-p'),\\&v=(p-p')^2.\end{aligned}\tag{4}$$

In our analysis, the Feyman amplitudes will in the first instance be expressed most directly in terms of the ten scalar invariants:

$$\begin{aligned}&W_0=-(k+p)^2,\quad W_1=-(p'+k')^2,\quad W_2=-(p'+k'')^2,\\&\bar{W}_0=-(p'-k)^2,\quad \bar{W}_1=-(p-k')^2,\quad \bar{W}_2=-(p-k'')^2,\\&\bar{z}_0=-(k'+k'')^2,\quad \bar{z}_1=-(k'-k)^2,\quad \bar{z}_2=-(k''-k)^2,\\&v=(p-p')^2.\end{aligned}\tag{5}$$

* Based on a thesis submitted to Princeton University in partial fulfillment of the requirement for a Ph.D. degree.

[1] J. C. Polkinghorne, Nuovo cimento 4, 216 (1956); T. W. B. Kibble, Proc. Roy. Soc. (London) **A244**, 355 (1958); A. A. Logunov and A. Tavkhelidze, Nuovo cimento **10**, 943 (1958).

[2] Y. S. Kim, Phys. Rev. **124**, 1241 (1961). This article will hereafter be called paper I.

[3] Y. S. Kim, Phys. Rev. Letters **6**, 313 (1961).

We denote these collectively by the symbol Z. These invariants are related to the five independent variables by

$$\begin{aligned}
W_0&=m^2+\mu_0^2+x+E,\\
\bar{W}_0&=m^2+\mu_0^2+x-E,\\
W_1&=m^2+\mu_1^2-x_1+\tfrac{1}{2}(1+\eta)E,\\
\bar{W}_1&=m^2+\mu_1^2-x_1-\tfrac{1}{2}(1+\eta)E,\\
\bar{W}_2&=m^2+\mu_2^2-x_2+\tfrac{1}{2}(1-\eta)E,\\
\bar{W}_2&=m^2+\mu_2^2-x_2-\tfrac{1}{2}(1-\eta)E,\\
\bar{z}_0&=\mu_0^2+2x-v,\\
\bar{z}_1&=\mu_2^2-2x_2-v,\\
\bar{z}_2&=\mu_1^2-2x_1-v,
\end{aligned}\tag{6}$$

where

$$x=x_1+x_2+v.$$

Among the ten invariants of Eq. (6), only W_i and $\bar{W}_i$ are dependent on the dispersion variable E. By reversing the sign of E, one interchanges W_i and $\bar{W}_i$. This operation is equivalent to interchanging p and $-p'$, and will be called nucleon conjugation. The invariants $\bar{z}_1$ and $\bar{z}_2$ are naturally negative whereas $\bar{z}_0$ is positive and greater than $(\mu_1+\mu_2)^2$.

III. ANALYTIC STRUCTURE OF THE PRODUCTION AMPLITUDES

In this section, we shall study the general structure of the Feynman amplitude for the processes under consideration. We shall first make a few remarks on the topological structure of Feynman diagrams and then proceed with a more detailed analysis. The discussion will be general and hence applicable to all orders of the perturbation series.

A diagram will be called weakly connected if it decomposes into two disconnected parts when an internal line is cut. Otherwise, the diagram is said to be strongly connected. Every Feynman diagram is either strongly connected or consists of several strongly connected parts connected together weakly. The four-momentum of a weakly connected internal line is uniquely determined by the external momenta. Such a line generates an isolated pole corresponding to its four-momentum on the mass shell. For the class of reactions under discussion, the amplitude as a whole will have poles associated with

$$W_i=m^2,\quad \bar{W}_i=m^2.\tag{7}$$

These are located at

$$\begin{aligned}
E&=\pm(\mu_0^2+x),\\
E&=[\pm 2/(1+\eta)](\mu_1^2-x_1),\\
E&=[\pm 2/(1-\eta)](\mu_2^2-x_2).
\end{aligned}\tag{8}$$

These pole-type singularities require no further discussions, and in the rest of our analysis we fix our attention on strongly connected diagrams.

We shall often discuss at once a whole group of Feynman diagrams. In such cases, observations and conclusions will refer to the sum of all amplitudes belonging to the group.

The amplitude corresponding to a diagram (strongly connected) with n internal lines and l degrees of freedom in internal momenta can be written as

$$F_{nl}=\lim_{\epsilon\to 0+}\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n\int d^4k_1\cdots \times\int d^4k_l\delta[1-\sum_{i=1}^n \alpha_i]/[\psi(\alpha,k_i,p_{\text{ext}})]^n,\tag{9}$$

where

$$\psi=\sum_{i=1}^n \alpha_i(q_i^2+m_i^2-i\epsilon);$$

q_i represents the four-momentum of the ith internal line and is linear in the k_i and the external momenta. m_i denotes the internal mass of the ith line. The small imaginary $-i\epsilon$ added to every m_i^2 ensures that causality is properly defined in the physical process. The quantities $\alpha_1,\ldots,\alpha_n$ are Feynman's parameters and are denoted collectively by α. Unless otherwise mentioned, we shall understand throughout the following discussion that α is in the region of integration:

$$\alpha_i\geqq 0,\quad \sum_{i=1}^n \alpha_i=1.\tag{10}$$

ψ of Eq. (9) is quadratic in the k_i:

$$\psi=\sum_{i,j=1}^l a_{ij}k_i\cdot k_j+2\sum_{j=1} b_j\cdot k_j+G(m_i^2,p_{\text{ext}}^2,i\epsilon),\tag{11}$$

where a_{ij} is linear in α and symmetric, $a_{ij}=a_{ji}$; b_i is linear in the external momenta; and $G(m_i^2, p_{\text{ext}}^2, i\epsilon)$ takes the form

$$G(m_i^2,p_{\text{ext}}^2,i\epsilon)=\sum_{i=1}^n \alpha_i(p_{\text{ext}})_i^2+\sum_{i=1}^n \alpha_i(m_i^2-i\epsilon),\tag{12}$$

where $(p_{\text{ext}})_i$ is linear in the external momenta. Now F_{nl} can be written as

$$F_{nl}=\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n \times\delta[1-\sum_{i=1}^n \alpha_i][C(\alpha)]^{n-2l-2}/[D_\epsilon(\alpha,Z)]^{n-2l},\tag{13}$$

where

$$D_\epsilon(\alpha,Z)=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1l} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2l} & b_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{1l} & a_{2l} & \cdots & a_{ll} & b_l \\ b_1 & b_2 & \cdots & b_l & G \end{vmatrix}, \tag{14}$$

and

$$C(\alpha)=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{12} & a_{22} & \cdots & a_{2l} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{1l} & a_{2l} & \cdots & a_{ll} \end{vmatrix}. \tag{15}$$

From the above expressions, one can draw the following conclusions:

Lemma 1. $C(\alpha)$ is positive.

Lemma 2. In $D_\epsilon(\alpha,Z)$, the coefficient of $-i\epsilon$ is $C(\alpha)$ and, therefore, is positive.

Lemma 3. D_ϵ is quadratic in the external momenta and has the form

$$D_\epsilon=\epsilon f(\eta,\alpha)+K(\alpha,x_1,x_2,v)-i\epsilon C(\alpha), \tag{16}$$

where$K$ (α,x_1,x_2,v) does not depend on the dispersion variable E.

It is clear that the analytic properties of the amplitude are determined by the D_ϵ function. In later discussions, we shall study the quantity D_ϵ without writing the integral expression for the amplitude. By the letter D, on such occasions, we shall denote "diagram", "discriminant" or determinant of Eq. (14), where the Feynman $i\epsilon$ is ignored.

Lemma 4. If there exists a positive E_0 such that D is positive for $|E|\leqq E_0$, then the amplitude associated with the diagram D is analytic in the entire complex E plane except possibly on the real axis where $|E|>E_0$. For real values of E, the amplitude F_{nl} is to be defined as

$$\begin{aligned} F_{nl}(E,\cdots)&=\lim_{\epsilon\to 0+} F_{nl}(E+i\epsilon,\cdots) \quad \text{for} \quad E>E_0, \\ F_{nl}(E,\cdots)&=\lim_{\epsilon\to 0+} F_{nl}(E-i\epsilon,\cdots) \quad \text{for} \quad E<-E_0. \end{aligned} \tag{17}$$

In the present work, we shall not attempt in all cases to find the largest possible value of E_0, i.e., the exact threshold for the cut-plane representation when it exists. Instead, we contend ourselves with establishing the existence of some E_0 where possible, i.e., the existence of a cut-plane representation.

Lemma 5. The coefficient of each m_i^2 is non-negative. Thus, if there exists an E_0 for a given diagram D, then the same conclusion holds for all diagrams which have the same internal configuration as in D but where each internal mass is equal to or greater than the corresponding mass in the original diagram D.

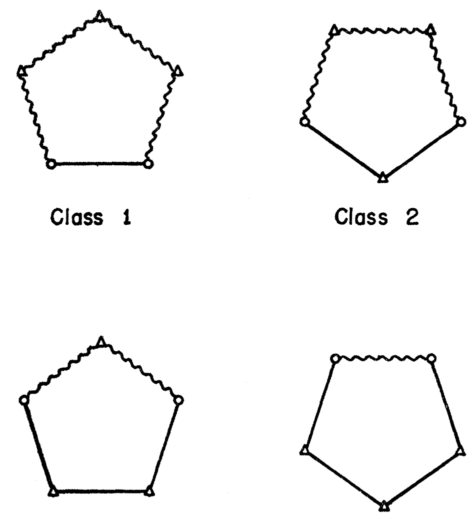

FIG. 1. Pentagons. The vertex symbol △ represents one of the three external bosons, whereas ○ is for the nucleon.

In later discussions, we shall reduce some of the internal masses in a given diagram before computing the value of E_0. We call such a procedure Operation H.

Not all the integrals in the perturbation series are convergent. The divergent integrals, however, are subject to total or partial subtraction according to the renormalization prescription. In investigating analyticity it is thus sufficient to study the integrand of the formal expression for the Feynman amplitude.

Lemma 6. Consider a diagram D having ν "parallel" lines joining a pair of vertices with their respective masses $m_1, \cdots m_\nu$. Next, consider another diagram D' where the ν parallel lines of D are replaced by a single line with mass $(m_1+m_2\cdots m_\nu)$. Suppose now that there exists an E_0 for the diagram D'. Then the same conclusion holds for the diagram D.

We call the procedure of replacing the ν parallel lines by the single-line Operation R.

IV. SINGLE-LOOP DIAGRAMS

We consider in this section diagrams having a single closed loop either originally or after the application of Operation R. Consider, in particular, loops whose sides are either pionic or nucleonic, the nucleonic sides being continuous according to the selection rules. Then these loops constitute the class of diagrams from which all single-loop diagrams can be obtained (through use, if need be, of Operation H).

The loops can be divided into several groups according to their geometrical configurations. For the present problem, there are pentagonal, square, triangular, and self-energy loops.

Let us first consider pentagons. Each of the five vertices is associated with one of the external momenta p, k, $-p'$, $-k'$, and $-k''$. We regard the external four-

vectors as directed inward. The two external nucleons are connected by a continuous nucleonic path which runs along the sides of the pentagon. One can divide these pentagonal loops into four classes according to the number of the nucleonic sides. See Fig. 1. The remaining three vertices are for the external bosons. It is easy to see that there are six crossing configurations for the three bosons, that is, each pentagon of Fig. 1 represents six different diagrams for a given process.

We label the internal lines by cyclic indices 1, 2, 3, 4, and 5; a vertex by a pair of numbers $(i, i+1)$; and the external momentum at the $(i, i+1)$ vertex by $p_{i,i+1}$. For all pentagons, the amplitude can be written as

$$F_5=\int_0^1 d\beta_1\cdots\int_0^1 d\beta_5[\sum_{i=1}^5 \beta_i/m_i]\delta(1-\sum_{i=1}^5 \beta_i)/\mathfrak{D}_5^3, \quad (18)$$

where

$$\mathfrak{D}_5=\sum_{i=1}^5 \beta_i^2-2\sum_{i=1}^5 y_{i,i+1}\beta_i\beta_{i+1} -2\sum_{i=1}^5 y_{i,i+2}\beta_i\beta_{i+2}, \quad (19)$$

$$y_{i,i+1}=-[p_{i,i+1}^2+m_i^2+m_{i+1}^2]/2m_im_{i+1},$$

$$y_{i,i+2}=-[(p_{i,i+1}+p_{i+1,i+2})^2+m_i^2+m_{i+2}^2]/2m_im_{i+2}.$$

The above expression for F_5 can be derived by the standard algebra.[4] The D function associated with F_5 is positive if and only if $\mathfrak{D}_5$ of Eq. (19) is positive. The quantities $y_{i,i+1}$ depend only on the external masses $(-p_{i,i+1}^2)^{\frac{1}{2}}$ and can take on the following values:

$$-\mu/2m,\ -1/2,\ -(1-\mu^2/2m^2),\ -1; \quad (20)$$

that is, every $y_{i,i+1}$ is negative. Since, however, the quantity $-(p_{i,i+1}+p_{i+1,i+2})^2$ is to be identified with one of the ten invariants in Eq. (5), $y_{i,i+2}$ is related to the five independent variables. Now we write $\mathfrak{D}_5$ as

$$\mathfrak{D}_5=\tfrac{1}{2}\sum_{i=1}^5 (\beta_i-\beta_{i+2})^2-2\sum_{i=1}^5 y_{i,i+1}\beta_i\beta_{i+1} +2\sum_{i=1}^5 (\tfrac{1}{2}-y_{i,i+2})^2\beta_i\beta_{i+2}. \quad (21)$$

Thus, in order that $\mathfrak{D}_5$ be positive, it is sufficient, but not necessary, that every $y_{i,i+2}$ be smaller than $\frac{1}{2}$. This can be stated in a more precise way. If

$$-(k'+k'')^2=(\mu_0^2+2x-v)<3\mu^2, \quad (22a)$$

then there exists an E_0, and

$$E_0=\min\left\{m\mu\left(1-\frac{x+\mu_0^2-\mu^2}{m\mu}\right), \frac{2m\mu}{1+\eta}\left(1+\frac{x_1-\mu_1^2+\mu^2}{m\mu}\right), \frac{2m\mu}{1-\eta}\left(1+\frac{x_2-\mu_2^2+\mu^2}{m\mu}\right)\right\}. \quad (22b)$$

The above statement establishes a cut-plane representation for the contributions from all pentagons with the restrictions implied by Eqs. (22a) and (22b). Since for the reactions under consideration $-(k'+k'')^2$ is greater than $(\mu_1+\mu_2)^2$, where at least one of the two masses is zero, the condition (22a) can be satisfied by some physical values of the fixed variables x_1, x_2, and v.

It should be conceded that the above E_0 is not necessarily the largest value obtainable. One may enlarge its value by making a careful analysis for each individual diagram. Nevertheless, the very existence of an E_0 is what we have set out to establish.

It is easy to see that the lower-order loops—squares, triangles and self-energies—can be generated from the pentagons by contraction of suitable sides. Since all possible end points have been considered in our discussion of the pentagons, these lower-order loops, in fact, have been studied, with the result that there exists an E_0 and its value is at least as large as given in Eq. (22).

V. GENERALIZATION TO HIGHER ORDERS

It has been seen that the amplitude associated with the single-loop diagrams satisfy a cut-plane representation in the variable E. In this section, we shall outline a general program of obtaining the same conclusion for all higher orders. Let us begin with some obvious remarks.

By $\{D_n\}$ we denote a set of diagrams $D_1, D_2, \cdots D_n$. Suppose that there exists a diagram D_0 such that

$$D_i\geqq D_0,\quad i=1, 2, \cdots, n; \quad (23)$$

and D_0 is positive for

$$-E_0<E<E_0. \quad (24)$$

Then the cut-plane representation is possible for the amplitude corresponding to the set $\{D_n\}$ as well as for D_0.

From the above remark, it is clear that the standard procedure for studying all Feynman amplitudes will be to divide the diagrams into a finite number of $\{D_n\}$ in such a way that for each $\{D_n\}$ there exists a diagram D_0 with the corresponding E_0. The number n is in general infinite.

The diagram D_0 is called the primitive of the set $\{D_n\}$. Proving the existence of E_0 for the primitive is, as one can see, an algebraic procedure. But establishing the inequality (23) requires a more elegant technique—majorization of Feynman diagrams. If the inequality (23) holds, it is said that the diagram D_0 majorizes the set $\{D_n\}$.

We now introduce the following two important theorems on majorization. Formal proofs of these theorems are given by Chernikov, *et al.*,[5] and are valid only if the external vectors are Euclidean, that is, if Gram's determinant is positive and definite.

[4] R. Karplus, C. Sommerfield, and E. Wichmann, Phys. Rev. **114**, 537 (1959).

[5] N. A. Chernikov, A. A. Logunov, and I. T. Todorov, Dubna report D-578 (unpublished).

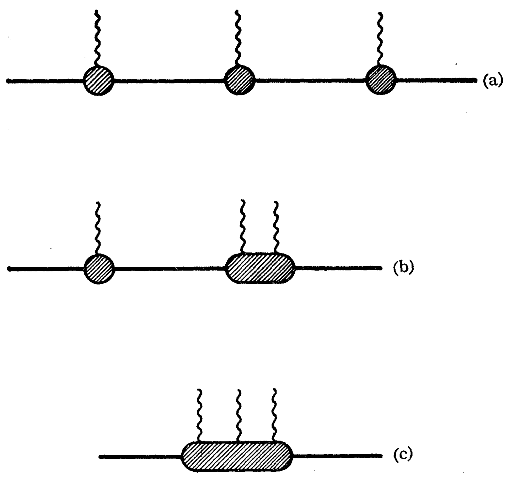

FIG. 2. Diagrams having two (a), one (b), and no (c) weakly connected nucleon lines.

Theorem I. Consider a diagram D_1 with one or more closed loops. Consider another diagram D_2 which is identical to the former except that it has additional internal lines which start and end at the sides of a given loop in the diagram D_1. Then the diagram D_1 majorizes the D_2.

Theorem II. Let a diagram contain a closed loop with $(n+1)$ vertices and sides, to n sides of which the mass m corresponds, and to one side the mass μ ($\mu<m$). Consider a new diagram obtained by interchanging the two masses. Then the new diagram majorizes the original one.

If the external momenta are Euclidean, these two theorems, in principle, enable us to obtain a finite number of $\{D_n\}$ for all reactions under consideration. But, due to the demand that the external vectors be Euclidean, applicability of the theorems is severely limited. In the following sections, we shall discuss first the class of diagrams for which the present formulation gives a complete solution. For the rest, only a plausibility argument will be given.

Before actually classifying the diagrams, let us make a remark on the self-energy parts. It has been seen earlier that in discussing the self-energy loop, one, in effect, considers all self-energy parts. Thus, without loss of generality, we can assume that no weakly connected line contains self-energy parts.

We now classify the diagrams according to the number of weakly connected nucleon lines—two, one, and zero. See Fig. 2. For the first group, our problem is to study vertex functions with two independent external momenta. For the rest, the problem is to investigate four- and five-point functions. We shall establish the validity of the cut-plane representation for the vertex functions associated with the first reduced diagram but, for the others, give only a plausibility argument and discuss the limitation of the present technique.

VI. VERTEX FUNCTIONS ASSOCIATED WITH THE FIRST REDUCED DIAGRAM

We study, in this section, the amplitude associated with one of the three parts in the first reduced diagram of Fig. 2. It is clear that to a given internal structure there correspond six crossing configurations for the external bosons.

We shall first find the Euclidean region in which the two theorems of the preceding section are shown to be valid, and then majorize all vertex diagrams under consideration. It will be shown that there exists an E_0 for the primitives and that the real interval $|E|<E_0$ overlaps the Euclidean region. This will establish the validity of the cut-plane representation for every amplitude corresponding to the diagram reducible to the first configuration of Fig. 2.

Let us first choose the external nucleon momenta (to the vertex part) as the two independent vectors for the vertex part. Let those be denoted by Q and K. For the vertex at the middle of the reduced diagram of Fig. 2, both of the vectors are dependent on the dispersion variable while for the others, one of them should be on the mass shell. In both cases, we choose the new set of independent vectors I_1 and I_2 as

$$I_1=Q+K;\quad I_2=Q-K, \tag{25}$$

and regard the amplitude as a function of three independent scalars I_1^2, I_2^2, and $I_1\cdot I_2$. Then the D function, in general, is written as

$$D=I_1^2f_1(\alpha)+I_2^2f_2(\alpha)+2I_1\cdot I_2g_{12}(\alpha)+G(m_i^2,\alpha), \tag{26}$$

where $f_1(\alpha)$ and $f_2(\alpha)$, as one can show easily, are nonnegative.

The quantity I_2^2 is either zero or $-\mu^2$ depending on the mass of the external boson. But the existence of an E_0 for $I_2^2=-\mu^2$ will automatically imply the same conclusion for the other case. Thus one can regard

$$I_2^2=-\mu^2.$$

Next, we relate the quantities I_1^2 and $I_1\cdot I_2$ to the five independent variables E, η, x_1, x_2, and v.

$$\begin{aligned} I_1^2&=2(Q^2+K^2)+\mu_2^2,\\ I_1\cdot I_2&=Q^2-K^2, \end{aligned} \tag{27}$$

where $(-Q^2,-K^2)=(W_0,\bar{W}_1)$, $(W_0,\bar{W}_2)$, $(W_1,\bar{W}_2)$, and their nucleon conjugations for the vertex at the middle. For the others, $(-Q^2,-K^2)=(W_i,m^2)$, $(\bar{W}_i,m^2)$; and $\mu_i^2=0$ or μ^2.

With this preparation, we can obtain the Euclidean region. Expanding the Gram determinant, one can express the condition for the vectors to be Euclidean as

$$-\mu^2(2Q^2+2K^2+\mu^2)-(Q^2-K^2)^2>0. \tag{28}$$

The above expression is quadratic in E, the coefficient of E being negative. Thus one can easily find the upper and lower limits of the Euclidean region, which, in general, are dependent on the fixed variables η, x_1, x_2,

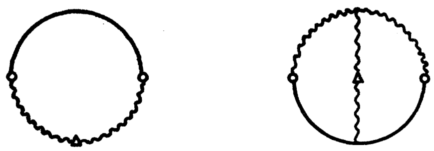

FIG. 3. Primitive diagrams for vertex parts under consideration.

and v. We understand here that the four fixed variables are so restricted that the upper limit is positive and the lower negative. Since the system of diagrams is invariant under nucleon conjugation, both limits must be equal in magnitude. Thus it is sufficient to indicate the upper limit by a positive quantity E_e.

Considering all possibilities for (Q^2,K^2) we finally obtain the boundary of the Euclidean region.

$$E_e=\min\{\mathcal{E}_1,\mathcal{E}_2,\cdots,\mathcal{E}_6\},$$

$$\mathcal{E}_1=2m\mu\left(1-\frac{|\bar{x}|}{2m\mu}\right),\quad \mathcal{E}_2=\frac{4m\mu}{1+\eta}\left(1-\frac{|\bar{x}_1|}{2m\mu}\right),$$

$$\mathcal{E}_3=\frac{4m\mu}{1-\eta}\left(1-\frac{|\bar{x}_2|}{2m\mu}\right),$$

$$\mathcal{E}_4=2m\mu\left\{\left[1+\frac{\mu^2}{4m^2}(3+\eta^2)-\frac{\mu^2}{2m^2}(\bar{x}_2-\bar{x}_1)\eta\right]^{\frac{1}{2}}-\frac{|\bar{x}_2-\bar{x}_1-\mu^2\eta|}{2m\mu}\right\},\tag{29}$$

$$\mathcal{E}_5=\frac{4m\mu}{(1+\eta)^2}\left\{\left[(1+\eta)^2+\frac{\mu^2}{2m^2}(3+\eta^2)-\frac{(1+\eta)[(1-\eta)\bar{x}+2\bar{x}_2]}{m^2}\right]^{\frac{1}{2}}-\frac{|\mu^2(3-\eta)-(1+\eta)(\bar{x}+\bar{x}_2)|}{2m\mu}\right\},\tag{30}$$

$\mathcal{E}_6=\mathcal{E}_5$, where $\bar{x}_2$ and $-\eta$ are, respectively, replaced by $\bar{x}_1$ and η; $\bar{x}=x-\mu^2+\mu_0^2$, $\bar{x}_1=x_1+\mu^2-\mu_1^2$, $\bar{x}_2=x_2+\mu^2-\mu_2^2$.

Next, we have to majorize the vertex diagrams and find an E_0 for the primitives. From the two theorems of the preceding section, it is clear that all vertex parts under consideration are majorized by the primitive diagrams of Fig. 3. The discriminants for those diagrams are calculated in Appendix A. The calculation shows that the E_0 for the two primitives is

$$E_p=\min\left\{m\mu\left(1-\frac{\mu}{4m}-\frac{\bar{x}}{m\mu}\right),\frac{2m\mu}{1+\eta}\left(1-\frac{\mu}{4m}+\frac{\bar{x}_1}{m\mu}\right),\frac{2m\mu}{1-\eta}\left(1-\frac{\mu}{4m}+\frac{\bar{x}_2}{m\mu}\right)\right\}.\tag{31}$$

Thus for all orders, there exists an E_0, and $E_0=\min\times(E_e,E_p)$, that is, the cut-plane representation is possible for all vertex functions under consideration.

VII. OTHER DIAGRAMS

For the vertex functions of the preceding section, it has been shown that the cut-plane representation is possible in every order. The present formalism, however, does not give us any easy method of completing the proof for the rest of the diagrams. We shall not describe here the situation in full detail but discuss briefly how far one can go and where the difficulty lies.

Our majorization procedure works only if the external vectors are Euclidean. In the case of vertex functions we have considered, it was not necessary to bring the fixed variables to unphysical values. This is due to the fact that Gram's inequality does not contain any fixed invariants of Eq. (5), $\bar{z}_0$, $\bar{z}_1$, $\bar{z}_2$, and v. If the rank of the Gram determinant is higher than two, for instance, in the case of four- or five-point functions associated with the diagrams in Fig. 2, Gram's condition depends explicitly on one or more of those fixed invariants. In such cases, one cannot find any finite Euclidean region in the variable E unless the fixed invariants are unphysical. This is the difficulty which prevents us from making such a straight-forward proof as in the preceding section.

It can be shown easily that for each fixed invariant, there exists an unphysical interval in which a finite Euclidean region can be obtained for dispersion variable E. But, in order to complete the proof, one has to justify that the fixed invariants in question can be brought back to their physical values. Along this direction, a progress has been made for the four-point functions under the assumption that the external bosons have the mass μ.[6] Various complicated primitive diagrams have been worked out. At present, the result gives a favorable outlook.

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APPENDIX A

It is easy to see that all vertex parts under consideration are majorized by the two primitive diagrams in Fig. 3. We now compute the D functions.

[6] Y. S. Kim, Thesis, Princeton University (unpublished).

For the first diagram,

$$D_a=(\mu^2+I_2^2)[\tfrac{1}{4}(\alpha_1\alpha_2+\alpha_1\alpha_3)+\alpha_2\alpha_3]$$
$$+(Q^2+m^2+\mu^2-\mu^2/4+3m\mu/2)\alpha_1\alpha_2$$
$$+(K^2+m^2+\mu^2-\mu^2/4+3m\mu/2)\alpha_1\alpha_3$$
$$+\tfrac{1}{8}[(2m\alpha_1-2\mu\alpha_2-\mu\alpha_3)^2$$
$$+(2m\alpha_1-2\mu\alpha_3-\mu\alpha_2)^2+3\mu^2(\alpha_2^2+\alpha_3^2)].$$

For the second diagram,

$$D_b=(\mu^2+I_2^2)[\tfrac{1}{4}(A(\alpha)+B(\alpha)+C(\alpha)]$$
$$+[Q^2+m^2+\mu^2+m\mu-\mu^2/4]A(\alpha)$$
$$+[K^2+m^2+\mu^2+m\mu-\mu^2/4]B(\alpha)$$
$$+(\alpha_3+\alpha_5+\alpha_6)[(m\alpha_1-\mu\alpha_2)^2+m\mu\alpha_1\alpha_2]$$
$$+(\alpha_1+\alpha_3+\alpha_4)[(m\alpha_5-\mu\alpha_2)^2+m\mu\alpha_2\alpha_5]$$
$$+\alpha_2[\sqrt{2}m\alpha_5-(\alpha_3+\alpha_4+\alpha_6)\mu]^2/2$$
$$+\alpha_6[\sqrt{2}m\alpha_1-(\alpha_1+\alpha_3+\alpha_4)\mu]^2/2$$
$$+\tfrac{1}{2}\mu^2[\alpha_2(\alpha_3^2+\alpha_4^2+\alpha_6^2)+\alpha_6(\alpha_2^2+\alpha_3^2+\alpha_4^2)]$$
$$+2m^2\alpha_1\alpha_5(\alpha_3+\alpha_4)$$
$$+\mu^2[(\alpha_3^2+\alpha_4^2)(\alpha_1+\alpha_5)+\alpha_3\alpha_4(\alpha_1+\alpha_5)]$$
$$+\mu^2[\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_6+(\alpha_3+\alpha_4)(\alpha_2\alpha_5+\alpha_1\alpha_6)],$$

where

$$A(\alpha)=\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_4\alpha_5,$$
$$B(\alpha)=\alpha_5\alpha_6(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_1\alpha_4\alpha_6+\alpha_2\alpha_3\alpha_5,$$
$$C(\alpha)=\alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5+\alpha_6)+\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_3\alpha_5.$$

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Relativistic Schrödinger's Equation for the Two-Nucleon System*

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The relationship between the relativistic and nonrelativistic forms of Schrödinger's equation for two nucleons in their center-of-mass system is investigated, and the velocity dependence of the relativistic one-pion exchange nuclear potential is discussed.

1. INTRODUCTION

THE nonrelativistic Schrödinger equation for two nucleons in the center-of-mass system is usually expressed as (taking $c=\hbar=1$)

$$(\nabla^2+k^2-MV)\psi=0, \tag{1}$$

where M is the nucleon mass, k^2/M is the energy of the system, V is the static potential, and ψ is the wave function involving the large components of the nucleon spinors. However, one often has to deal with physical problems like the proton-proton scattering up to about 300 Mev in the laboratory system, where the relativistic corrections are small but not entirely negligible. One must then not only find the velocity-dependent corrections to V but also replace the nonrelativistic operator ∇^2+k^2 appearing in (1) by the relativistic one. Assuming that a velocity-dependent potential V for the two-nucleon system can be obtained, we shall discuss the problem of formulating the relativistic Schrödinger equation. We shall then consider the two-nucleon Schrödinger equation with one-pion exchange potential, and compare our result with those of Breit and Hull[1] and of Sugawara and Okubo.[2]

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[1] G. Breit and M. H. Hull, Nuclear Phys. **15**, 216 (1960).

[2] M. Sugawara and S. Okubo, Phys. Rev. **117**, 605 (1960).

2. RELATIVISTIC SCHRÖDINGER'S EQUATION

The relativistic Schrödinger equation for the two-nucleon system can be written as

$$[(M^2+\mathbf{p}_1^2)^{\frac{1}{2}}+(M^2+\mathbf{p}_2^2)^{\frac{1}{2}}+V]\phi=2E\phi, \tag{2}$$

where the potential V is a function of the relative coordinates of the two nucleons as well as other variables like momentum, spin, isospin etc., $2E$ is the energy of the system including the rest energy of the nucleons, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the nucleon momentum operators

$$\mathbf{p}_1=-i\boldsymbol{\nabla}_1,\quad \mathbf{p}_2=-i\boldsymbol{\nabla}_2, \tag{3}$$

and a square root involving the differential operators is to be regarded as an infinite series in powers of the differential operators, so that

$$(M^2+\mathbf{p}^2)^{\frac{1}{2}}\equiv M+\frac{\mathbf{p}^2}{2M}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2}\frac{(\frac{1}{2})(\frac{3}{2})\cdots(n-\frac{1}{2})}{(n+1)!}\frac{\mathbf{p}^{2n+2}}{M^{2n+1}}. \tag{4}$$

The center-of-mass system can be defined by the condition

$$(\mathbf{p}_1+\mathbf{p}_2)\phi=0, \tag{5}$$

which is compatible with (2) due to the fact that V is a function of the relative coordinates. Then, (2) can be