For the first diagram,

$$
\begin{aligned}
D_a = {} & (\mu^2+I_2^2)[\tfrac{1}{4}(\alpha_1\alpha_2+\alpha_1\alpha_3)+\alpha_2\alpha_3] \\
& +(Q^2+m^2+\mu^2-\mu^2/4+3m\mu/2)\alpha_1\alpha_2 \\
& +(K^2+m^2+\mu^2-\mu^2/4+3m\mu/2)\alpha_1\alpha_3 \\
& +\tfrac{1}{8}[(2m\alpha_1-2\mu\alpha_2-\mu\alpha_3)^2 \\
& \qquad +(2m\alpha_1-2\mu\alpha_3-\mu\alpha_2)^2+3\mu^2(\alpha_2^2+\alpha_3^2)].
\end{aligned}
$$

For the second diagram,

$$
\begin{aligned}
D_b = {} & (\mu^2+I_2^2)[\tfrac{1}{4}(A(\alpha)+B(\alpha)+C(\alpha)] \\
& +[Q^2+m^2+\mu^2+m\mu-\mu^2/4]A(\alpha) \\
& +[K^2+m^2+\mu^2+m\mu-\mu^2/4]B(\alpha) \\
& +(\alpha_3+\alpha_5+\alpha_6)[(m\alpha_1-\mu\alpha_2)^2+m\mu\alpha_1\alpha_2] \\
& +(\alpha_1+\alpha_3+\alpha_4)[(m\alpha_5-\mu\alpha_2)^2+m\mu\alpha_2\alpha_5] \\
& +\alpha_2[\sqrt{2}m\alpha_5-(\alpha_3+\alpha_4+\alpha_6)\mu]^2/2 \\
& +\alpha_6[\sqrt{2}m\alpha_1-(\alpha_1+\alpha_3+\alpha_4)\mu]^2/2 \\
& +\tfrac{1}{2}\mu^2[\alpha_2(\alpha_3^2+\alpha_4^2+\alpha_6^2)+\alpha_6(\alpha_2^2+\alpha_3^2+\alpha_4^2)] \\
& +2m^2\alpha_1\alpha_5(\alpha_3+\alpha_4) \\
& +\mu^2[(\alpha_3^2+\alpha_4^2)(\alpha_1+\alpha_5)+\alpha_3\alpha_4(\alpha_1+\alpha_5)] \\
& \qquad +\mu^2[\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_6+(\alpha_3+\alpha_4)(\alpha_2\alpha_5+\alpha_1\alpha_6)],
\end{aligned}
$$

where

$$
\begin{aligned}
A(\alpha) &= \alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_4\alpha_5, \\
B(\alpha) &= \alpha_5\alpha_6(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_1\alpha_4\alpha_6+\alpha_2\alpha_3\alpha_5, \\
C(\alpha) &= \alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5+\alpha_6)+\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_3\alpha_5.
\end{aligned}
$$

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Relativistic Schrödinger's Equation for the Two-Nucleon System*

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The relationship between the relativistic and nonrelativistic forms of Schrödinger's equation for two nucleons in their center-of-mass system is investigated, and the velocity dependence of the relativistic one-pion exchange nuclear potential is discussed.

1. INTRODUCTION

THE nonrelativistic Schrödinger equation for two nucleons in the center-of-mass system is usually expressed as (taking $c=\hbar=1$)

$$(\nabla^2+k^2-MV)\psi=0, \tag{1}$$

where M is the nucleon mass, k^2/M is the energy of the system, V is the static potential, and ψ is the wave function involving the large components of the nucleon spinors. However, one often has to deal with physical problems like the proton-proton scattering up to about 300 Mev in the laboratory system, where the relativistic corrections are small but not entirely negligible. One must then not only find the velocity-dependent corrections to V but also replace the nonrelativistic operator ∇^2+k^2 appearing in (1) by the relativistic one. Assuming that a velocity-dependent potential V for the two-nucleon system can be obtained, we shall discuss the problem of formulating the relativistic Schrödinger equation. We shall then consider the two-nucleon Schrödinger equation with one-pion exchange potential, and compare our result with those of Breit and Hull[1] and of Sugawara and Okubo.[2]

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[1] G. Breit and M. H. Hull, Nuclear Phys. **15**, 216 (1960).

[2] M. Sugawara and S. Okubo, Phys. Rev. **117**, 605 (1960).

2. RELATIVISTIC SCHRÖDINGER'S EQUATION

The relativistic Schrödinger equation for the two-nucleon system can be written as

$$[(M^2+\mathbf{p}_1^2)^{\frac{1}{2}}+(M^2+\mathbf{p}_2^2)^{\frac{1}{2}}+V]\phi=2E\phi, \tag{2}$$

where the potential V is a function of the relative coordinates of the two nucleons as well as other variables like momentum, spin, isospin etc., $2E$ is the energy of the system including the rest energy of the nucleons, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the nucleon momentum operators

$$\mathbf{p}_1=-i\boldsymbol{\nabla}_1, \quad \mathbf{p}_2=-i\boldsymbol{\nabla}_2, \tag{3}$$

and a square root involving the differential operators is to be regarded as an infinite series in powers of the differential operators, so that

$$(M^2+\mathbf{p}^2)^{\frac{1}{2}}\equiv M+\frac{\mathbf{p}^2}{2M} +\sum_{n=1}^{\infty}\frac{(-1)^n}{2}\frac{(\frac{1}{2})(\frac{3}{2})\cdots(n-\frac{1}{2})}{(n+1)!}\frac{\mathbf{p}^{2n+2}}{M^{2n+1}}. \tag{4}$$

The center-of-mass system can be defined by the condition

$$(\mathbf{p}_1+\mathbf{p}_2)\phi=0, \tag{5}$$

which is compatible with (2) due to the fact that V is a function of the relative coordinates. Then, (2) can be

expressed as

$$[2(M^2+\mathbf{p}^2)^{\frac{1}{2}}+V]\phi=2E\phi, \tag{6}$$

with

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2). \tag{7}$$

Further, putting

$$E=(M^2+k^2)^{\frac{1}{2}}, \tag{8}$$

and

$$\phi=\psi(x,y,z)\chi(X,Y,Z), \tag{9}$$

where

$$\begin{aligned} x&=x_1-x_2, & y&=y_1-y_2, & z&=z_1-z_2,\\ X&=\tfrac{1}{2}(x_1+x_2), & Y&=\tfrac{1}{2}(y_1+y_2), & Z&=\tfrac{1}{2}(z_1+z_2), \end{aligned} \tag{10}$$

we obtain

$$\mathbf{p}\psi=-i\nabla\psi,\quad \mathbf{p}\chi=0, \tag{11}$$

and (6) reduces to the relativistic Schrödinger equation for relative motion,

$$[2(M^2-\nabla^2)^{\frac{1}{2}}-2(M^2+k^2)^{\frac{1}{2}}+V]\psi=0. \tag{12}$$

Multiplying (12) by $-\frac{1}{2}(M^2-\nabla^2)^{\frac{1}{2}}$, we get after some simplification

$$[\nabla^2+k^2-\tfrac{1}{2}(M^2-\nabla^2)^{\frac{1}{2}}V-\tfrac{1}{2}(M^2+k^2)^{\frac{1}{2}}V]\psi=0, \tag{13}$$

which can be written as

$$(\nabla^2+k^2-MV')\psi=0, \tag{14}$$

with

$$V'=\frac{(M^2-\nabla^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}}{2M}V, \tag{15}$$

where ∇^2 operates on V and all the factors following it. This shows that the relativistic Eq. (12) can be expressed also in the usual nonrelativistic form, provided that V is replaced by V'.

3. GREEN'S FUNCTION FOR RELATIVISTIC SCHRÖDINGER'S EQUATION

The Green's function and the Born approximation method for the nonrelativistic form (14) of the Schrödinger equation are well known. It would be interesting to derive the Green's function for the relativistic form (12), and show that it leads to the same result up to any order in the Born approximation.

Let us write (12) as

$$(\Omega-MV)\psi=0, \tag{16}$$

with

$$\Omega=-2M[(M^2-\nabla^2)^{\frac{1}{2}}-(M^2+k^2)^{\frac{1}{2}}], \tag{17}$$

and express ψ as

$$\psi=\psi_0+\psi_1+\psi_2+\cdots, \tag{18}$$

where ψ_0 is the incident wave function, and $\psi_1,\psi_2,\cdots$ are the scattered wave functions in the various Born approximations. Then ψ_0 satisfies the equation

$$\Omega\psi_0=0, \tag{19}$$

and has solutions of the usual form

$$\psi_0(\mathbf{r})=\psi_0(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{20}$$

while $\psi_1,\psi_2,\cdots$ satisfy the equation

$$\Omega\psi_{n+1}=MV\psi_n, \tag{21}$$

where $n=0, 1, 2, \cdots$.

The above equation has the solution

$$\psi_{n+1}=-\int G(\mathbf{r},\mathbf{r}')MV(\mathbf{r}')\psi_n(\mathbf{r}')d\tau', \tag{22}$$

where the Green's function $G(\mathbf{r},\mathbf{r}')$, satisfying the equation

$$\Omega G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'), \tag{23}$$

is given by[3]

$$\begin{aligned} G(\mathbf{r},\mathbf{r}')&=\frac{1}{(2\pi)^3}\int\frac{1}{2M[(M^2+k'^2)^{\frac{1}{2}}-(M^2+k^2)^{\frac{1}{2}}]}\times e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{k}'\\ &=\frac{1}{(2\pi)^3}\int\frac{(M^2+k'^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}}{2M(k'^2-k^2)}\times e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{k}'. \end{aligned} \tag{24}$$

Since the integral (24) cannot be evaluated in a simple manner due to the presence of the term $(M^2+k'^2)^{\frac{1}{2}}$ in the numerator, we express it as

$$G(\mathbf{r},\mathbf{r}')=\frac{1}{(2\pi)^3}\int\frac{(M^2-\nabla^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}}{2M(k'^2-k^2)}\times e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}d\mathbf{k}', \tag{25}$$

and, extracting the outgoing wave function,

$$G(\mathbf{r},\mathbf{r}')=[(M^2-\nabla^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}]\times e^{ik|\mathbf{r}-\mathbf{r}'|}/8\pi M|\mathbf{r}-\mathbf{r}'|. \tag{26}$$

From (22) and (26) we obtain

$$\psi_{n+1}=-\int[(M^2-\nabla^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}]\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{8\pi|\mathbf{r}-\mathbf{r}'|}\times V(\mathbf{r}')\psi_n(\mathbf{r}')d\tau', \tag{27}$$

or

$$\psi_{n+1}=-\int\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{8\pi|\mathbf{r}-\mathbf{r}'|}[(M^2-\nabla'^2)^{\frac{1}{2}}+(M^2+k^2)^{\frac{1}{2}}]\times V(\mathbf{r}')\psi_n(\mathbf{r}')d\tau', \tag{28}$$

so that, using (15),

$$\psi_{n+1}=-\int\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}MV'(\mathbf{r}')\psi_n(\mathbf{r}')d\tau', \tag{29}$$

which is the same as the usual result derived from (14).

[3] For a comparison with the Green's function for the nonrelativistic case, see L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955).

4. SCHRÖDINGER'S EQUATION WITH ONE-PION EXCHANGE POTENTIAL

Using the general method[4] of deriving the potential from the unitary expansion of the scattering operator, we have discussed the relativistic one-pion exchange nuclear potential in an earlier paper.[5] Since the meaning of a potential is perhaps somewhat ambiguous, we would like to clarify our definition of a potential before we compare our result with those of other authors. We define the one-pion exchange potential as an interaction term in the two-nucleon Schrödinger equation, which is chosen in such a way as to produce the one-pion exchange K-matrix element in the first Born approximation. It is then evident that our one-pion exchange potential will contain only terms of the second order in the pion-nucleon coupling constant. Moreover, the velocity-dependent part of the potential will involve some arbitrariness, as we shall discuss below.

Let us first express the velocity-dependence of the one-pion exchange potential entirely in terms of the incident momentum $\mathbf{k}$ of one of the nucleons. Then, the relativistic potential is[5]

$$V=(1+k^2/M^2)^{-1}V_S, \tag{30}$$

where V_S is the well-known one-pion exchange static potential.

However, we can also express the velocity-dependence in terms of the momentum operator $\mathbf{p}=-i\boldsymbol{\nabla}$. We can then express V in many different Hermitian forms such as

$$V=(1+\mathbf{p}^2/M^2)^{-\frac{1}{2}}V_S(1+\mathbf{p}^2/M^2)^{-\frac{1}{2}}, \tag{31}$$

or as

$$V=\tfrac{1}{2}[(1+\mathbf{p}^2/M^2)^{-1}V_S+V_S(1+\mathbf{p}^2/M^2)^{-1}], \tag{32}$$

[4] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).
[5] S. N. Gupta, Nuclear Phys. **24**, 160 (1961).

which agree with each other in the first Born approximation but differ in the higher approximations. This arbitrariness is related to the fact that velocity-dependent terms with vanishing matrix elements on the energy shell can be converted into higher-order potential terms.

Since in the first Born approximation (15) is equivalent to

$$V'=(1+k^2/M^2)^{\frac{1}{2}}V, \tag{33}$$

we can take V' corresponding to (30) as

$$V'=(1+k^2/M^2)^{-\frac{1}{2}}V_S, \tag{34}$$

which, on substitution in (14), agrees with the Breit equation[1] for the two-nucleon system. On the other hand, according to Sugawara and Okubo,[2]

$$V'=V_S-[(\mathbf{p}^2/2M^2)V_S+V_S(\mathbf{p}^2/2M^2)], \tag{35}$$

where higher powers of $\mathbf{p}^2/M^2$ have been neglected. In the first Born approximation (35) is equivalent to

$$V'=(1-k^2/M^2)V_S, \tag{36}$$

which does not agree with (34) within the limits of approximation. This discrepancy is due to the fact that they have used the nonrelativistic expressions for nucleon energies in reducing the relativistic Schrödinger equation to the nonrelativistic form, which is equivalent to equating V and V' appearing in (12) and (14). The present investigation, however, shows that V and V' are related as in (15), and therefore the result of Sugawara and Okubo needs modification.

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