Instabilities of a Cylindrical Electron-Hole Plasma in a Magnetic Field

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The stability of an electron-hole plasma carrying current in a parallel magnetic field is investigated. The theory employed is a generalization of the treatment Kadomtsev and Nedospasov applied to gas discharges. In its generalized form it is applicable to plasmas produced either by injection or by impact ionization and no restrictions are placed upon the magnitude of the magnetic field, provided the classical, linearized treatment is appropriate. The possibility of finite plasma densities at the surface of the crystal is included in an approximate manner. Instabilities are predicted, and the calculated values of the electric field, magnetic field, and lowest oscillation frequency which occur at the onset of instability are compared with available observations, including those of the "Oscillistor." Agreement is reasonably satisfactory.

INTRODUCTION

THERE have been a number of observations[1–4] recently of spontaneous oscillations of the current passing through semiconductor plasmas made up of electrons and holes, placed in a magnetic field. It has been suggested[5] that these oscillations are the result of inherent instabilities in such plasmas, similar to those suggested as the cause of the rapid diffusion across a magnetic field observed in gaseous plasmas.[6] In this paper we develop a theory for the time-dependent behavior of electron-hole plasmas in semiconductors, carrying current in an external, longitudinal magnetic field, and compare the predicted instabilities with the available experimental evidence. The results presented here amplify an earlier discussion,[5] and include a treatment of the problem with one of the assumptions made earlier removed. It will be seen that the conditions predicted for the onset of instabilities are in good agreement with experimental observations.

THEORY

We deal with a plasma made up of electrons and holes, free to move about inside a semiconductor cylinder of radius a. The cylinder is assumed sufficiently long to make end effects negligible so that we may simplify the z variation of the properties of interest, i.e., the plasma density and carrier velocities. In the steady state the plasma will occupy the available volume with a distribution in density which depends on the sources and sinks for the plasma. The sink for the plasma will be assumed to be primarily at the surface, with some volume recombination also allowable, so that the plasma density will have a maximum at the center of the cylinder and fall off to some smaller value at the surface. The treatment will be applied to the two types of sources of interest: injection of electrons and holes from end contacts, and impact ionization in the bulk.

A plasma of this distribution, carrying an axial current, will be stable in form at low currents where the self-magnetic field of the current applies a force much smaller than the diffusion force which maintains the steady state. Any perturbations will be opposed by diffusion, and, since no large driving force is available, they will be damped. If an external magnetic field in the axial direction (H_z) is applied, however, instabilities may occur. Consider a helical perturbation in density which causes a helical distortion in the current with azimuthal component Δj_ϕ. This current will now cause a driving force of magnitude $\Delta j_\phi H_z c^{-1}$, in a direction either towards or away from the surface, depending on the sense of the helical perturbation with respect to the direction of H_z. The strength of this driving force will depend on j and H, and we may expect that at sufficiently large values of the product jH_z this force will be larger than the diffusion force and the perturbations will grow. We shall investigate the conditions for the growth of perturbations of helical form and show that instabilities set in at values of j and H_z above thresholds which are interdependent.

We use cgs units throughout. The plasma may be described by the following set of equations, where n and p are the electron and hole densities, respectively:

$$\partial n/\partial t+\nabla\cdot(n\mathbf{v}_e)=\gamma n, \quad (1)$$

$$\partial p/\partial t+\nabla\cdot(p\mathbf{v}_h)=\gamma p, \quad (2)$$

$$\frac{kT_e}{m_e^*n}\nabla n=-\frac{e}{m_e^*c}\mathbf{v}_e\times\mathbf{H}+\frac{e}{m_e^*}\nabla V-\frac{\mathbf{v}_e}{\tau_e}, \quad (3)$$

$$\frac{kT_h}{m_h^*p}\nabla p=\frac{e}{m_h^*c}\mathbf{v}_h\times\mathbf{H}-\frac{e}{m_h^*}\nabla V-\frac{\mathbf{v}_h}{\tau_h}, \quad (4)$$

$$\nabla^2V=-(4\pi/\kappa)(p-n). \quad (5)$$

The subscripts e and h denote electrons and holes, respectively; $\mathbf{v}$ is the velocity, T the temperature, m^* the effective mass, and τ the scattering time of the carriers. V and $\mathbf{H}$ are the electric potential and mag-

[1] I. L. Ivanov and S. W. Ryvkin, J. Tech. Phys. (U.S.S.R.) **28**, 774 (1958); Sov. Phys.-Tech. Phys. **3**, 722 (1958).
[2] R. D. Larrabee and M. C. Steele, J. Appl. Phys. **31**, 1519 (1960).
[3] J. Bok and R. Veilex, Compt. rend. **248**, 2300 (1959).
[4] M. Glicksman and R. A. Powlus, Phys. Rev. **121**, 1659 (1961).
[5] M. Glicksman, Bull. Am. Phys. Soc. **6**, 116 (1961).
[6] B. B. Kadomtsev and A. V. Nedospasov, J. Nuclear Energy **1**, 230 (1960).

netic field, respectively, and κ is the dielectric constant of the semiconductor.

Equations (1) and (2) are written in a form which includes both volume recombination and bulk plasma generation in the terms containing γ, which would be, respectively, negative and positive. Since we assume that the recombination is dominantly at the surface, γ will be positive in the case of bulk plasma generation. In the case of an injected plasma with dominant recombination at the surface, γ will be approximately 0. Recombination at the surface appears in the boundary conditions. In Eqs. (1)–(4) we will set $n=p$, since $n-p$ will always be much smaller than n in the cases of interest.

Equations (3) and (4) may be solved for the carrier velocities, and we do so for $\mathbf{H}$ in the z direction. The collision times should be replaced by appropriate averages over the carrier distribution functions. For simplicity we shall neglect the distinction between Hall and drift mobilities, and write for the mobility μ (in esu) and the diffusion coefficient D

$$y_e\equiv(e\tau_e/m_e^*c)H=\mu_eH/c,\quad D_e=(kT_e/e)\mu_e, \tag{7}$$

$$y_h\equiv(e\tau_h/m_h^*c)H=\mu_hH/c,\quad D_h=(kT_h/e)\mu_h. \tag{8}$$

In terms of the unit vector in the z direction, $\hat{z}$, we have

$$\mathbf{v}_e=\frac{y_e^2}{1+y_e^2}\left\{\left(\mu_e\frac{\partial V}{\partial z}-\frac{D_e}{n}\frac{\partial n}{\partial z}\right)\hat{z}-\frac{1}{y_e}\left(\frac{D_e}{n}\hat{z}\times\nabla n-\mu_e\hat{z}\times\nabla V\right)+\frac{1}{y_e^2}\left(\mu_e\nabla V-\frac{D_e}{n}\nabla n\right)\right\}, \tag{9}$$

$$\mathbf{v}_h=\frac{y_h^2}{1+y_h^2}\left\{\left(-\mu_h\frac{\partial V}{\partial z}-\frac{D_h}{n}\frac{\partial n}{\partial z}\right)\hat{z}+\frac{1}{y_h}\left(\frac{D_h}{n}\hat{z}\times\nabla n+\mu_h\hat{z}\times\nabla V\right)+\frac{1}{y_h^2}\left(-\mu_h\nabla V-\frac{D_h}{n}\nabla n\right)\right\}. \tag{10}$$

The continuity equations (1) and (2) are then written:

$$\frac{\partial n}{\partial t}+\frac{y_e^2}{1+y_e^2}\left\{\frac{\partial}{\partial z}\left(n\mu_e\frac{\partial V}{\partial z}-D_e\frac{\partial n}{\partial z}\right)+\frac{1}{y_e}\mu_e\hat{z}\cdot\nabla V\times\nabla n+\frac{1}{y_e^2}\mu_e\nabla\cdot(n\nabla V)-\frac{D_e}{y_e^2}\nabla^2 n\right\}=\gamma n, \tag{11}$$

$$\frac{\partial n}{\partial t}+\frac{y_h^2}{1+y_h^2}\left\{\frac{\partial}{\partial z}\left(-n\mu_h\frac{\partial V}{\partial z}-D_h\frac{\partial n}{\partial z}\right)+\frac{1}{y_h}\mu_h\hat{z}\cdot\nabla V\times\nabla n-\frac{1}{y_h^2}\mu_h\nabla\cdot(n\nabla V)-\frac{D_h}{y_h^2}\nabla^2 n\right\}=\gamma n. \tag{12}$$

Equations (11) and (12) are to be solved for n and V as functions of r, z, ϕ, and t. We need not treat differently the two cases of impact ionization in the bulk, and injection, (γ positive and 0, respectively) since we shall see that, with quite reasonable assumptions, the two cases yield similar results.

(a) The Steady State

When $\partial n/\partial t=0$, the values of density and potential are described by the functions n_0 and V_0. Equations (11) and (12) may then be written in the form:

$$\frac{1}{1+y_e^2}\left\{\frac{\mu_e}{r}\frac{\partial}{\partial r}\left(n_0r\frac{\partial V_0}{\partial r}\right)-\frac{D_e}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)\right\}+\mu_e\frac{\partial n_0}{\partial z}\frac{\partial V_0}{\partial z}-D_e\frac{\partial^2 n_0}{\partial z^2}=\gamma n_0, \tag{13}$$

$$\frac{1}{1+y_h^2}\left\{-\frac{\mu_h}{r}\frac{\partial}{\partial r}\left(n_0r\frac{\partial V_0}{\partial r}\right)-\frac{D_h}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)\right\}-\mu_h\frac{\partial n_0}{\partial z}\frac{\partial V_0}{\partial z}-D_h\frac{\partial^2 n_0}{\partial z^2}=\gamma n_0, \tag{14}$$

where we have introduced the assumption that the field E_0 along the length of the cylinder is constant, i.e.,

$$\partial^2V_0/\partial z^2=0,\quad -\partial V_0/\partial z=E_0, \tag{15}$$

and we have also assumed that n_0 and V_0 are not functions of the angular variable ϕ. The term involving $\partial V_0/\partial r$ may be eliminated between Eqs. (13) and (14), with the result

$$\mu_e\frac{\partial n_0}{\partial z}\frac{\partial V_0}{\partial z}\left(1-\frac{1+y_h^2}{1+y_e^2}\right)-\frac{\partial^2 n_0}{\partial z^2}\left[D_e+\frac{bD_h(1+y_h^2)}{1+y_e^2}\right]-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n_0}{\partial r}\right)\left[\frac{D_e}{1+y_e^2}+\frac{bD_h}{1+y_e^2}\right]=\gamma n_0\left[1+\frac{b(1+y_h^2)}{1+y_e^2}\right], \tag{16}$$

where b is the ratio of electron and hole mobilities, μ_e/μ_h.

Equation (16) has a solution of the form

$$n_0=N_0J_0(\beta_0 r)Z_0(z);$$

$$G+\gamma\left[1+\frac{b(1+y_h^2)}{1+y_e^2}\right]=\frac{bD_h+D_e}{1+y_e^2}\beta_0^2, \tag{17}$$

with G the separation constant. We need not detail the function Z_0 for the rest of our investigation. We wish it to be a weak function of z to conform with our

model of a cylindrical plasma of approximately constant density in the z direction. How weak the variation with z needs to be will be seen in the next section; here we can say that it needs to be weak enough to justify our assumption of a constant E_0.

In order for the plasma density to fall to zero at the surface,[7] β_0 should have the value α_0/a, where α_0 is the first root of J_0 and is equal to 2.4048. When we set the left sides of Eqs. (13) and (14) equal to each other and apply the requirement that $\partial n_0/\partial r$ and $\partial V_0/\partial r$ must be regular at the origin, we can show that

$$\frac{\partial V_0}{\partial r}=\frac{D_e(1+y_h^2)-D_h(1+y_e^2)}{\mu_e(1+y_h^2)+\mu_h(1+y_e^2)}\frac{1}{n_0}\frac{\partial n_0}{\partial r}. \tag{18}$$

(b) Instabilities in the Density

We use a perturbation approach to treat the possible growth of instabilities in the plasma density. n and V are assumed expressible in the forms

$$n=n_0+n_1,\quad V=V_0+V_1. \tag{19}$$

We assume that n_1 and V_1 are small and may be expressed in the form

$$\begin{aligned}n_1&=f(r,z)e^{im\phi+ikz-i\omega t},\\ V_1&=F(r)e^{im\phi+ikz-i\omega t}.\end{aligned} \tag{20}$$

These expressions are substituted into Eqs. (11) and (12), and terms of higher order than 1 in the perturbation quantities are dropped. We then have

$$\begin{aligned}&\left\{-i\omega+ik\mu_e\frac{\partial V_0}{\partial z}+D_ek^2+\frac{i\mu_e y_e}{1+y_e^2}\frac{m}{r}\frac{\partial V_0}{\partial r}+\frac{D_e}{1+y_e^2}\frac{m^2}{r^2}-\gamma\right\}f-2ikD_e\frac{\partial f}{\partial z}+\mu_e\frac{\partial V_0}{\partial z}\frac{\partial f}{\partial z}-D_e\frac{\partial^2 f}{\partial z^2}-\frac{D_e}{1+y_e^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\\ &+\left\{ik\mu_e\frac{\partial n_0}{\partial z}-n_0\mu_ek^2-\frac{n_0m^2}{r^2}\frac{\mu_e}{1+y_e^2}-\frac{y_e}{1+y_e^2}\mu_e\frac{im}{r}\frac{\partial n_0}{\partial r}\right\}F+\frac{\mu_e}{1+y_e^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rf\frac{\partial V_0}{\partial r}+rn_0\frac{\partial F}{\partial r}\right)\right\}=0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\left\{-i\omega-\gamma-ik\mu_h\frac{\partial V_0}{\partial z}+D_hk^2+i\mu_h\frac{y_h}{1+y_h^2}\frac{m}{r}\frac{\partial V_0}{\partial r}+\frac{D_h}{1+y_h^2}\frac{m^2}{r^2}\right\}f-2ikD_h\frac{\partial f}{\partial z}-\mu_h\frac{\partial V_0}{\partial z}\frac{\partial f}{\partial z}-D_h\frac{\partial^2 f}{\partial z^2}-\frac{D_h}{1+y_h^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\\ &-\left\{ik\mu_h\frac{\partial n_0}{\partial z}-n_0\mu_hk^2-n_0\frac{m^2}{r^2}\frac{\mu_h}{1+y_h^2}+\frac{y_h}{1+y_h^2}\mu_h\frac{im}{r}\frac{\partial n_0}{\partial r}\right\}F-\frac{\mu_h}{1+y_h^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rf\frac{\partial V_0}{\partial r}+rn_0\frac{\partial F}{\partial r}\right)\right\}=0.\end{aligned} \tag{22}$$

In order to remove the explicit dependence on z from these two equations, we wish to ignore the term $ik\mu(\partial n_0/\partial z)F$ which appears in each of them. This may be justified if

$$\left|\frac{\partial n_0}{\partial z}\right|\ll\left|\frac{y}{1+y^2}\frac{m}{kr}\frac{\partial n_0}{\partial r}\right|; \tag{23}$$

m will be integral and equal to 1 or larger; the right-hand side will have its smallest value at $r=a$, so that (23) becomes

$$\left|\frac{\partial n_0}{\partial z}\right|\ll\frac{y}{1+y^2}\frac{1.25N_0}{ka^2}. \tag{24}$$

As we shall see when the results are applied to a practical example, this is a very mild restriction which is satisfied easily in the experiments referred to earlier.[1–4] In keeping with the perturbation approximation, we also assume that the perturbed density n_1 has the same loss rate in the z direction as did n_0, i.e., in Eqs. (21) and (22),

$$f(r,z)=f_1(r)Z_0(z). \tag{25}$$

Terms ignored are of second order. With these conditions, Eqs. (21) and (22) are separable in the r and z dependence and may be written in the form

$$\begin{aligned}&\left\{-i\omega-\gamma'+ikv_0+D_ek^2+\frac{im}{r}\frac{y_e}{1+y_e^2}\mu_e\frac{\partial V_0}{\partial r}\right\}f_1-\frac{D_e}{1+y_e^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_1}{\partial r}\right)-\left\{J_0\mu_ek^2+\frac{y_e}{1+y_e^2}\frac{\mu_e im}{r}\frac{dJ_0}{dr}+J_0\frac{m^2}{r^2}\frac{\mu_e}{1+y_e^2}\right\}FN_0\\ &+\frac{\mu_e}{1+y_e^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rf_1\frac{\partial V_0}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(rN_0J_0\frac{\partial F}{\partial r}\right)\right\}=0,\end{aligned} \tag{26}$$

$$\begin{aligned}&\left\{-i\omega-\gamma'-\frac{ikv_0}{b}+D_hk^2+\frac{im}{r}\frac{y_h}{1+y_h^2}\mu_h\frac{\partial V_0}{\partial r}\right\}f_1-\frac{D_h}{1+y_h^2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_1}{\partial r}\right)+\left\{J_0\mu_hk^2-\frac{y_h}{1+y_h^2}\mu_h\frac{im}{r}\frac{dJ_0}{dr}+J_0\frac{m^2}{r^2}\frac{\mu_h}{1+y_h^2}\right\}FN_0\\ &-\frac{\mu_h}{1+y_h^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(rf_1\frac{\partial V_0}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(rN_0J_0\frac{\partial F}{\partial r}\right)\right\}=0,\end{aligned} \tag{27}$$

[7] We relax this requirement later, in the discussion of surface effects.

where

$$\gamma'=\gamma+\frac{(1+y_e^2)G}{1+y_e^2+b(1+y_h^2)}=\frac{\beta_0^2(bD_h+D_e)}{1+y_e^2+b(1+y_h^2)},\tag{28}$$

$$v_0=\mu_e\partial V_0/\partial z.\tag{29}$$

We look for solutions to Eqs. (26) and (27) which satisfy our boundary conditions: n_1 and V_1 must be regular at the origin; n goes to zero at the surface; the flows of electrons and holes to the surface are equal, since there can be no net radial current. The last condition can be satisfied as long as the potential remains finite at the surface.

We follow the approach of Kadomtsev and Nedospasov,[6] who treated a simpler pair of equations in an approximate manner by assuming solutions of the form $f_1=n'J_1(\beta_1r)$; $F=V'J_1(\beta_1r)$, where $\beta_1=(\alpha_1/a)$, and α_1 is the first root of the Bessel function J_1, equal to 3.8317. We substitute for f_1 and F in Eqs. (26) and (27), multiply through by $J_1(\beta_1r)rdr$ and integrate with respect to r. In this approach we set $m^2=1$, since we seek the lowest order instabilities, and the smallest value of m should define this lowest region.

We then have two algebraic equations whose coefficients are functions of the parameters describing the plasma and the wave parameters ω and k only.

$$\begin{aligned}\Big\{&-i\omega-\gamma'+ikv_0+D_ek^2\\&-\frac{imy_e}{1+y_e^2}L\beta_0^2\frac{[D_e(1+y_h^2)-D_h(1+y_e^2)]}{1+y_h^2+(1/b)(1+y_e^2)}\\&+\frac{D_e\beta_1^2}{1+y_e^2}-\frac{Q\beta_0^2}{1+y_e^2}\,\frac{[D_e(1+y_h^2)-D_h(1+y_e^2)]}{1+y_h^2+(1/b)(1+y_e^2)}\Big\}\frac{n'}{\bar{N}_0}\\&-\left\{\mu_ek^2-\frac{imy_e}{1+y_e^2}R\mu_e\beta_0^2+\frac{P\beta_1^2\mu_e}{1+y_e^2}\right\}V'=0,\end{aligned}\tag{30}$$

$$\begin{aligned}\Big\{&-i\omega-\gamma'-ik\frac{v_0}{b}+D_hk^2\\&-\frac{imy_h}{1+y_h^2}L\beta_0^2\frac{[D_e(1+y_h^2)-D_h(1+y_e^2)]}{b(1+y_h^2)+1+y_e^2}\\&+\frac{D_h\beta_1^2}{1+y_h^2}+\frac{Q\beta_0^2}{1+y_h^2}\,\frac{[D_e(1+y_h^2)-D_h(1+y_e^2)]}{b(1+y_h^2)+1+y_e^2}\Big\}\frac{n'}{\bar{N}_0}\\&+\left\{\mu_hk^2+\frac{imy_h}{1+y_h^2}R\mu_h\beta_0^2+\frac{P\beta_1^2\mu_h}{1+y_h^2}\right\}V'=0.\end{aligned}\tag{31}$$

The coefficients L, P, Q, R, and $\bar{N}_0$ are expressed as integrals of Bessel functions. If

$$S=\int_0^a[J_1(\beta_1r)]^2rdr,\tag{32}$$

$$L=\frac{1}{\beta_0S}\int_0^aJ_1(\beta_0r)[J_0(\beta_0r)]^{-1}[J_1(\beta_1r)]^2dr,\tag{33}$$

$$\begin{aligned}Q=1&+\frac{1}{S}\int_0^a[J_1(\beta_0r)]^2[J_0(\beta_0r)]^{-2}[J_1(\beta_1r)]^2rdr\\&-L+\frac{\beta_1}{\beta_0S}\int_0^aJ_1(\beta_0r)[J_0(\beta_0r)]^{-1}\\&\qquad\times J_0(\beta_1r)J_1(\beta_1r)rdr,\end{aligned}\tag{34}$$

$$R=\left\{\int_0^aJ_1(\beta_0r)[J_1(\beta_1r)]^2dr\right\}\times\left\{\beta_0\int_0^aJ_0(\beta_0r)[J_1(\beta_1r)]^2rdr\right\}^{-1},\tag{35}$$

$$P=1+\frac{\beta_0}{\beta_1}\left\{\int_0^aJ_0(\beta_1r)J_1(\beta_0r)J_1(\beta_1r)rdr\right\}\times\left\{\int_0^aJ_0(\beta_0r)[J_1(\beta_1r)]^2rdr\right\}^{-1}-\left(\frac{\beta_0}{\beta_1}\right)^2R,\tag{36}$$

$$\bar{N}_0=\frac{N_0}{S}\int_0^aJ_0(\beta_0r)[J_1(\beta_1r)]^2rdr.\tag{37}$$

These expressions have the numerical values

$$S=0.08114a^2,\quad L=0.7478,\quad Q=1.4888,\quad R=0.6656,\quad P=0.80305,\quad \bar{N}_0/N_0=0.5879.\tag{38}$$

We obtain the dispersion relation for $\omega(k)$ by setting the determinant of Eqs. (30) and (31) equal to zero. Since we seek instabilities, i.e., growth in time in this case, k is a real quantity, and we must investigate the behavior of ω, a complex quantity. We assume that the electrons and holes are in thermal equilibrium with each other, i.e., $T_e=T_h$, and go over to a simplified, dimensionless notation:

$$\begin{aligned}&D\equiv D_e=bD_h,\quad y\equiv y_e=by_h,\\&X\equiv ka,\quad \mathcal{E}\equiv v_0a/D,\\&\Omega\equiv a^2\omega/D,\quad \Gamma\equiv a^2\gamma'/D,\\&U\equiv\frac{b^2+y^2-b(1+y^2)}{b^2+y^2+b(1+y^2)}.\end{aligned}\tag{39}$$

The equation for Ω may then be written

$$\left(-i\Omega-\Gamma+i\mathcal{E}X+X^2-\frac{4.3246imyU}{1+y^2}+\frac{14.682}{1+y^2}-\frac{8.6099U}{1+y^2}\right)\left(\frac{X^2}{b}+\frac{3.8492imy}{b^2+y^2}+\frac{11.790b}{b^2+y^2}\right)+\left(X^2+\frac{11.790}{1+y^2}-\frac{3.8492imy}{1+y^2}\right)$$
$$\times\left(-i\Omega-\Gamma-\frac{i\mathcal{E}X}{b}+\frac{X^2}{b}-\frac{4.3246imyU}{b^2+y^2}+\frac{14.682b}{b^2+y^2}+\frac{8.6099b^2U}{b^2+y^2}\right)=0, \quad (40)$$

and in explicit form for Ω as

$$\Omega=\frac{[(X^2/b)+B](-iA-iX^2+\mathcal{E}X)-(X^2+C)(iW+iX^2/b+\mathcal{E}X/b)}{(1+1/b)X^2+B+C}, \quad (41)$$

where the coefficients are defined as

$$\begin{aligned} A&=-\Gamma+\frac{14.682}{1+y^2}-\frac{8.6099U}{1+y^2}-\frac{4.3246imyU}{1+y^2},\\ B&=\frac{11.790b}{b^2+y^2}+\frac{3.8492imy}{b^2+y^2},\\ C&=\frac{11.790}{1+y^2}-\frac{3.8492imy}{1+y^2},\\ W&=-\Gamma+\frac{14.682b}{b^2+y^2}+\frac{8.6099b^2U}{b^2+y^2}-\frac{4.3246imyU}{b^2+y^2}. \end{aligned} \quad (42)$$

The imaginary part of Ω may be expressed in the form

$$-\frac{\lambda_1X^6+\lambda_2X^4+\lambda_3X^2+\lambda_4-\mathcal{E}mX(\lambda_5+\lambda_6X^2)}{\lambda_7+\lambda_8X^2+\lambda_9X^4}, \quad (43)$$

and for stability (ImΩ negative) we then require that

$$\frac{\lambda_1X^6+\lambda_2X^4+\lambda_3X^2+\lambda_4}{\lambda_5+\lambda_6X^2}\geqslant m\mathcal{E}X. \quad (44)$$

The λ_i are coefficients containing only b and y. Since the left-hand side of the inequality (44) is a positive quantity, the plasma will be stable for $m=-1$, but may be unstable for $m=1$. This agrees with our physical picture for the source of the instability; helical waves with the $m=-1$ variation have a $\Delta j_\phi H$ force which is in the direction to aid restoration of the steady state, rather than to oppose it.

The threshold for the onset of instability will occur when (44) becomes an equality, and Ω is then real. The parameters of the system are of course the magnetic and electric fields applied to the plasma, which are proportional, respectively, to y and $\mathcal{E}$. For any fixed value of X, which is proportional to the wave number of the disturbance, there will be a corresponding curve of $y(\mathcal{E})$, which will represent the onset of growing oscillations of that wave number.

In most of the experiments performed, the wavelengths were not necessarily fixed by the arrangement used. In this case we should consider X as free, and seek the *smallest* values of y and $\mathcal{E}$ which represent the onset of instabilities. These will occur when the curve representing the right-hand side of (44) touches the curve representing the left-hand side, for the first time as $\mathcal{E}$ is increased from zero. At this first point of contact, the slopes of the two curves with respect to X will also be equal. In searching for this intersection and the corresponding values of y and $\mathcal{E}$, we must then look for the smallest slope for which the slopes and the values of the two sides are equal.

For given values of b and y, we may solve the equations derived from (44) for the point X which gives the smallest value of $\mathcal{E}$ for which instabilities may occur. We may in this way plot a curve of $y(\mathcal{E})$ which represents the threshold dividing the stable and unstable regions of operation of the plasma. Such curves are shown in Fig. 1, for values of b of 0.5, 2.2, 10, and 50. The general behavior is as expected from the physical model, with larger magnetic fields required to cause instabilities when lower electric fields are applied. For large b we note that the curves approach one another.

For a given system, i.e., some value of b, the magnetic field and electric field enter only through the terms y and $\mathcal{E}$. In the case where the electron drift velocity may be expressed in terms of the same mobility that appears

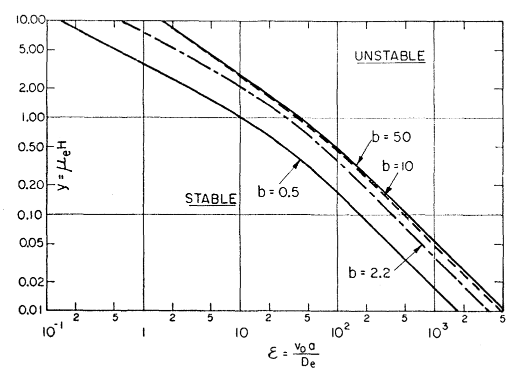

FIG. 1. The dimensionless magnetic field as a function of the dimensionless electric field necessary for onset of oscillation. μ_e is the electron mobility (in emu), v_0 the electron drift velocity in the direction of the electron field, a the radius of the plasma cylinder, and D_e the electron diffusion coefficient. b is the ratio of electron to hole mobilities.

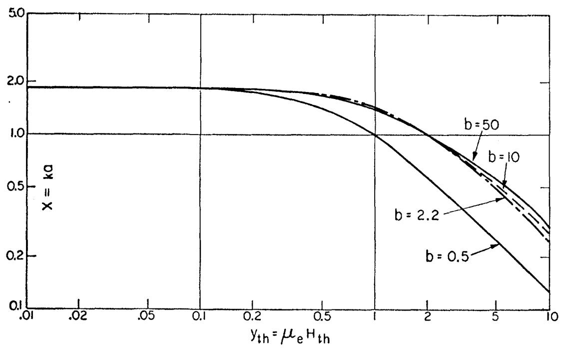

FIG. 2. The dimensionless wave number as a function of the dimensionless threshold magnetic field.

in the Einstein relation,

$$D_e = \mu_e k T_e / e, \tag{45}$$

$\mathcal{E}$ may be rewritten in the form

$$= a e E_0 / k T_e. \tag{46}$$

Thus for a given value of b, the critical magnetic field for a given electric field is inversely proportional to the electron mobility, and the critical electric field, for a given $\mu_e H$, is proportional to kT/a.

The values of X for which the thresholds of instability occur are plotted as a function of the threshold magnetic field, y_{th}, in Fig. 2, for values of b of 0.5, 2.2, 10, and 50. We see that X varies little as y_{th} is increased to about 0.5, and falls rapidly with y_{th} above 1. These values of X are the lowest for which instability will develop, provided the geometry of the plasma will allow waves of the calculated wavelength. When this is not the case, i.e., (L is the length of the plasma column)

$$L/a < \pi/X, \tag{47}$$

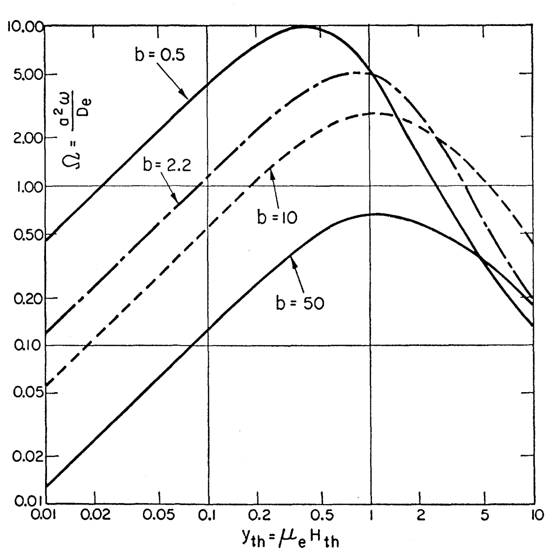

FIG. 3. The dimensionless frequency as a function of the dimensionless threshold magnetic field.

the instabilities will develop only for larger values of X, and correspondingly larger values of y and $\mathcal{E}$, than those given in Fig. 1. We would then calculate these by solving the inequality (44) for a specified value or set of values for X: $X = 2\pi a s/L$, where s is an integer or half-integer, depending on the end boundary conditions. We would then expect that the conditions for onset of the instability would depend on the length of the specimen until the inequality (47) is reversed to $(L/a) \gg (\pi/X)$.

At threshold, Ω is a real quantity. We may derive two expressions for the value of the frequency at the threshold, Ω_{th}, by requiring that the real and imaginary parts of Eq. (41) be separately satisfied. One of the expressions is

$$\Omega_{\mathrm{th}} = \frac{1}{\mathrm{Im}(B+C)} \{ \mathcal{E} X \,\mathrm{Im}(B - C/b) - 2X^4/b - X^2 \,\mathrm{Re}(A/b + B + C/b + W) + \mathrm{Re}(AB) - \mathrm{Re}(CW) \}, \tag{48}$$

and it may be used to calculate the lowest frequencies which should appear when the plasma becomes unstable. Ω_{th} is plotted as a function of y_{th} in Fig. 3, for the values $b = 0.5$, 2.2, 10, and 50. For $b = 1$, the equations yield a minimum frequency of 0, although all of the other curves show proper intermediate behavior between the cases $b = 0.5$ and 2.2. For $b < 1$, the values of the frequency are negative, which indicates a change in direction of the propagating waves, as is expected since the current is now dominated by the hole rather than the electron conduction. For small values of y_{th}, the frequency varies linearly with y. We note also that the frequency appears in a combination such that for fixed conditions of b and y_{th}, ω should be proportional to D_e/a^2.

TABLE I. Values of coefficients in Eq. (51) for different surface conditions.

Relative density at the surface δ	0	0.2	0.5	0.8	0.95	0.98
$(\beta_0' a)^2$	5.783	4.169	2.317	0.844	0.2022	0.0819
$(\beta_1' a)^2$	14.682	10.583	5.833	2.142	0.5134	0.2079
$(\beta_0' a)^2 Q'$	8.610	6.208	3.457	1.264	0.3043	0.1230
$(\beta_0' a)^2 L'$	4.325	2.946	1.417	0.453	0.1044	0.0414
$(\beta_1' a)^2 P'$	11.790	9.012	5.914	2.432	0.6063	0.2478
$(\beta_0' a)^2 R'$	3.849	2.737	1.393	0.4523	0.1039	0.0412

(c) Influence of the Surface

In the foregoing calculations we have assumed that the plasma density in the steady state is zero at the surface. For many of the semiconductors to which the theory will be applied, this approximation is inappropriate, since the density at the surface may be quite large. The calculations can be generalized in

terms of the parameter

$$\delta = n_0(a)/n_0(0), \tag{49}$$

by redefining the values of β_0 and the various integrals in Eqs. (32)–(37). The modified values, β_0', are simple to calculate, since they come from the new condition

$$J_0(\beta_0' a) = \delta. \tag{50}$$

For the rest, however, we must choose some treatment for the perturbed density which is not too arbitrary. We assume that the function $J_1(\beta_1' r)$ in n_1 falls to zero at the same point in space $(r > a)$ as does J_0; there is then a finite perturbed density at the surface. We rewrite Eqs. (42) in terms of the new coefficients, which are labeled with primes.

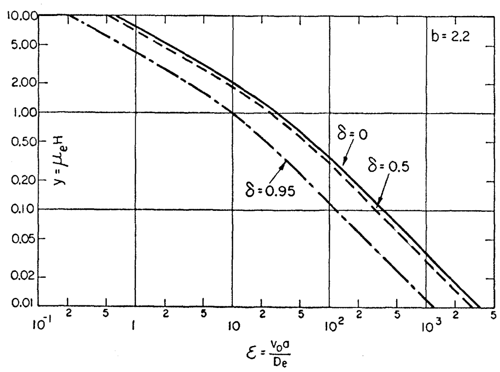

FIG. 4. The dimensionless magnetic field as a function of the dimensionless electric field necessary for oscillation, calculated for a plasma with electron-to-hole mobility ratio 2.2. δ is the ratio of plasma density at the surface to the plasma density on the axis of the cylinder.

$$\begin{aligned}
\Gamma' &= (\beta_0' a)^2 \frac{2b}{b(1+y^2)+b^2+y^2},\\
A' &= -\Gamma' + \frac{(\beta_1' a)^2}{1+y^2} - \frac{(\beta_0' a)^2 Q' U}{1+y^2} - \frac{(\beta_0' a)^2 L' m y U}{1+y^2} i,\\
B' &= \frac{(\beta_1' a)^2 P' b}{b^2+y^2} + \frac{(\beta_0' a)^2 R' i m y}{b^2+y^2},\\
C' &= \frac{(\beta_1' a)^2 P'}{1+y^2} - \frac{(\beta_0' a)^2 i m R' y}{1+y^2},\\
W' &= -\Gamma' + \frac{(\beta_1' a)^2 b}{b^2+y^2} + \frac{(\beta_0' a)^2 Q' b U}{b^2+y^2} - \frac{(\beta_0' a)^2 L' m y U}{b^2+y^2} i.
\end{aligned} \tag{51}$$

The values of the coefficients which appear in the Eqs. (51) are presented in Table I, for parameters δ in the range 0 to 0.98. Solutions for Eq. (44) with the modified coefficients have been obtained, and some of these are plotted in the series of Figs. (4)–(6), which show the effects for the one value $b = 2.2$, which corresponds most closely to the case of the germanium "Oscillistor" which has been studied in detail in recent experiments.[2,8]

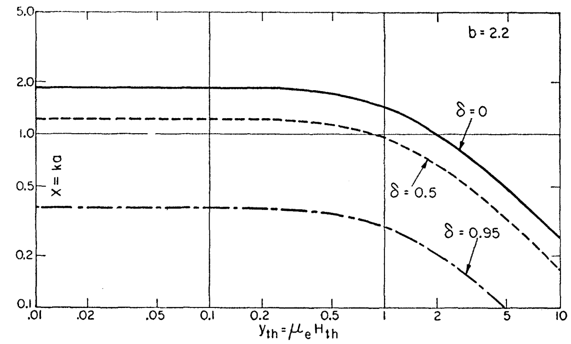

FIG. 5. The dimensionless wave number as a function of the dimensionless threshold magnetic field, for a plasma with electron-to-hole mobility ratio 2.2.

We note that as the surface concentration is increased, the threshold for instability decreases to smaller values of $\mathcal{E}$ and y. This seems reasonable, since with higher surface concentrations the diffusion force is reduced, and it is this force which must be overcome by the $j_\phi H_z$ forces to cause the growth of oscillations. How much the magnitude of the effect calculated is influenced by our choice of boundary conditions for n_1 is not clear. The general form chosen should be correct, although the most appropriate value for n_1 at the surface may not be the one which was used. However, we do not expect that the plasma will become unstable at vanishing values of the fields for $\delta = 1$. Our assumed form for n_1 is clearly not appropriate in this

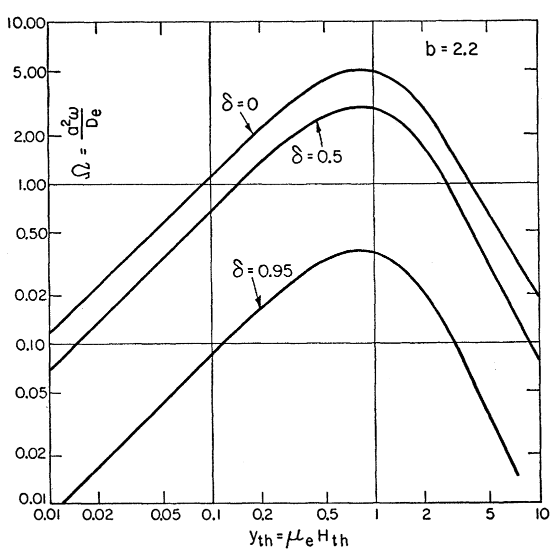

FIG. 6. The dimensionless frequency as a function of the dimensionless threshold magnetic field, for a plasma with electron-to-hole mobility ratio 2.2.

[8] R. D. Larrabee (private communication).

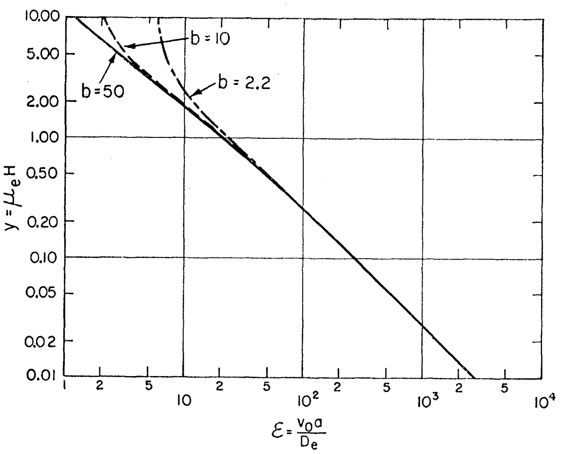

FIG. 7. The dimensionless magnetic field as a function of the dimensionless electric field necessary for onset of oscillation calculated under the assumption that the hole temperature is much less than the electron temperature.

case, since it yields zero perturbation. For a finite perturbation there will still be a diffusion force $-D\nabla n_1$, and no other forces if the fields are zero; the plasma will thus be stable.

The decrease in the values of ka and $\omega a^2/D_e$ is even greater than the change in the threshold fields. We thus expect that changes in the surface should have a considerable effect on threshold fields, and a stronger effect on the observed frequencies, provided that the observations involve the *lowest* possible modes of oscillation.

(d) Effect of Hole Diffusion

The theory presented has treated both carriers in the plasma on an equivalent basis, and assumed that the electron and hole temperatures were equal. The equations have also been investigated when this is not the case, i.e., when some of the terms involving the hole diffusion may be neglected. This may be justified only when $T_e \gg T_h$, a case which is expected to be rare in semiconductor plasmas, but is included here for completeness, and because it corresponds to the case treated by Kadomtsev and Nedospasov,[6] with their assumption that $\mu_e H \gg 1$ removed.

We use Eq. (11) but replace Eq. (12) by

$$\partial n/\partial t - \mu_h \nabla \cdot (n \nabla V) = \gamma n. \tag{52}$$

The treatment is the same; the resulting values for the thresholds for oscillation are plotted in Fig. 7. For small y, the threshold fields are independent of b; at large y, the threshold fields increase with decreasing b. For large b and large y, the curve approaches the curve shown in Fig. 1 for $b=50$.

COMPARISON WITH EXPERIMENT

Although there have been a number of observations of "spontaneous" oscillations which occur in semiconductor plasmas, the data available for comparison with the theory proposed are only semiquantitative. Within the limitations of these data, fruitful comparison may be made to investigate the validity of the proposed basic mechanism of oscillation.

The instabilities we have discussed occur in the presence of a magnetic field in the direction of current flow, and must be compared with experiments which also involve a magnetic field. This was the case for the work of Ivanov and Ryvkin[1] and of the other authors referred to[2–4,8] although other factors did appear to influence the character and properties of the oscillations. In all cases it is quite probable that the semiconductor contained an electron-hole plasma, produced either by injection or by across-the-gap ionization.[4] In the most detailed investigations, namely references 1, 2, and 8 dealing with germanium, it was clear that there was a threshold current (or electric field) and a corresponding threshold longitudinal magnetic field which had to be exceeded to induce the oscillations. The surfaces of the samples were important in affecting the properties of the oscillating plasma, although no quantitative study is yet available relating the surface density to the threshold fields and observed frequencies.

All of these qualitative features are just what we expect if the oscillations observed were of the class of hydromagnetic instabilities calculated in this paper. The amplitude of oscillation observed in the experiments was very large, and for this reason, we might expect that our perturbation approach should provide good quantitative agreement only very near the thresholds. The properties of the particular semiconductor plasmas which have been investigated are summarized in Table II. The threshold electric and magnetic fields, and the lowest frequencies which should appear have been calculated and are listed in Table III, together with an estimate of the values which have been observed. Although there is a paucity of data, what is available is in good agreement in most cases. In spite of the fact that all observations were made for relatively large amplitude of oscillation, there is no indication of a violent disagreement between the spread

TABLE II. Properties of electron hole plasmas.

Material	Temperature (°K)	Electron mobility (cm²/v-sec) (emu)	Mobility ratio b	Electron diffusion coefficient (cm²/sec)
Germanium	300	3.6×10^{-5}	2.2	93
Germanium	77	$2\text{–}4\times10^{-4}$	1–1.5	130–260
Silicon	77	$1\text{–}1.5\times10^{-4}$	≈ 2	66–100
Indium antimonide	77	$2\text{–}7\times10^{-3}$	20–70	1300–4650
Indium antimonide	220[a]	2.8×10^{-3}	≈ 30	5230

[a] Lattice at 77°K, plasma at 220°K.

TABLE III. Oscillation thresholds.

Material (temperature)	Threshold magnetic fields $y=\mu_e H$	Threshold electric fields aE (volts) Calculated $\delta=0$	$\delta=0.95$	Observed	Frequencies $a^2 f$ (cm²/sec) Calculated $\delta=0$	$\delta=0.95$	Observed
Germanium (300°K)	0.1	9.3	3.0	2[a]	16.3	1.27	
	0.36	2.47	0.8	0.7[a] 0.4[b]	53.5	4.0	5+[a] 56[b]
Germanium (77°K)	0.1	≈2	≈0.65				
	1.0	≈0.15	≈0.05				
Silicon (77°K)	0.1	≈2.3	≈0.75		12–19	1–1.5	
	1.0	≈0.19	≈0.06		55–85	5–6.5	
Indium antimonide (77°K)	0.02	15.9	5.2		≈12.5	0.9	
	0.1	3.25	1.1		60–70	5	
	1.0	0.25	0.08		330–400	30	
Indium antimonide (carriers at 220°K)	0.03	32	10	≈5[c]	50	3.5	⩽400[c]

[a] R. D. Larrabee, references 2 and 8.
[b] Ivanov and Ryvkin, reference 1.
[c] Glicksman and Powlus, reference 4.

of observations by a number of workers (with unknown conditions of the surface) and the theory presented.

In our treatment we neglected a term involving the rate of decrease of the density along the plasma cylinder axis [Eq. (24)]. In the germanium experiments the right-hand side of this inequality is about equal to N_0 cm^{-4}, essentially independent of the value of δ. The inequality is therefore satisfied as long as the plasma injected at one end reaches the other end only slightly decreased in density: Our error will be about 10% in the term retained if the density falls off 10% as it passes down the semiconductor. The experiments which showed oscillations had plasma of high density throughout; in cases where the density fell off rapidly, oscillations were not observed.[2]

Our thresholds are valid and independent of the specimen length as long as the condition discussed above, i.e., $(L/a)\gg(\pi/X)$ is satisfied. In most of the experiments, L/a was of the order of 10–30, while X is calculated to be of the order of 0.4 to 2, depending on the surface conditions. We see that some length dependence might be expected when L/a is only about 10 and the surfaces are "good," since in the latter case X at threshold has its lowest values.

We note that the conditions for onset of oscillation do not depend explicitly on the plasma density. The use of external light radiation to change the plasma density[1,2] would be expected to affect the observations when the light was necessary to produce the plasma, or when the light affected the ratio of surface to interior density of plasma.

The oscillations calculated are in the form of helical waves which travel down the plasma at a phase velocity ω/k. As such they would not produce a measurable ac current or voltage modulation unless the conditions at the ends of the specimen provide effective reflection which can result in a standing pattern of these waves. This must be the case; oscillations were observed to last for very long times[2] at apparently stable conditions. We must then expect that some regular relation of the kind discussed with respect to the specimen length and the oscillation wavelength will exist, and for the wavelengths calculated we expect that the number of such wavelengths present is small, of the order of 10 or less, in the experiments performed.

The theory presented predicts a variety of dependences of the threshold fields and frequencies on the geometry and the surface conditions, in addition to the normally expected dependence on the transport properties of the plasma particles. The strong inverse-square dependence of the frequency on the transverse dimension a, and the inverse dependence of the threshold electric field on a should be capable of experimental verification. However, it is necessary to have precise knowledge of the sample conditions, such as the relative concentrations at the center and surfaces, in order to provide a proper test of the calculations.

CONCLUSIONS

We have investigated the conditions affecting the growth of hydromagnetic instabilities in electron-hole plasmas. The conditions predicted for the onset of these instabilities coincide reasonably well with the observed critical conditions necessary to induce oscillations in such plasmas carrying a current in a longitudinal magnetic field. We therefore believe that these instabilities provide the basic mechanism for current oscillations in a plasma operating under these conditions (e.g., the "Oscillistor"[2]). The theory predicts that the threshold electric field for oscillation should vary inversely as the plasma radius a, and that the lowest frequency of oscillation should vary inversely as the square of a, provided that in both cases, the other

properties of the plasma are maintained constant. We have discussed the conditions of applicability of the theory as presented and have shown that experiments performed to date should be describable within the approximations employed.

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Statistical Mechanics of Dimers on a Plane Lattice

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This paper considers the statistical mechanics of hard rigid dimers distributed on a lattice (each dimer occupying two nearest neighbor lattice sites). The problem is solved in exact closed form for a finite $m\times n$ plane square lattice with edges which is completely filled with $\frac{1}{2}mn$ dimers (close-packed limit). In terms of the activities x and y of horizontal and vertical dimers, the configurational partition function $Z_{mn}(x,y)$ is given in the limit of a large lattice by

$$\lim_{m,n\to\infty}(mn)^{-1}\ln Z_{mn}(x,y)=\tfrac{1}{2}\ln y+(1/\pi)\int_0^{x/y}(1/v)\tan^{-1}v\,dv.$$

It follows that the free energy and entropy of the system are smooth continuous functions of the densities of horizontal and vertical dimers. The number of ways of filling the lattice with dimers is calculated exactly for $m=n=8$ and is given asymptotically by $[\exp(2G/\pi)]^{\frac{1}{2}mn}=(1.791\,623)^{\frac{1}{2}mn}$. The results are derived with the aid of operator techniques which reduce the partition function to a Pfaffian and hence to a determinant. Some results are also presented for the more general case with monomers present.

1. INTRODUCTION

One of the simplest models of a system containing diatomic molecules is that of lattice gas (or solution) of N_d *rigid dimers*, each of which fills two nearest neighbor sites of a space lattice of N sites. The remaining $N-2N_d$ sites of the lattice may be regarded as occupied by N_0 "holes" (or "monomers"). This model has been used by many authors to discuss the thermodynamics of adsorbed films and mixed solutions.[1–5] It is also interesting in connection with the theory of the condensation of gases.[6] All the thermodynamic properties can be derived from the configurational grand partition function and it is the calculation of this which constitutes the main theoretical problem. Since (in the simplest form of the model) there are no interactions other than "hard core" infinite repulsive forces between dimers, the problem reduces to the determination of the number of ways of placing N_d identical dimers on the lattice so that no two overlap. This is an unsolved combinatorial problem of considerable interest in its own right,[7] and is comparable to the well-known topological aspects of the Ising model[8] first elucidated by Kac and Ward.[9,10]

For a one-dimensional lattice (linear chain) the partition function (or generating function) can be evaluated quite easily in closed form[11] (see Sec. 8) but for two- or three-dimensional lattices no exact results are available. (The Bethe approximation and low-density series expansions have been employed in the main.[1–4]) This paper considers the problem on the plane square (or rectangular) lattice and the partition function is evaluated exactly for the case when the dimers completely fill the lattice (close-packed or high-density limit, $N_d=\frac{1}{2}N$). Our results are exact even for a finite $n\times m$ rectangular lattice with edges so that both bulk and boundary terms in the free energy of a large lattice can be determined.

The partition function is calculated with the aid of operator techniques and the argument follows the lines used recently by Hurst and Green[12] in rederiving Onsager's solution of the plane square Ising model.[8] In

[1] R. H. Fowler and G. S. Rushbrooke, Trans. Faraday Soc. 33, 1272 (1937).
[2] T. S. Chang, Proc. Roy. Soc. (London) A169, 512 (1939); Proc. Cambridge Phil. Soc. 35, 265 (1939).
[3] J. K. Roberts and A. R. Miller, Proc. Cambridge Phil. Soc. 35, 293 (1939).
[4] G. S. Rushbrooke, H. I. Scoins, and A. J. Wakefield, Discussions Faraday Soc. No. 15, 57 (1953).
[5] H. S. Green and R. Leipnik, Revs. Modern Phys. 32, 129 (1960); but see reference 11.
[6] C. N. Yang and T. D. Lee, Phys. Rev. 87, 410 (1952).
[7] F. Harary, "Feynman's simplification of the Kac-Ward treatment of the two-dimensional Ising problem"; (planographed preprint) University of Michigan, June 12, 1958; to appear as a chapter in a book on graph theory.
[8] L. Onsager, Phys. Rev. 65, 117 (1944).
[9] M. Kac and J. C. Ward, Phys. Rev. 88, 1332 (1952).
[10] S. Sherman, J. Math. Phys. 1, 202 (1960).
[11] M. E. Fisher and H. N. V. Temperley, Revs. Modern Phys. 32, 1029 (1960).
[12] C. A. Hurst and H. S. Green, J. Chem. Phys. 33, 1059 (1960).