plier Λ, add the product to $\ln\mathcal{G}_b+\ln\mathcal{G}_s$ and then maximize the sum with respect to P_{ij}. Thus we obtain

$$P_{ij}=\tau\Theta_{ij}p_i(t)e^{\Lambda(E_i-E_j)}+O(\tau^2). \quad \text{(AII.4)}$$

The value of Λ is determined as follows. Using (AII.4), the change of $p_i(t)$ in the heat bath is derived as

$$\begin{aligned}\dot{p}_i(t)&=\sum_j (P_{ji}-P_{ij})/\tau\\&=\sum_j [p_je^{\Lambda(E_j-E_i)}-p_ie^{\Lambda(E_i-E_j)}]\Theta_{ij}.\end{aligned} \quad \text{(AII.5)}$$

In the derivation, the quantum-mechanical reciprocity, $\Theta_{ij}=\Theta_{ji}$, was used. Since the definition of the heat bath requires that $\dot{p}_i(t)$ is arbitrarily small even when the system is interacting with it, (AII.5) vanishes. In the sum of (AII.5), Θ_{ij} is the kinetic quantity related to the transition among levels, whereas the equilibrium statistical mechanics tells us that the equilibrium values of p_i's are independent of Θ_{ij}. Therefore the coefficients of Θ_{ij}'s vanish individually:

$$p_je^{\Lambda(E_j-E_i)}-p_ie^{\Lambda(E_i-E_j)}=0. \quad \text{(AII.6)}$$

On the other hand, we know for the equilibrium distribution,

$$p_j/p_i=e^{-\beta(E_j-E_i)}, \quad \text{(AII.7)}$$

where $\beta=1/kT$. From (AII.6) and (AII.7) we can identify

$$\Lambda=\beta/2. \quad \text{(AII.8)}$$

Thus, as far as the variations with respect to the quantities of the system, P_i for $i=2$, 3, 4 and 5, are concerned, we may add only $L\epsilon\beta(P_4-P_3)/2$ to (AII.2) and need not worry about the detail of the heat bath. This is the origin of the Boltzmann-like factors in (AI.1) and (AI.2).

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Path Integral in Irreversible Statistical Dynamics

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Extending the path probability method for irreversible statistical dynamics proposed previously, the two-gate probability distribution for a stochastic Markoffian system is written in the form of a path integral. The most probable path is derived as a solution of the variational Euler-Lagrange equation. Introducing the concept of the anticausal path, a method to calculate the one-gate probability distribution function (for fluctuations away from the nonequilibrium steady state) is explained. The distribution function derived recently by Mathews, Shapiro, and Falkoff for a system of many levels is shown to follow. The concept of the pseudo-entropy $\tilde{S}$ is introduced. $\tilde{S}$ always increases in time as the system approaches its steady state. It is shown that the rate $d\tilde{S}/dt$, rather than dS/dt of Prigogine, is minimum in the steady state, although the latter can serve as an approximation when the steady state is near equilibrium.

I. INTRODUCTION

In a previous publication by one of the authors[1] a theorem was proved which shows that a function (of state probability parameters[2] $\{p_i\}$) called the persistency, $\mathcal{G}^{(\mathrm{III})}[\{p_i\},\{p_i\}]$, takes its maximum value in the nonequilibrium steady state of a Markoffian system. This theorem has a wider range of validity than Prigogine's principle of minimum entropy production[3] $[\min(dS/dt)]$, and the variation function $\mathcal{G}^{(\mathrm{III})}$ is derived from the path probability of irreversible statistical dynamics proposed previously.[4] In the present paper, the path probability is exploited further, and the relation between the proposed theorem and some other related principles will be discussed.

When the fluctuation from equilibrium is small, the two-gate probability distribution has been written in the form of the path integral.[5–7] Section II derives the path integral representation of the two-gate probability for the case when the irreversible steady state may not be close to the equilibrium state. In Sec. III the concept of the anticausal path is used to calculate the probability for a specific fluctuation from the steady-state distribution. The relation between the maximum persistency theorem of VS and Prigogine's principle of $\min(dS/dt)$ will be discussed in later sections with the help of the concept of pseudo-entropy.

II. PATH INTEGRAL REPRESENTATION

In deriving the maximum persistency theorem in the previous paper, first the path probability $\mathcal{G}^{(\mathrm{I})}$ is written in terms of the path parameters[1] $\{P_{ijr}(t;t+\tau)\}$. Then $\mathcal{G}^{(\mathrm{I})}$ is maximized keeping the end states $\{p_i^{(1)}(t)\}$ and

[1] R. Kikuchi, preceding paper [Phys. Rev. **124**, 1682 (1961)]. This paper will be called VS hereafter.
[2] The curly brackets indicate a set of numbers. $\{p_i\}$ may be written in the vector form as **p**, p_i being regarded as the ith component of this vector. See Sec. II of VS.
[3] I. Prigogine, *Etude Thermodynamique des Phénomènes Irréversibles* (Maison Desoer, Liège, 1947).
[4] R. Kikuchi, Ann. Phys. **10**, 127 (1960).
[5] L. Onsager and S. Machlup, Phys. Rev. **91**, 1505 (1953).
[6] N. Hashitsume, Progr. Theoret. Phys. (Kyoto) **15**, 369 (1956).
[7] D. Falkoff, Ann. Phys. **4**, 325 (1958).

$\{p_j^{(2)}(t+\tau)\}$ fixed to arrive at $\mathcal{G}^{(\mathrm{II})}[\mathbf{p}^{(1)};\mathbf{p}^{(2)}]$. The probability that the assembly follows a certain sequence of states $\mathbf{p}^{(0)}(t)$, $\mathbf{p}^{(1)}(t+\tau)$, $\mathbf{p}^{(2)}(t+2\tau)$, $\cdots$ for a finite time interval is written, as in (2.2) of the previous paper VS, as

$$\mathcal{G}[\mathbf{p}^{(0)}(t);\mathbf{p}^{(1)}(t+\tau);\cdots]=\prod_{\nu=0}\mathcal{G}^{(\mathrm{II})}[\mathbf{p}^{(\nu)};\mathbf{p}^{(\nu+1)}]$$
$$=\exp\{\sum_{\nu=0}\tau L\Gamma[\mathbf{p}^{(\nu)},\dot{\mathbf{p}}^{(\nu)}]\}. \quad (2.1)$$

As was seen in (3.7) and (4.14) of VS, when τ is small $\ln\mathcal{G}^{(\mathrm{II})}[\mathbf{p}^{(\nu)};\mathbf{p}^{(\nu+1)}]$ is proportional to τ and to the number of systems L in the assembly, so that it was written in (2.1) as

$$\ln\mathcal{G}^{(\mathrm{II})}[\mathbf{p}^{(\nu)};\mathbf{p}^{(\nu+1)}]=\tau L\Gamma[\mathbf{p}^{(\nu)},\dot{\mathbf{p}}^{(\nu)}], \quad (2.2)$$

where

$$\dot{\mathbf{p}}^{(\nu)}=\lim_{\tau\to 0}(\mathbf{p}^{(\nu+1)}-\mathbf{p}^{(\nu)})/\tau \quad (2.3)$$

is defined along the path specified arbitrarily. Thus for the limit of infinitesimal τ, the probability that the assembly follows a certain path $p(t)$ for $t_1\leqslant t\leqslant t_2$ is written, as an extension of (2.1),

$$\mathcal{G}[\mathbf{p}(t);t_1\leqslant t\leqslant t_2]=\exp\left\{L\int_{t_1}^{t_2}dt\Gamma[\mathbf{p}(t),\dot{\mathbf{p}}(t)]\right\}. \quad (2.4)$$

The so-called two-gate probability of going from the initial state $\mathbf{p}(t_1)$ to the final state $\mathbf{p}(t_2)$ after a finite time interval is obtained in the path integral form[8] from (2.4) as

$$\mathcal{G}[\mathbf{p}(t_1);\mathbf{p}(t_2)]$$
$$=\int_{\mathbf{p}(t_1)}^{\mathbf{p}(t_2)}\exp\left\{L\int_{t_1}^{t_2}du\,\Gamma[\mathbf{p}(u),\dot{\mathbf{p}}(u)]\right\}\mathfrak{D}\mathbf{p}(u), \quad (2.5)$$

where $\int\mathfrak{D}\mathbf{p}(u)$ indicates to integrate over all paths starting from $\mathbf{p}(t_1)$ and ending at $\mathbf{p}(t_2)$.

In many examples of the path integral method treated so far,[8] the function Γ is quadratic in its arguments; this is because the path integral can be evaluated analytically only for this case. In the present treatment, however, we can include more complicated functions for Γ because we limit ourselves to the case when L, the number of systems in the assembly, is very large. This condition of a large L is analogous to the technique familiar in equilibrium statistical mechanics, and allows us to neglect terms of the order L^{-1} with respect to unity. Under this condition we can replace the path integral (2.5) by the most probable path, that is the path which maximizes the exponent:

$$\mathcal{G}[\mathbf{p}(t_1);\mathbf{p}(t_2)]=\exp\left\{\max L\int_{t_1}^{t_2}du\,\Gamma[\mathbf{p}(u),\dot{\mathbf{p}}(u)]\right\}. \quad (2.6)$$

[8] The path integral in equilibrium statistical mechanics has been used successfully by Feynman and others. See, for instance, the review article, S. B. Brush, Revs. Modern Phys. **33**, 79 (1961).

The most probable path is to be determined from the Euler-Lagrange equation. Because of the analogy between (2.6) and Hamilton's principle in classical mechanics, we will call the function $\Gamma[\mathbf{p}(u),\dot{\mathbf{p}}(u)]$ the Lagrangian for the path.

III. MANY-LEVEL SYSTEM

The approach outlined in the previous section is to be applied here to the problem treated by Mathews *et al.*[9] A system has many states $i=1, 2, 3, \cdots$, and it obeys Boltzmann statistics. An assembly made of L systems will be considered. In order to make the treatment specific the system is assumed to interact with two heat baths of temperatures T' and T''. The transition from the state i to j can occur by interchanging energy with either of the heat baths. The quantum mechanical transition probability from i to j is written as θ_{ij}' and θ_{ij}'' where a single (a double) prime means the energy exchange with the first (the second) heat bath.

The state parameter for the state i at time t is written as $p_i(t)$ and the path parameter of going from the state i at t to j at $t+\tau$ is written as $P_{ij}'(t;t+\tau)$ or $P_{ij}''(t;t+\tau)$, depending on the heat bath to which the energy is exchanged. They satisfy the compatibility relations:

$$p_i(t)=P_{ii}(t;t+\tau)+\textstyle\sum_j'(P_{ij}'+P_{ij}''), \quad (3.1)$$

$$p_i(t+\tau)-p_i(t)=\textstyle\sum_j'(P_{ji}'+P_{ji}''-P_{ij}'-P_{ij}''), \quad (3.2)$$

where P_{ij}' stands for $P_{ij}'(t;t+\tau)$, and $P_{ii}(t;t+\tau)$ is the probability that a system remains in the state i without making any transitions during τ. The prime on $\sum_j'$ excludes the state i.

The first form of the path probability $\mathcal{G}^{(\mathrm{I})}\{P_{ij}(t;t+\tau)\}$ is written, extending the formula (3.3) of the previous paper, as

$$\begin{aligned}L^{-1}\ln&\mathcal{G}^{(\mathrm{I})}\{P_{ij}(t;t+\tau)\}\\ &=\textstyle\sum_i p_i(t)\ln p_i(t)-\sum_i P_{ii}\ln P_{ii}\\ &\quad-\textstyle\sum_i\sum_j'[P_{ij}'\ln P_{ij}'+P_{ij}''\ln P_{ij}'']\\ &\quad+\textstyle\sum_i\sum_j'[P_{ij}'\ln(\tau\theta_{ij}')+P_{ij}''\ln(\tau\theta_{ij}'')]\\ &\quad+\textstyle\sum_i P_{ii}\ln(1-\tau\theta_i)\\ &\quad+\textstyle\sum_i\sum_j'(P_{ij}'\beta'+P_{ij}''\beta'')(E_i-E_j)/2,\end{aligned} \quad (3.3)$$

where $\beta'=(kT')^{-1}$ and E_i is the energy of the ith state of the system. θ_i is the *a priori* probability that a system in the state i makes any transition out of the state i during τ and is determined by

$$\tau\theta_i=\textstyle\sum_j'[P_{ij}'^{(n)}(t;t+\tau)+P_{ij}''^{(n)}(t;t+\tau)]/p_i(t), \quad (3.4)$$

where the superscript (n) indicates the natural path defined in VS.

As the next step, the second form of the path probability $\mathcal{G}^{(\mathrm{II})}$ is derived by maximizing (3.3) with respect to $\{P_{ij}(t;t+\tau)\}$ keeping both $\{p_i(t)\}$ and $\{p_j(t+\tau)\}$

[9] P. M. Mathews, I. I. Shapiro, and D. L. Falkoff, Phys. Rev. **120**, 1 (1960).

fixed. With the use of a Lagrange multiplier $\lambda_i(t)$ for (3.2), the maximization of (3.3) leads to

$$P_{ij}'(t;t+\tau)=\tau\theta_{ij}'p_i(t)\exp[\beta'(E_i-E_j)/2]\times\exp[\lambda_i(t)-\lambda_j(t)]+O(\tau^2), \quad (3.5)$$

and the corresponding expression for P_{ij}''. The natural path is derived from this by putting $\lambda_i(t)=0$, since there is no restriction on $\{p_i(t+\tau)\}$ for the natural path. Thus

$$P_{ij}'^{(n)}(t;t+\tau)=\tau\theta_{ij}'p_i(t)\exp[\beta'(E_i-E_j)/2]+O(\tau^2), \quad (3.6)$$

and the corresponding expression for $P_{ij}''^{(n)}$. From (3.6) and (3.4) we can determine θ_i as

$$\theta_i=\textstyle\sum_j{}'\,\Theta_{ij}, \quad (3.7)$$

where

$$\Theta_{ij}=\theta_{ij}'\exp[\beta'(E_i-E_j)/2]+\theta_{ij}''\exp[\beta''(E_i-E_j)/2]. \quad (3.8)$$

The second form of the path probability

$$\mathcal{G}^{(\mathrm{II})}[\{p_i(t)\};\{p_j(t+\tau)\}],$$

or the Lagrangian Γ, for the path is derived by substituting (3.5) in (3.3) as

$$\begin{aligned}\Gamma[\{p_i(t)\},\{\dot{p}_i(t)\}]&=(\tau L)^{-1}\ln\mathcal{G}^{(\mathrm{II})}[\{p_i(t)\};\{p_j(t+\tau)\}]\\&=\textstyle\sum_i\sum_j{}'\,p_i(t)\Theta_{ij}\{\exp[\lambda_i(t)-\lambda_j(t)]-1\}\\&\quad+\textstyle\sum_i\lambda_i(t)\dot{p}_i(t),\end{aligned} \quad (3.9)$$

where $\lambda_i(t)$ is determined from (3.2) combined with (3.5) as

$$\dot{p}_i(t)=\textstyle\sum_j{}'\,\{p_j(t)\Theta_{ji}\exp[\lambda_j(t)-\lambda_i(t)]-p_i(t)\Theta_{ij}\exp[\lambda_i(t)-\lambda_j(t)]\}. \quad (3.10)$$

This is seen to be an extension of VS (3.8b).

The most probable path is determined by varying $\int\Gamma dt$ with respect to $\{p_i(t)\}$. In order to derive a convenient equation, the following procedure is used. Substituting (3.10) for $\dot{p}_i$ in (3.9), and then taking the variation, we obtain

$$\begin{aligned}\delta\int\Gamma[\{p_i(t)\},\{\dot{p}_i(t)\}]dt&=\int\textstyle\sum_i\sum_j{}'\,\{\Theta_{ij}[(\lambda_j-\lambda_i)\exp(\lambda_i-\lambda_j)\\&\quad+\exp(\lambda_i-\lambda_j)-1]\delta p_i-p_i\Theta_{ij}(\lambda_i-\lambda_j)\\&\quad\times\exp(\lambda_i-\lambda_j)\delta(\lambda_i-\lambda_j)\}dt.\end{aligned} \quad (3.11)$$

To evaluate the last term of this expression, we write the variation of (3.10) as

$$\begin{aligned}\textstyle\sum_i\lambda_i\delta\dot{p}_i=\sum_i\sum_j{}'\,[&(\lambda_j-\lambda_i)\Theta_{ij}\exp(\lambda_i-\lambda_j)\delta p_i\\&-p_i\Theta_{ij}(\lambda_i-\lambda_j)\exp(\lambda_i-\lambda_j)\delta(\lambda_i-\lambda_j)].\end{aligned} \quad (3.12)$$

Eliminating $\delta(\lambda_i-\lambda_j)$ from (3.11) and (3.12) and partially integrating once to change $\delta\dot{p}_i$ to δp_i, we arrive at the following Euler-Lagrange equation:

$$\dot{\lambda}_i(t)=\textstyle\sum_j{}'\,\Theta_{ij}\{\exp[\lambda_i(t)-\lambda_j(t)]-1\}. \quad (3.13)$$

The most probable path $\{p_i(t)\}$, and $\{\lambda_i(t)\}$ associated with it, must satisfy (3.10) and (3.13) for given initial and final conditions. When (3.13) is used in (3.9), the latter is simplified to

$$\Gamma[\{p_i(t)\},\{\dot{p}_i(t)\}]=d[\textstyle\sum_i\lambda_i(t)p_i(t)]/dt, \quad (3.14)$$

and from (2.6) we arrive at

$$\mathcal{G}[\mathbf{p}(t_1);\mathbf{p}(t_2)]=\exp\{L\textstyle\sum_i[\lambda_i(t_2)p_i(t_2)-\lambda_i(t_1)p_i(t_1)]\}. \quad (3.15)$$

Although it is not easy to solve the path in general, two special cases are of importance. The first case is the natural path for which $\lambda_i=0$ as we have seen before. It is obvious that this satisfies (3.13) and for this case the familiar kinetic equation for the natural path results from (3.10):

$$\dot{p}_i^{(n)}(t)=\textstyle\sum_j{}'\,[p_j^{(n)}(t)\Theta_{ji}-p_i^{(n)}(t)\Theta_{ij}]. \quad (3.16)$$

This is an integral of the simultaneous equations (3.10) and (3.13) and corresponds to the boundary conditions that $p_i(t)$ approaches the stationary value, which is denoted by $p_{i\infty}$, as t goes to $+\infty$. The steady state is a special case of (3.16) and satisfies

$$\textstyle\sum_j{}'\,[p_{j\infty}\Theta_{ji}-p_{i\infty}\Theta_{ij}]=0. \quad (3.17)$$

It should be noted that detailed balance:

$$p_{j\infty}\Theta_{ji}=p_{i\infty}\Theta_{ij}, \quad (3.18)$$

does not necessarily hold.

One special feature of the natural path is that when $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ are connected by the natural path,

$$\mathcal{G}[\mathbf{p}(t_1);\mathbf{p}(t_2)]=1, \quad (3.19)$$

which is derived from (3.15) by setting $\lambda_i(t)=0$. To be more precise, an additional term of the order of L^{-1} is neglected in (3.19); this term dropped when we replaced the path integral of (2.5) by the most probable path in (2.6).

Another case of special significance is the following solution of (3.13):

$$\exp(\lambda_i(t))=p_{i\infty}/p_i^{(a)}(t). \quad (3.20)$$

It can be shown that (3.20) satisfies (3.13) and (3.10) because of (3.17). When (3.20) holds, (3.10) becomes

$$\dot{p}_i^{(a)}(t)=\textstyle\sum_j{}'\,[p_i^{(a)}(t)\Theta_{ji}p_{j\infty}/p_{i\infty}-p_j^{(a)}(t)\Theta_{ij}p_{i\infty}/p_{j\infty}]. \quad (3.21)$$

In order to understand the meaning of this path, let us consider a special case when detailed balance (3.18) holds; then we see that $\dot{p}_i^{(a)}(t)$ or (3.21) is the negative of $\dot{p}_i^{(n)}(t)$ of (3.16). This suggests that even when

detailed balance does not hold, the path governed by (3.21) is roughly obtained from the natural path by reversing the time, or we may interpret that the path of (3.21) starts from the steady state at time $-\infty$ and deviates from it to approach a fluctuated state $\{p_i(t)\}$ at t. We will call this the path the *anticausal* path[10] and indicate it with a superscript (a). It might seem that $\dot{p}_i(a)$ for the anticausal path should be the negative of $\dot{p}_i^{(n)}$ of the natural path (3.16):

$$\dot{p}_i^{(a)}(t) = -\sum_j' [p_j^{(a)}(t)\Theta_{ji} - p_i^{(a)}(t)\Theta_{ij}]. \quad (3.22)$$

rather than (3.21). However, this inference does not hold in general, since there is no solution of (3.10) and (3.13) of the form of (3.22) except when detailed balance (3.18) holds.

When the anticausal path is known, we can calculate the so-called one-gate probability that the fluctuated state $\{p_i\}$ occurs in the assembly which is in the over-all steady state. This probability $G\{p_i\}$ is calculated by considering a path which goes from the steady state $\{p_{i\infty}\}$ at $t=-\infty$ to the fluctuated state $\{p_i\}$ at t, since we can identify

$$G\{p_i(t)\} = \mathcal{G}[\{p_{i\infty}(-\infty)\};\{p_i(t)\}]. \quad (3.23)$$

The right-hand side quantity, which is of the form of (2.5), can be calculated using the most probable path which is the anticausal path in this case.[11] Thus we obtain, by combining (3.23), (3.15), and (3.20),

$$G\{p_i\} = \exp[L \sum_i p_i \ln(p_{i\infty}/p_i)]. \quad (3.24)$$

Using Stirling's approximation, we may write (3.24) as

$$G\{p_i\} = \frac{L!}{\prod_i (Lp_i)!} \prod_i (p_{i\infty})^{Lp_i}. \quad (3.25)$$

This is the formula proved by Mathews, Shapiro, and Falkoff[9] by means of a different method.

IV. PSEUDO-ENTROPY

The expression (3.25) suggests a simple interpretation of $G\{p_i\}$. If the system is interacting with only one heat bath and it is in thermal equilibrium, we know that $p_{i\infty} \propto \exp(-\beta E_i)$ and $-kT \ln G\{p_i\}$ is the free energy for the fluctuated state $\{p_i\}$, or $+k \ln G\{p_i\}$ is the entropy (of the fluctuated state) of the universe including the system and the heat bath. In analogy with this property in equilibrium, we introduce a quantity $\tilde{S}\{p_i\}$, which has the characteristics of entropy, by

$$\tilde{S}\{p_i\} = k \ln G\{p_i\}, \quad (4.1)$$

and call it the *pseudo-entropy* (of the fluctuated state) of the universe, i.e., of the system plus the heat baths.

The importance of the notion of the pseudo-entropy is that in the steady state it plays the role that the entropy plays in equilibrium. First we will show that $\tilde{S}\{p_i(t)\}$ increases as the state of the assembly follows the natural path. When the state $\mathbf{p}(t_1)$ changes into $\mathbf{p}(t_2)$ along the natural path, use of (3.23), (2.6), and (3.19) leads to

$$\begin{aligned} G\{\mathbf{p}(t_2)\} &= \mathcal{G}[\mathbf{p}_\infty(-\infty), \mathbf{p}(t_2)] \\ > \mathcal{G}[\mathbf{p}_\infty(-\infty), \mathbf{p}(t_1), \mathbf{p}(t_2)] &= \mathcal{G}[\mathbf{p}_\infty(-\infty), \mathbf{p}(t_1)] \\ &\times \mathcal{G}[\mathbf{p}(t_1), \mathbf{p}(t_2)] = G\{\mathbf{p}(t_1)\}. \quad (4.2) \end{aligned}$$

Thus

$$\tilde{S}\{\mathbf{p}(t_2)\} > \tilde{S}\{\mathbf{p}(t_1)\}. \quad (4.3)$$

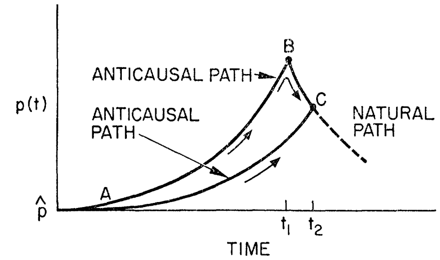

FIG. 1. Schematic diagram for Eq. (4.2). $G\{\mathbf{p}(t_1)\} = \mathcal{G}[A \to B]$, $G\{\mathbf{p}(t_2)\} = \mathcal{G}[A \to C]$, and $\mathcal{G}[A \to C] > \mathcal{G}[A \to B \to C]$. The stationary state $\hat{\mathrm{p}}$ is written as $\mathbf{p}_\infty$ in the text.

Figure 1 explains (4.2) graphically. This result implies the following theorem: (I) *The pseudo-entropy is a maximum in the steady state*, or in equation form

$$\tilde{S}\{p_{i\infty}\} \geqslant \tilde{S}\{p_i\}. \quad (4.4)$$

In analogy with the entropy production, we may introduce the concept of the pseudo-entropy production. It is defined along the natural path of the change of state. Then using (4.3), we derive the second theorem[12]: (II) *The rate of pseudo-entropy production vanishes in the steady state, and is positive otherwise.*

These two theorems concerning the pseudo-entropy were pointed out by Klein.[13] As he commented, the "principle of minimum pseudo-entropy production" cannot replace Prigogine's principle of "minimum entropy production" in deriving the steady state, because the pseudo-entropy needs the knowledge of the steady state in its definition as is shown in (3.24) or (3.25). The two theorems in this section are useful, however, since the pseudo-entropy production is proportional to the persistency in some cases as will be shown in Sec. VI.

[10] The anticausal path was discussed by E. N. Adams [Phys. Rev. **120**, 675 (1960)] in a different context.

[11] H. B. Callen [Phys. Rev. **111**, 367 (1958)] calculated the one-gate probability using a different approach. His expression, however, has a close similarity with the integration along the anticausal path explained here. When the fluctuation is close to the equilibrium state, to which case Callen limits his treatment, the anticausal path is the negative of the natural path since the detailed balance holds in the equilibrium state. In Callen's Eq. (45), $\langle d\tilde{Q}_i(t)/dt\rangle_{\tilde{Q}_k(0)}$ is the time derivative of $\tilde{Q}_i$ along the natural path at time t when $\tilde{Q}_k(0)$ at $t=0$ is given; this is equal to the negative of the time derivative of $\tilde{Q}_k$ along the anticausal path at time $-t$ when the value of $\tilde{Q}_k(0)$ at $t=0$ is to be specified. Thus Callen's Eq. (45) can be interpreted as the integration along the anticausal path except for the time dependence of the driving force $V_i(t)$.

[12] For a special case in which the steady state is the equilibrium state, these theorems express the second law of thermodynamics in the language of the path.

[13] M. J. Klein, *Transport Processes in Statistical Mechanics*, edited by I. Prigogine (Interscience Publishers, Inc., New York, 1958), Sec. IV, p. 311.

V. SMALL DEVIATION FROM THE STEADY STATE

In order to illustrate some of the key relations, the case of small deviation from the steady state will be discussed. Going back to the problem of Sec. III, let us define the deviation α_i from the steady state as

$$p_i(t)=p_{i\infty}(1+\alpha_i(t)). \tag{5.1}$$

Use of (5.1) in (3.24) and (4.1) leads to

$$\tilde{S}\{p_i\}=-kL\sum_i p_{i\infty}\alpha_i^2/2+O(\alpha^3). \tag{5.2}$$

This clearly shows the theorem (I) in the previous section that the pseudo-entropy is maximum at the steady state.

The two-gate probability distribution, (2.6) or (3.15), is next examined. When α_i's are small, λ_i's in (3.10) are also small (since $\lambda_i=0$ when all α_i's vanish), so that we can expand (3.10) and (3.13) as

$$p_{i\infty}\dot{\alpha}_i=\sum_j{}'\left[p_{j\infty}\Theta_{ji}(\alpha_j+\lambda_j-\lambda_i)-p_{i\infty}\Theta_{ij}(\alpha_i+\lambda_i-\lambda_j)\right], \tag{5.3a}$$

$$\dot{\lambda}_i=\sum_j{}'\,\Theta_{ij}(\lambda_i-\lambda_j). \tag{5.3b}$$

These equations can be regarded as a set of linear simultaneous equations for the variables α_i's and λ_i's. It can be shown by simple substitution that (5.3) is equivalent to the following set of equations for the variables $\alpha_i^{(n)}$'s and $\alpha_i^{(a)}$'s, which are linear combinations of α_i's and λ_i's as shown in (5.5) and (5.6):

$$\begin{aligned}p_{i\infty}\dot{\alpha}_i^{(n)}&=\sum_j{}'\left[p_{j\infty}\alpha_j^{(n)}\Theta_{ji}-p_{i\infty}\alpha_i^{(n)}\Theta_{ij}\right],\\ p_{i\infty}\dot{\alpha}_i^{(a)}&=\sum_j{}'\left[p_{j\infty}\alpha_i^{(a)}\Theta_{ji}-p_{i\infty}\alpha_j^{(a)}\Theta_{ij}\right],\end{aligned} \tag{5.4}$$

$$\alpha_i(t)=\alpha_i^{(n)}(t)+\alpha_i^{(a)}(t). \tag{5.5}$$

$$\lambda_i(t)=-\alpha_i^{(a)}(t). \tag{5.6}$$

On the other hand the two equations in (5.4) are the linearized forms of (3.16) and (3.21), so that (5.5) is interpreted as the decomposition of $\alpha_i(t)$ into the natural component $\alpha_i^{(n)}(t)$ and the anticausal component $\alpha_i^{(a)}(t)$. Notice that (5.6) is the linearized version of (3.20). It must be emphasized here that the fraction of $\alpha_i(t)$ belonging to each of the two components depends on the initial and the final states of the path.

Since we want to retain the square terms of α_i's correctly in the two-gate probability, the direct use of (3.15) is not convenient; we go back to (3.9) and expand it using (5.1), (5.5), and (5.6) to obtain

$$\Gamma[\boldsymbol{\alpha}(t),\dot{\boldsymbol{\alpha}}(t)]=\sum_i\sum_j{}'\,p_{i\infty}\Theta_{ij}\times\left[\alpha_i(\alpha_j^{(a)}-\alpha_i^{(a)})+\tfrac{1}{2}(\alpha_j^{(a)}-\alpha_i^{(a)})^2\right]-\sum_i p_{i\infty}\alpha_i^{(a)}(\dot{\alpha}_i^{(n)}+\dot{\alpha}_i^{(a)}). \tag{5.7}$$

It can be shown by use of (5.4) that the first sum is equal to $\sum_i p_{i\infty}\dot{\alpha}_i^{(n)}\alpha_i^{(a)}$ which cancels with a part of the last sum, so that after integration of (2.6) we obtain

$$\mathcal{G}[\boldsymbol{\alpha}(t_1);\boldsymbol{\alpha}(t_2)]=\exp\{-L\sum_i p_{i\infty}[(\alpha_i^{(a)}(t_2))^2-(\alpha_i^{(a)}(t_1))^2]/2\}. \tag{5.8}$$

This is the two-gate probability when α_i's are small; it should be noticed that only the anticausal components appear in it.

Two cases of (5.8) are of special interest. When a natural path is decomposed as in (5.5), all the anticausal components vanish so that for a natural path (5.8) results in

$$\mathcal{G}[\boldsymbol{\alpha}(t_1);\boldsymbol{\alpha}(t_2)]=1, \tag{5.9}$$

which is a simplified form of (3.19). For an anticausal path, all the natural components vanish to yield

$$\alpha_i^{(a)}(t)=\alpha_i(t). \tag{5.10}$$

When t_1 is taken as $-\infty$ so that $\alpha_i^{(a)}(-\infty)=\alpha_i(-\infty)=0$, (5.8) reduces to the one-gate probability or to the expression of the pseudo-entropy (5.2) as is expected.

The two-gate probability (5.8) can be used in deriving the correlation function. For this purpose we express $\alpha_i^{(a)}$ in terms of the end states. We notice that for $\alpha_i^{(a)}$ and $\alpha_i^{(n)}$ of (5.4), without loss of accuracy, we can require

$$\sum_i p_{i\infty}\alpha_i^{(n)}=\sum_i p_{i\infty}\alpha_i^{(a)}=0, \tag{5.11}$$

since, if for instance

$$\sum_i p_{i\infty}\alpha_i^{(a)}=c\neq 0$$

or

$$\sum_i p_{i\infty}(\alpha_i^{(a)}-c)=0,$$

we see that $\alpha_i^{(a)}-c$ satisfies the same set of equations as (5.4) because of (3.17) so that we may take $\alpha_i^{(a)}-c$ as the new $\alpha_i^{(a)}$. Equation (5.11) is used in expressing one of the components of $\alpha_i^{(a)}$, say $\alpha_1^{(a)}$, in terms of the rest of $\alpha_i^{(a)}$'s. We introduce a vector $\mathbf{z}^{(a)}$ having $\alpha_i^{(a)}$ $(i=2,3,\cdots)$ as its components; notice that $i=1$ is not included in it so that all the components of the vector $\mathbf{z}^{(a)}$ are independent. Similarly we define $\mathbf{z}^{(n)}$. Use of (5.11) enables us to eliminate $\alpha_1^{(a)}$ and $\alpha_1^{(n)}$ from (5.4) and to write the latter in the vector form as

$$\begin{aligned}\dot{\mathbf{z}}^{(n)}&=-\mathbf{N}\mathbf{z}^{(n)},\\ \dot{\mathbf{z}}^{(a)}&=\mathbf{A}\mathbf{z}^{(a)}.\end{aligned} \tag{5.12}$$

Here $\mathbf{N}$ and $\mathbf{A}$ are the coefficient matrices derived from (5.4); the explicit form of the matrix elements, however, is not necessary for the present discussion. Using the formal integrals of (5.12) in (5.4), we can write

$$\mathbf{z}(t-t_1)=\exp[-\mathbf{N}(t-t_1)]\mathbf{z}^{(n)}(t_1)+\exp[\mathbf{A}(t-t_1)]\mathbf{z}^{(a)}(t_1). \tag{5.13}$$

Equation (5.13) can be written for $t=t_1$ and t_2 giving two vector equations which can be solved to yield the expression of the anticausal component in terms of the end states:

$$\begin{aligned}\mathbf{z}^{(a)}(t_1)&=\{\exp[\mathbf{A}(t_2-t_1)]-\exp[-\mathbf{N}(t_2-t_1)]\}^{-1}\times\{\mathbf{z}(t_2)-\exp[-\mathbf{N}(t_2-t_1)]\mathbf{z}(t_1)\}\\ \mathbf{z}^{(a)}(t_2)&=\exp[\mathbf{A}(t_2-t_1)]\mathbf{z}^{(a)}(t_1).\end{aligned} \tag{5.14}$$

On the other hand, $\alpha_1^{(a)}$ can be eliminated from (5.8) and (5.11) to obtain the expression for the two-gate probability in terms of $\mathbf{z}^{(a)}$ as

$$\mathcal{G}[\mathbf{z}(t_1);\mathbf{z}(t_2)]=\exp\{-(L/2)[\mathbf{z}^{(a)}(t_2)\cdot\mathbf{S}\cdot\mathbf{z}^{(a)}(t_2)-\mathbf{z}^{(a)}(t_1)\cdot\mathbf{S}\cdot\mathbf{z}^{(a)}(t_1)]\}. \tag{5.15}$$

The matrix **S** is now nondiagonal. We can substitute (5.14) into (5.15) to write the latter in terms of the end states. The resulting expression is not easy to simplify because **N** and **A** do not commute in general. However, it is apparent from the form of (5.14) that the two-gate probability does lead to the expected correlation similar to that given by Lax[14]:

$$\langle \mathbf{z}(t_1)\mathbf{z}(t_2)\rangle = \exp[-\mathbf{N}(t_2-t_1)]\langle \mathbf{z}(t_1)\mathbf{z}(t_1)\rangle. \quad (5.16)$$

VI. PSEUDO-ENTROPY PRODUCTION

When the deviation from the steady state is small, the production of pseudo-entropy is derived either from (5.2) or from its modification corresponding to (5.15) as

$$d\tilde{S}\{\mathbf{z}(t)\}/dt = -kL[\dot{\mathbf{z}}(t)\cdot\mathbf{S}\cdot\mathbf{z}(t)+\mathbf{z}(t)\cdot\mathbf{S}\cdot\dot{\mathbf{z}}(t)]/2. \quad (6.1)$$

Here $\dot{\mathbf{z}}(t)$ is the change of $\mathbf{z}$ following the natural path starting from $\mathbf{z}(t)$, so that from the first equation of (5.12), we have

$$\dot{\mathbf{z}}(t) = -\mathbf{N}\mathbf{z}(t). \quad (6.2)$$

On the other hand the persistency, $\mathcal{G}^{(\mathrm{III})}$, which was defined in VS and was proved to take its maximum value in the steady state, is written from (5.15) as

$$(\tau L)^{-1}\ln\mathcal{G}^{(\mathrm{III})}[\mathbf{z}(t);\mathbf{z}(t+\tau)] = -\tfrac{1}{2}[\dot{\mathbf{z}}^{(a)}(t)\cdot\mathbf{S}\cdot\mathbf{z}^{(a)}+\mathbf{z}^{(a)}(t)\cdot\mathbf{S}\cdot\dot{\mathbf{z}}^{(a)}], \quad (6.3)$$

where $\mathbf{z}^{(a)}(t)$ and its time derivative are to be calculated for the path for which

$$\dot{\mathbf{z}}(t) = \dot{\mathbf{z}}^{(n)}(t)+\dot{\mathbf{z}}^{(a)}(t) = 0. \quad (6.4)$$

Use of (5.12) and (5.5) in (6.4) leads to the relations

$$\begin{aligned}\mathbf{z}^{(a)}(t) &= (\mathbf{N}+\mathbf{A})^{-1}\mathbf{N}\mathbf{z}(t),\\ \dot{\mathbf{z}}^{(a)}(t) &= \mathbf{A}(\mathbf{N}+\mathbf{A})^{-1}\mathbf{N}\mathbf{z}(t).\end{aligned} \quad (6.5)$$

The equations in (6.5) depend on the detail of **N** and **A**; a simplification results, however, when detailed balance (3.18) holds in the steady state. In that case $\mathbf{N}=\mathbf{A}$, so that from (6.5),

$$\begin{aligned}\mathbf{z}^{(a)}(t) &= \mathbf{z}(t)/2,\\ \dot{\mathbf{z}}^{(a)}(t) &= \mathbf{N}\mathbf{z}(t)/2.\end{aligned} \quad (6.6)$$

When (6.6) and (6.2) are used in (6.3) and (6.1), respectively, we see that the latter two are connected by

$$d\tilde{S}\{\mathbf{z}(t)\}/dt = -4(k/\tau)\ln\mathcal{G}^{(\mathrm{III})}[\mathbf{z}(t);\mathbf{z}(t+\tau)]. \quad (6.7)$$

It is a simple matter to verify this relation for the example of the two-level atoms treated in Sec. III of VS, since detailed balance holds in the steady state of the two-level problem. The comparison of (3.12) and (3.13) of VS with the help of (6.7) clearly shows the difference between dS/dt and $d\tilde{S}/dt$, the latter being a rigorous minimum in the steady state while the former is only an approximate minimum.

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Correlation in the Fluctuating Outputs from Two Square-Law Detectors Illuminated by Light of Any State of Coherence and Polarization*

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By treating light fluctuations as a random process (stationary and Gaussian to second order), an expression is derived for the correlation in the output fluctuations of two square-law detectors which are illuminated by a plane light wave of any state of coherence and polarization. The expression takes a particularly simple form when, as is usually the case, the light is spectrally pure in the sense of a definition introduced elsewhere. The solution yields, as a special case, the basic formula relating to the Hanbury Brown-Twiss effect. The generalization discussed here is of particular interest for correlation experiments performed with light beams from optical maser sources. Moreover, the present analysis appears to be simpler than other treatments previously given in connection with more restricted cases.

1. INTRODUCTION

In a series of well-known papers,[1] Hanbury Brown and Twiss have shown that when a light wave illuminates two photoelectric detectors P_1, P_2, the outputs from the detectors are in general correlated and that this correlation is proportional to the square of the degree of coherence of the light vibrations at P_1 and P_2. This effect, which has also been studied by many other authors,[2–7] has an important bearing on the problem of

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