The matrix **S** is now nondiagonal. We can substitute (5.14) into (5.15) to write the latter in terms of the end states. The resulting expression is not easy to simplify because **N** and **A** do not commute in general. However, it is apparent from the form of (5.14) that the two-gate probability does lead to the expected correlation similar to that given by Lax[14]:

$$\langle \mathbf{z}(t_1)\mathbf{z}(t_2)\rangle = \exp[-\mathbf{N}(t_2-t_1)]\langle \mathbf{z}(t_1)\mathbf{z}(t_1)\rangle. \quad (5.16)$$

VI. PSEUDO-ENTROPY PRODUCTION

When the deviation from the steady state is small, the production of pseudo-entropy is derived either from (5.2) or from its modification corresponding to (5.15) as

$$d\tilde{S}\{\mathbf{z}(t)\}/dt = -kL[\dot{\mathbf{z}}(t)\cdot\mathbf{S}\cdot\mathbf{z}(t)+\mathbf{z}(t)\cdot\mathbf{S}\cdot\dot{\mathbf{z}}(t)]/2. \quad (6.1)$$

Here $\dot{\mathbf{z}}(t)$ is the change of $\mathbf{z}$ following the natural path starting from $\mathbf{z}(t)$, so that from the first equation of (5.12), we have

$$\dot{\mathbf{z}}(t) = -\mathbf{N}\mathbf{z}(t). \quad (6.2)$$

On the other hand the persistency, $\mathcal{G}^{(\mathrm{III})}$, which was defined in VS and was proved to take its maximum value in the steady state, is written from (5.15) as

$$(\tau L)^{-1}\ln\mathcal{G}^{(\mathrm{III})}[\mathbf{z}(t);\mathbf{z}(t+\tau)] = -\tfrac{1}{2}[\dot{\mathbf{z}}^{(a)}(t)\cdot\mathbf{S}\cdot\mathbf{z}^{(a)}+\mathbf{z}^{(a)}(t)\cdot\mathbf{S}\cdot\dot{\mathbf{z}}^{(a)}], \quad (6.3)$$

where $\mathbf{z}^{(a)}(t)$ and its time derivative are to be calculated for the path for which

$$\dot{\mathbf{z}}(t) = \dot{\mathbf{z}}^{(n)}(t)+\dot{\mathbf{z}}^{(a)}(t) = 0. \quad (6.4)$$

Use of (5.12) and (5.5) in (6.4) leads to the relations

$$\begin{aligned}\mathbf{z}^{(a)}(t) &= (\mathbf{N}+\mathbf{A})^{-1}\mathbf{N}\mathbf{z}(t),\\ \dot{\mathbf{z}}^{(a)}(t) &= \mathbf{A}(\mathbf{N}+\mathbf{A})^{-1}\mathbf{N}\mathbf{z}(t).\end{aligned} \quad (6.5)$$

The equations in (6.5) depend on the detail of **N** and **A**; a simplification results, however, when detailed balance (3.18) holds in the steady state. In that case $\mathbf{N}=\mathbf{A}$, so that from (6.5),

$$\begin{aligned}\mathbf{z}^{(a)}(t) &= \mathbf{z}(t)/2,\\ \dot{\mathbf{z}}^{(a)}(t) &= \mathbf{N}\mathbf{z}(t)/2.\end{aligned} \quad (6.6)$$

When (6.6) and (6.2) are used in (6.3) and (6.1), respectively, we see that the latter two are connected by

$$d\tilde{S}\{\mathbf{z}(t)\}/dt = -4(k/\tau)\ln\mathcal{G}^{(\mathrm{III})}[\mathbf{z}(t);\mathbf{z}(t+\tau)]. \quad (6.7)$$

It is a simple matter to verify this relation for the example of the two-level atoms treated in Sec. III of VS, since detailed balance holds in the steady state of the two-level problem. The comparison of (3.12) and (3.13) of VS with the help of (6.7) clearly shows the difference between dS/dt and $d\tilde{S}/dt$, the latter being a rigorous minimum in the steady state while the former is only an approximate minimum.

[14] M. Lax, Revs. Modern Phys. **32**, 25 (1960).

PHYSICAL REVIEW VOLUME 124, NUMBER 6 DECEMBER 15, 1961

Correlation in the Fluctuating Outputs from Two Square-Law Detectors Illuminated by Light of Any State of Coherence and Polarization*

L. Mandel
Department of Physics, Imperial College of Science and Technology, London University, London, England

AND

E. Wolf
Department of Physics and Astronomy, University of Rochester, Rochester, New York

(Received July 26, 1961)

By treating light fluctuations as a random process (stationary and Gaussian to second order), an expression is derived for the correlation in the output fluctuations of two square-law detectors which are illuminated by a plane light wave of any state of coherence and polarization. The expression takes a particularly simple form when, as is usually the case, the light is spectrally pure in the sense of a definition introduced elsewhere. The solution yields, as a special case, the basic formula relating to the Hanbury Brown-Twiss effect. The generalization discussed here is of particular interest for correlation experiments performed with light beams from optical maser sources. Moreover, the present analysis appears to be simpler than other treatments previously given in connection with more restricted cases.

1. INTRODUCTION

In a series of well-known papers,[1] Hanbury Brown and Twiss have shown that when a light wave illuminates two photoelectric detectors P_1, P_2, the outputs from the detectors are in general correlated and that this correlation is proportional to the square of the degree of coherence of the light vibrations at P_1 and P_2. This effect, which has also been studied by many other authors,[2–7] has an important bearing on the problem of

* This research was supported in part by the Air Force Office of Scientific Research (Office of Aerospace Research).

[1] R. Hanbury Brown and R. Q. Twiss, (a) Nature **177**, 27 (1956); (b) Proc. Roy. Soc. (London) **A242**, 300 (1957); (c) **243**, 291 (1957).

[2] E. M. Purcell, Nature **178**, 1449 (1956).

[3] L. Jánossy, Nuovo cimento **6**, 111 (1957); **12**, 369 (1959).

[4] E. Wolf, Phil. Mag. **2**, 351 (1957).

[5] F. D. Kahn, Optica Acta **5**, 93 (1958).

[6] L. Mandel, Proc. Phys. Soc. (London) **72**, 1037 (1958).

[7] L. Mandel, Proc. Phys. Soc. (London) **74**, 233 (1959).

measuring stellar diameters. It appears that the Hanbury Brown-Twiss effect can also be used for other purposes, for example to determine the velocity of light,[8] the profiles of spectral lines,[9,10] or the degree of polarization of a light beam.[11] Until recently it was difficult to demonstrate this effect, because of the very low value of the "degeneracy parameter" δ of light generated by ordinary sources. As has been pointed out by Mandel,[12] usual sources give light with $\delta \ll 1$, but the degeneracy parameter of radiation from an optical maser is of quite a different order of magnitude ($\delta \sim 10^8$). Thus one may now carry out with relative ease a number of experiments which make use of the Hanbury Brown-Twiss effect.

In the present paper, some of the previously obtained results are generalized by using an approach that appears to be simpler than that employed previously. An expression is derived for the correlation in the output from two square-law detectors illuminated by a plane light wave of any state of coherence and polarization. The formula takes a particularly simple form in the usual case of "spectrally pure" light.[13] The limiting form of the solution for detectors whose resolving time is either very long or very short compared with the coherence time of the light are also derived from the general solution. The former case is of interest in connection with the experiments of Hanbury Brown and Twiss and our solution describes in a simple manner the essential features of the effect observed by them. The latter is of interest in connection with similar experiments using light from an optical maser.

2. CORRELATION IN THE OUTPUTS FROM TWO SQUARE LAW DETECTORS ILLUMINATED BY CORRELATED LIGHT BEAMS

Consider a plane, quasi-monochromatic light wave propagated in the z direction. Let $\mathbf{E}^{(r)}(\mathbf{r},t)$ denote the electric vector at a point P, specified by the position vector $\mathbf{r}$, at time t. $\mathbf{E}^{(r)}$ is a rapidly fluctuating function of time and the fluctuations at any two points P_1 and P_2 (even when situated on the same "wavefront"), generally will not be the same. The correlation between the electric vectors $\mathbf{E}^{(r)}$ at P_1 and P_2 constitutes the phenomenon of partial coherence, while the correlation between the orthogonal components of $\mathbf{E}^{(r)}$ at any one point P constitutes the phenomenon of partial polarization.

It will be convenient to employ in place of $\mathbf{E}^{(r)}$ the associated complex analytic signal $\mathbf{E}$ (cf. Born and Wolf,[14] §10.2), as customary in current research on partial coherence and partial polarization:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}^{(r)}(\mathbf{r},t)+i\mathbf{E}^{(i)}(\mathbf{r},t), \tag{2.1}$$

where $\mathbf{E}^{(i)}$ is the function conjugate to $\mathbf{E}^{(r)}$ (its Hilbert transform),[15]

$$\mathbf{E}^{(i)}(\mathbf{r},t)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathbf{E}^{(r)}(\mathbf{r},t')}{t'-t}dt', \tag{2.2}$$

P denoting the Cauchy principal value at $t'=t$.

The instantaneous intensity $I(\mathbf{r},t)$ of the wave may be defined by[16]

$$I(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}^*(\mathbf{r},t). \tag{2.3}$$

Let suffixes x, y denote Cartesian components with respect to a fixed set of axes at right angles to the direction of propagation. Then (2.3) may be rewritten as

$$I(\mathbf{r},t)=I_x(r,t)+I_y(\mathbf{r},t), \tag{2.3a}$$

where

$$I_x(\mathbf{r},t)=|E_x(\mathbf{r},t)|^2,\quad I_y(\mathbf{r},t)=|E_y(\mathbf{r},t)|^2. \tag{2.4}$$

Suppose now that a square-law detector is placed at the point $P(\mathbf{r})$ and that this detector registers a time average of $I(\mathbf{r},t)$ taken over a time interval of duration T, which plays the role of a resolving time. (Most detectors used at present have a resolving time which is not shorter than about 10^{-9} sec.) An example of such a square-law detector would be a photoelectric detector in which the quantum nature of the incident light and of the photocurrent is ignored. The detector signal $S(\mathbf{r},t)$ is then given by

$$S(\mathbf{r},t)=\frac{\alpha}{T}\int_t^{t+T}I(\mathbf{r},t')dt', \tag{2.5}$$

where α is some constant which represents the efficiency of the square-law detector. $S(\mathbf{r})$ will fluctuate in the course of time. Let us consider the average value of S obtained from a large number of measurements. If stationarity of the process is assumed, this ensemble average will be equal to a time average, taken over a very long time interval, of a typical "sample." Let sharp brackets denote such an average. Then

$$\langle S(\mathbf{r},t)\rangle=\alpha\langle I(\mathbf{r},t)\rangle=\alpha\langle I_x(\mathbf{r},t)\rangle+\alpha\langle I_y(\mathbf{r},t)\rangle. \tag{2.6}$$

The cross-correlation between the signals from two

---

[8] J. H. Sanders, Nature **183**, 312 (1959).

[9] A. T. Forrester, J. Opt. Soc. Am. **51**, 253 (1961).

[10] L. Mandel, *Progress in Optics* [North-Holland Publishing Company, Amsterdam], Vol. II (to be published).

[11] E. Wolf, Proc. Phys. Soc. (London) **76**, 424 (1960).

[12] L. Mandel, J. Opt. Soc. Am. **51**, 797 (1961).

[13] L. Mandel, J. Opt. Soc. Am. (to be published).

[14] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, London and New York, 1959).

[15] For stationary fields, with which we shall mainly be concerned, the integral defining the conjugate field may diverge. To avoid this difficulty one may first assume that the field exists only for a finite time interval $-T \leqslant t \leqslant T$ and one can proceed to the limit $T \to \infty$ at the end of the calculations. The final results are formally identical with those obtained for fields for which the integral converges.

If the field is quasi-monochromatic, then

$$\mathbf{E}^{(i)}(\mathbf{r},t)\sim\mathbf{E}^{(r)}(\mathbf{r},\, t+1/4\bar{\nu}),$$

$\bar{\nu}$ being the mid-frequency of the wave. This relation follows immediately from Eq. (14), p. 494 of reference 14.

[16] Using elementary properties of the analytic signals, the intensity so defined may be shown to be proportional to the local value of the electric energy density, averaged over a time interval of duration equal to a few mean periods of the wave.

detectors placed at points P_1 and P_2 is given by

$$\begin{aligned}\langle S(\mathbf{r}_1, t+\tau)S(\mathbf{r}_2,t)\rangle &= \frac{\alpha^2}{T^2}\int_t^{t+T}\int_t^{t+T}\langle I(\mathbf{r}_1, t'+\tau)I(\mathbf{r}_2,t'')\rangle dt'dt'' \\ &= \frac{\alpha^2}{T^2}\sum_{i,j}\int_0^T\int_0^T R_{ij}(\mathbf{r}_1, \mathbf{r}_2, t'-t''+\tau)dt'dt'' \\ &= \frac{\alpha^2}{T^2}\sum_{i,j}\int_{-T}^{T}(T-|\tau'|)R_{ij}(\mathbf{r}_1, \mathbf{r}_2, \tau+\tau')d\tau', \end{aligned} \tag{2.7}$$

where

$$R_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle I_i(\mathbf{r}_1, t+\tau)I_j(\mathbf{r}_2,t)\rangle, \quad (i, j=x, y), \tag{2.8}$$

denotes the cross-correlation function between the two intensity contributions. In the first line of (2.7) the order of the two averaging processes was interchanged; in going from the first to the second line, stationarity was assumed, and in going from the second to the third line the double integral was reduced to a single integral by a suitable change of variables (cf. Rice[17]). The parameter τ represents the time delay which may be introduced in one of the channels connecting the two detectors to a correlator.

So far no assumptions has been made about the probability distributions which govern the fluctuations. It will now be assumed that the joint probability distribution of $E_i(\mathbf{r}_1, t+t')$ and of $E_j(\mathbf{r}_2, t+t'')$ is *Gaussian*. It would be difficult to give a rigorous justification of this assumption; however, if the field arises from the superposition of a large number of independently generated elementary fields as is usually the case, one may readily give a plausability argument by an appeal to the central limit theorem of probability theory (cf. Blanc-Lapierre and Dumontet,[18] p. 18). With this assumption and with the assumption that the means of E_i and E_j are zero, the correlation functions R_{ij} may be expressed in the following form, derived in the Appendix:

$$R_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \mathcal{E}_{ii}(\mathbf{r}_1,\mathbf{r}_1,0)\mathcal{E}_{jj}(\mathbf{r}_2,\mathbf{r}_2,0)+|\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)|^2, \tag{2.9}$$

where $\mathcal{E}_{ij}$ are the elements of the electric correlation matrix

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle E_i(\mathbf{r}_1, t+\tau)E_j^*(\mathbf{r}_2,t)\rangle, \tag{2.10}$$

which was introduced (for the more general 3-dimensional case) by Wolf[19] and studied in detail by Roman and Wolf.[20] Note that in terms of $\mathcal{E}$, the average intensity of each component is

$$I_i(\mathbf{r})\equiv\langle I_i(\mathbf{r},t)\rangle=\mathcal{E}_{ii}(\mathbf{r},\mathbf{r},0), \quad (i=x, y). \tag{2.11}$$

[17] S. O. Rice, Bell System Tech. J. **24**, 46 (1945).
[18] A. Blanc-Lapierre and P. Dumontet, Rev. Opt. **34**, 1 (1955).
[19] E. Wolf, Nuovo cimento **12**, 884 (1954).
[20] P. Roman and E. Wolf, (a) Nuovo cimento **17**, 462 (1960); (b) **17**, 477 (1960); P. Roman, Nuovo cimento **20**, 759 (1961).

Using (2.9) it readily follows from (2.7), (2.6), and (2.11) that the correlation

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle = \langle[S(\mathbf{r}_1, t+\tau)-m_1][S(\mathbf{r}_2,t)-m_2]\rangle,$$

where $m_1=\langle S(r_1,t)\rangle$, $m_2=\langle S(r_2,t)\rangle$, is given by

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle = \frac{\alpha^2}{T^2}\sum_{i,j}\int_{-T}^{T}(T-|\tau'|)|\mathcal{E}_{ij}(\mathbf{r}_1, \mathbf{r}_2, \tau+\tau')|^2 d\tau'. \tag{2.12}$$

The formula (2.12) is a general expression for the correlation in the fluctuations in the outputs from two square-law detectors, each of "resolving time" T, illuminated by a plane light wave characterized by the coherency matrix $\mathcal{E}$.

The right-hand side of (2.12) must evidently be invariant with respect to a rotation of axes about the z-direction (direction of propagation of the wave). It may readily be verified that this is the case. To show this consider the matrix $\mathcal{E}\mathcal{E}^\dagger$, where $\mathcal{E}^\dagger$ is the Hermitian conjugate of $\mathcal{E}$. The (ij) element of this matrix is given by

$$[\mathcal{E}(\mathbf{r}_1,\mathbf{r}_2,\tau)\mathcal{E}^\dagger(\mathbf{r}_1,\mathbf{r}_2,\tau)]_{ij} = \sum_k \mathcal{E}_{ik}(\mathbf{r}_1,\mathbf{r}_2,\tau)\mathcal{E}_{jk}^*(\mathbf{r}_1,\mathbf{r}_2,\tau), \tag{2.13}$$

and evidently $\mathcal{E}\mathcal{E}^\dagger$ is a Hermitian matrix. (In the special case when $\mathbf{r}_1=\mathbf{r}_2$ and $\tau=0$, discussed by Wolf,[21] $\mathcal{E}$ itself is Hermitian.) The trace of this matrix is

$$\mathrm{Tr}(\mathcal{E}\mathcal{E}^\dagger)=\sum_{i,j}\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)\mathcal{E}_{ij}^*(\mathbf{r}_1,\mathbf{r}_2,\tau), \tag{2.14}$$

and this is precisely the quantity appearing on the righthand side of (2.12). Since the trace is invariant with respect to the rotation of axes, the formula (2.12) is independent of the particular choice of axes, as expected.

It is of interest to consider two limiting cases of (2.12). Suppose first that the "resolving time" T is short compared to the coherence time (which is of the order of $1/\Delta\nu$, where $\Delta\nu$ is the effective spectral range of the light). This case may prove of interest when light from an optical maser is used.[21a] Under these circumstances $|\mathcal{E}_{ij}(\mathbf{r}_1, \mathbf{r}_2, \tau+\tau')|^2$ is effectively equal to $|\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)|^2$ throughout the interval of integration and (2.12) reduces to

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \sim \alpha^2\sum_{ij}|\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)|^2. \tag{2.15}$$

Note that this correlation depends on the time shift τ but is independent of T.

[21] E. Wolf, Nuovo cimento **13**, 1165 (1959).
[21a] The output of an optical maser is not necessarily a stationary random process, but it seems likely that it is stationary for time intervals long compared with the coherence time. During such an interval the maser can be regarded as a feedback amplifier, and if the input light is describable by a stationary random process, so is the output.

Next consider the other (more common) extreme case when the "resolving time" T is large compared to both the coherence time and the delay time τ. In this case one has from (2.12),

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \sim \frac{\alpha^2}{T}\sum_{i,j}\int_{-\infty}^{\infty} |\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau')|^2 d\tau'. \quad (2.16)$$

The correlation is now independent of τ but varies inversely with the resolving time T of the detector.

3. REDUCTION FORMULAS FOR SPECTRALLY PURE BEAMS

In deriving the main formula (2.12), no restrictive assumptions have been made about the light beyond those implied in the assumption that the underlying random process is stationary and Gaussian to second order. In some cases of practical interest, including those studied by Hanbury Brown and Twiss[1(c)] certain simplifying assumptions may be made about the two beams that are incident on the detectors. These assumptions have been expressed by a number of authors, in a rather formal way, equivalent to the assertion that the cross-correlation function $\mathcal{E}_{ii}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ is expressible as the product of two functions, one depending on $\mathbf{r}_1$ and $\mathbf{r}_2$ alone, the other depending on τ alone. The physical significance of this requirement has recently been studied in detail by Mandel[13] who introduced the term *spectrally pure beams* for light beams of this type and he called the product formula just referred to the *reduction formula.* It was shown in reference 13 that if the light vibrations at $\mathbf{r}_1$ and $\mathbf{r}_2$ have the same normalized spectral distribution, and if, moreover, the reduction formula holds, then the light obtained by superposing these vibrations will have an identical normalized spectral distribution.

Mandel's work referred to light that is adequately described by a single scalar function (e.g., unpolarized or linearly polarized light). In the present section reduction formulas relating to light that must be represented by vector wave functions (e.g., partially polarized light) will be briefly considered and will be used in Sec. 4 to simplify, in appropriate cases, the expression (2.12) for the correlation in the fluctuations of the detector signals.

Consider first the propagation of a quasi-monochromatic beam of light from a plane extended source Σ (not necessarily an incoherent source) *in vacuo.*[22] Let the xy-axes be chosen in the plane of the source. If $\mathcal{E}_{ij}(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,\tau)$, $(i, j=x, y)$, is a typical element of the electric correlation matrix which represents the correlation at points $Q_1(\boldsymbol{\varrho}_1)$ and $Q_2(\boldsymbol{\varrho}_2)$ of the source, the elements of the correlation matrix at any two points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ in the field (not to close to the source) may be expressed by means of the following propagation law strictly analogous to that derived by Wolf[23] (see also Born and Wolf,[14] p. 532) for the scalar case:

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \iint_{\Sigma} \frac{\mathcal{E}_{ij}(\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, \tau-(R_1-R_2)/c)}{R_1R_2}\Lambda_1\Lambda_2{}^* d\boldsymbol{\varrho}_1 d\boldsymbol{\varrho}_2, \quad (3.1)$$

where $R_1=|\mathbf{r}_1-\boldsymbol{\varrho}_1|$, $R_2=|\mathbf{r}_2-\boldsymbol{\varrho}_2|$ are the distances Q_1P_1 and Q_2P_2, respectively, and Λ_1, Λ_2 are inclination factors.

We will now make a number of assumptions about the source and the geometry. First assume the source to be *incoherent.* Then $\mathcal{E}_{ij}(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_2,\tau)$ will be zero unless the point Q_2 is very close to $Q_1(\boldsymbol{\varrho}_2\sim\boldsymbol{\varrho}_1)$ and it follows on integrating on the right-hand side of (3.1) that

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \sigma\int_{\Sigma} \frac{\mathcal{E}_{ij}(\boldsymbol{\varrho}, \boldsymbol{\varrho}, \tau-(R_1-R_2)/c)}{R_1R_2}\Lambda_1\Lambda_2{}^* d\boldsymbol{\varrho}, \quad (3.2)$$

where σ is the very small effective area of coherence associated with each source point.

Next let us suppose that the difference in the *distances of the points P_1 and P_2 from each source point are small compared with the coherence length*[24] of the light, i.e., that $|R_1-R_2|\ll c/\Delta\nu$, where $\Delta\nu$ is the effective frequency range of the light. Under these circumstances, we have to a good approximation [cf. Born and Wolf[14] p. 505, Eq. (10b)].

$$\mathcal{E}_{ij}(\boldsymbol{\varrho}, \boldsymbol{\varrho}, \tau-(R_1-R_2)/c) = \mathcal{E}_{ij}(\boldsymbol{\varrho},\boldsymbol{\varrho},\tau)\exp[2\pi i\bar{\nu}(R_1-R_2)/c]. \quad (3.3)$$

Next let us assume that the source is *homogeneous as regards its normalized spectral density distribution and polarization.* This requirement is conveniently expressed in terms of the normalized correlation matrix

$$\gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau) = \frac{\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)}{[\mathcal{E}_{ii}(\mathbf{r}_1,\mathbf{r}_1,0)]^{\frac{1}{2}}[\mathcal{E}_{jj}(\mathbf{r}_2,\mathbf{r}_2,0)]^{\frac{1}{2}}} = \frac{\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)}{[I_i(\mathbf{r}_1)]^{\frac{1}{2}}[I_j(\mathbf{r}_2)]^{\frac{1}{2}}}, \quad (3.4)$$

whose Fourier τ-transform is the normalized (cross) spectral density distribution. The assumptions of homogeneity just made are equivalent to assuming that $\gamma_{ij}(\boldsymbol{\varrho},\boldsymbol{\varrho},\tau)$ is independent of $\boldsymbol{\varrho}$, so that it may be set equal to $\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)$ where $\boldsymbol{\varrho}_0$ refers to some fixed reference point on the source. Under these circumstances it

[22] Strictly speaking, since plane waves only are being considered, in the present investigation, the source Σ should be at infinity. The concept of spectral purity applies, however, to more general situations and it is illustrative to derive the appropriate reduction formulas without limiting the derivation to the special case of plane waves.

[23] E. Wolf, Proc. Roy. Soc. (London) **A230**, 246 (1955).

[24] Strictly there are in general several coherence lengths since with each element $\mathcal{E}_{ij}$ of the correlation matrix an effective spectral range $\Delta\nu_{ij}$ may be associated and these $\Delta\nu_{ij}$ may be all different. We shall consider the usual case when the coherence lengths are effectively the same.

follows from (3.2), (3.3), and (3.4) that

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)\mathfrak{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2), \tag{3.5}$$

where $\mathfrak{M}_{ij}$ is the integral

$$\mathfrak{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2)=\sigma\int_\Sigma \frac{[I_i(\boldsymbol{\varrho})]^{\frac{1}{2}}[I_j(\boldsymbol{\varrho})]^{\frac{1}{2}}}{R_1R_2} \times\exp[2\pi i\bar{\nu}(R_1-R_2)/c]\Lambda_1\Lambda_2{}^*d\boldsymbol{\varrho}. \tag{3.6}$$

The formula (3.5) expresses $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ as product of two functions, one depending on τ, the other on $\mathbf{r}_1$ and $\mathbf{r}_2$ and is an example of a "reduction formula."

A further simplification is obtained if we assume that the spectral distribution of the vibrations is the same in all directions, i.e., if $\gamma_{xx}(\tau)$ is *invariant under rotation of axes* around the direction of propagation of the wave. Most quasi-monochromatic partially polarized light beams, produced from incoherent sources by the introduction of thin polarizers will very nearly obey this condition.

Now a rotational transformation of $E_x(\mathbf{r},t)$ and $E_y(\mathbf{r},t)$ to a new coordinate frame making an angle θ with the original one is expressed by

$$\left.\begin{aligned}E_x'(\mathbf{r},t)&=E_x(\mathbf{r},t)\cos\theta+E_y(\mathbf{r},t)\sin\theta,\\ E_y'(\mathbf{r},t)&=-E_x(\mathbf{r},t)\sin\theta+E_y(\mathbf{r},t)\cos\theta.\end{aligned}\right\} \tag{3.7}$$

It is seen that $E_x'(\mathbf{r},t)$ and $E_y'(\mathbf{r},t)$ are obtained from $E_x(\mathbf{r},t)$ and $E_y(\mathbf{r},t)$ by a superposition with zero time shift. It follows from the analysis of Mandel[13] that such superposition does not, in general, reproduce the original spectral distribution unless $E_x(\mathbf{r},t)$ and $E_y(\mathbf{r},t)$ are *spectrally pure "in the strong sense,"* expressed by the reduction formula

$$\gamma_{ij}(\mathbf{r},\mathbf{r},\tau)=\gamma_{ij}(\mathbf{r},\mathbf{r},0)\gamma_{ii}(\mathbf{r},\mathbf{r},\tau). \tag{3.8}$$

Let us assume that this condition is satisfied. If we now substitute from (3.8) into (3.5) where $\mathbf{r}$ refers to the source point $\boldsymbol{\varrho}_0$ and divide by the corresponding equation with $\tau=0$, we obtain the formula:

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\gamma_{ii}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,0). \tag{3.9}$$

This formula is analogous to one found by Mandel[13] for spectrally pure unpolarized light.

4. CORRELATION ARISING FROM SPECTRALLY PURE LIGHT BEAMS

4.1 Beams Obeying the Reduction Formula (3.5)

Suppose that two beams of light which obey the reduction formula (3.5) illuminate two square law detectors situated at P_1 and P_2. It immediately follows from (2.12) and (3.5) that the correlation in the fluctuations of the outputs from the detectors is given by

$$\langle\Delta S(\mathbf{r}_1,t+\tau)\Delta S(\mathbf{r}_2,t)\rangle =\frac{\alpha^2}{T}\sum_{i,j}|\mathfrak{M}_{ij}(\mathbf{r}_1,\mathbf{r}_2)|^2\xi_{ij}(T,\tau), \tag{4.1}$$

where

$$\xi_{ij}(T,\tau)=\frac{1}{T}\int_{-T}^{T}(T-|\tau'|) \times|\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau+\tau')|^2d\tau'. \tag{4.2}$$

The quantity $\xi_{ij}(T,\tau)$ is a generalization of the parameter ξ which was found by Mandel[6,7] to play a central role in the analysis of related problems. By similar arguments as given in connection with (2.15) and (2.16) it follows that

if T is small compared with $1/\Delta\nu$, then

$$\xi_{ij}(T,\tau)\sim T|\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)|^2; \tag{4.3}$$

if T is large compared with $1/\Delta\nu$ *and* τ, then

$$\xi_{ij}(T,\tau)\sim\int_{-\infty}^{\infty}|\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau')|^2d\tau'. \tag{4.4}$$

In the latter case, [given by (4.4)], $\xi_{xx}(\infty,0)$ and $\xi_{yy}(\infty,0)$ may be interpreted as the coherence times of the x and y vibrations, respectively (cf. reference 7, p. 238 and reference 10, Sec. 3.2).

Suppose that in addition to obeying the spectral purity condition expressed by the reduction formula (3.5), the two beams are *linearly polarized*, with their electric vectors in the x direction. Then all the elements of the $\mathcal{E}$ matrix except $\mathcal{E}_{xx}$ vanish and (4.1) reduces to

$$\langle\Delta S(\mathbf{r}_1,t+\tau)\Delta S(\mathbf{r}_2,t)\rangle=\frac{\alpha^2}{T}|\mathfrak{M}_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2\xi_{xx}(T,\tau). \tag{4.5}$$

Also from (2.6), (2.11), and (3.5), one has

$$\langle S(\mathbf{r},t)\rangle=\alpha\mathfrak{M}_{xx}(\mathbf{r},\mathbf{r}), \tag{4.6}$$

since $\gamma_{xx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)=1$. Hence (4.5) may be expressed in the form

$$\langle\Delta S(\mathbf{r}_1,t+\tau)\Delta S(\mathbf{r}_2,t)\rangle =\frac{\xi_{xx}(T,\tau)}{T}\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle|\mu_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2, \tag{4.7}$$

where

$$\mu_{xx}(\mathbf{r}_1,\mathbf{r}_2)=\frac{\mathfrak{M}_{xx}(\mathbf{r}_1,\mathbf{r}_2)}{[\mathfrak{M}_{xx}(\mathbf{r}_1,\mathbf{r}_1)]^{\frac{1}{2}}[\mathfrak{M}_{xx}(\mathbf{r}_2,\mathbf{r}_2)]^{\frac{1}{2}}} \tag{4.8}$$

is the normalized spatial coherence factor which may readily be calculated from the formula (3.6). If $\tau=0$, (4.7) is the classical analog of a formula derived by Mandel, Sec. 5.[6]

Two limiting cases of (4.8) are of special interest: When $\tau=0$ and T is large compared to the coherence time $1/\Delta\nu$ (which are the conditions usually encountered in practice), one has from (4.7) using (4.4),

$$\langle\Delta S(\mathbf{r}_1,t)\Delta S(\mathbf{r}_2,t)\rangle =[\xi(\infty,0)/T]\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle|\mu_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2, \tag{4.9}$$

where $\xi(\infty,0)=\int_{-\infty}^{\infty}|\gamma_{xx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)|^2d\tau$ is the coherence

time of the light. The formula (4.9) is the basis formula in the theory of the Hanbury Brown-Twiss experiment (cf. Mandel[6]).

Another limiting case, which is of particular interest in connection with optical maser sources, is represented by the other extreme condition: namely T small compared with $1/\Delta\nu$. From the experiment of Javan, Bennett, and Herriott[25] it appears that coherence time of the order 10^{-4} or 10^{-5} sec can now be achieved, while, as already mentioned, T may be as short as 10^{-9} sec. Under such conditions ($T\ll 1/\Delta\nu$), (4.7) with the help of (4.3) reduces to

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle |\gamma_{xx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)|^2|\mu_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2. \quad (4.10)$$

It is to be noted that this correlation depends on τ only through the normalized autocorrelation function γ_{xx} of the light. Since γ_{xx} is the Fourier transform of the spectral density of the light, (4.10) implies that one may obtain information about the spectral distribution of the light from measurements of the correlation $\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle$ over a range of τ values.

Let us now turn attention to *unpolarized* (*natural*) *light.* For such light one may assume

$$\begin{aligned} \mathcal{E}_{xx}(\mathbf{r},\mathbf{r},\tau)&=\mathcal{E}_{yy}(\mathbf{r},\mathbf{r},\tau),\\ \mathcal{E}_{xy}(\mathbf{r},\mathbf{r},\tau)&=\mathcal{E}_{yx}(\mathbf{r},\mathbf{r},\tau)=0. \end{aligned} \quad (4.11)$$

(The less general case with $\tau=0$ is considered in Wolf[21] (p. 1176) and Born and Wolf,[14] pp. 545–546). Then (3.4) implies that $\gamma_{xx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)=\gamma_{yy}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)$, $\gamma_{xy}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)=\gamma_{yx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau)=0$ so that (4.2) then gives $\xi_{xx}(T,\tau)=\xi_{yy}(T,\tau)$, $\xi_{xy}(T,\tau)=\xi_{yx}(T,\tau)=0$. Moreover, (3.6) and (2.11) now give $\mathfrak{M}_{xx}(\mathbf{r}_1,\mathbf{r}_2)=\mathfrak{M}_{yy}(\mathbf{r}_1,\mathbf{r}_2)$ and the formula (4.1) reduces to

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =(2\alpha^2/T)|\mathfrak{M}_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2\xi_{xx}(T,\tau). \quad (4.12)$$

Also from (2.6), (2.11), and (3.5), since $\gamma_{xx}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)=1$,

$$\langle S(\mathbf{r},t)\rangle=2\alpha\mathfrak{M}_{xx}(\mathbf{r},\mathbf{r}), \quad (4.13)$$

so that (4.12) may be expressed in the form

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =\tfrac{1}{2}[\xi_{xx}(T,\tau)/T]\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle|\mu_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2. \quad (4.14)$$

Comparison with (4.7) shows that the correlation is now one half of that obtained with linearly polarized light beams. Again one may consider the limiting cases of large or small resolving time. One then finds, of course, that in both cases the resulting expressions for the correlation each differ by a factor $\frac{1}{2}$ from the expressions appropriate to linearly polarized light.

[25] A. Javan, W. R. Bennett, Jr., and D. R. Herriott, Phys. Rev. Letters **6**, 106 (1961).

4.2 Beams Obeying the Reduction Formula (3.9)

If the light incident on the detectors obeys the reduction formula (3.9), the general expression (2.12) for the correlation takes the form

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =\frac{\alpha^2}{T}\sum_{i,j}|\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,0)|^2\xi_{ii}(T,\tau), \quad (4.15)$$

where $\xi_{ii}(T,t)$ is again given by (4.2).

Suppose further that the light is completely coherent, but of any state of polarization. According to a result recently derived by Mandel and Wolf,[26] the assumption of complete coherence implies that

$$\gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\gamma_{ij}(\boldsymbol{\varrho}_0,\ \boldsymbol{\varrho}_0+\tau_{01}-\tau_{02}), \quad (4.16)$$

where τ_{01} and τ_{02} are certain time delays depending on the positions of the points $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$ relative to a fixed reference point $Q_0(\rho_0)$, and in general, of course, also on i and j. If one only considers points such that the time delay $\tau_{01}-\tau_{02}$ is small compared to the coherence time [cf. (3.3)] then one may neglect this delay on the righthand side of (4.16) and (3.4) gives

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=[I_i(\mathbf{r}_1)]^{\frac{1}{2}}[I_j(\mathbf{r}_2)]^{\frac{1}{2}}\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,\tau). \quad (4.17)$$

Now the directions along which the x and y axes were chosen have so far been arbitrary (apart from the assumption that they lie in a plane at right angles to the direction of propagation of the wave). Let us now choose them so that $I_x(\boldsymbol{\varrho})=I_y(\boldsymbol{\varrho})$. It has been shown by Wolf[21] that this can be always done. If we choose the axes in this way and denote them by $O\bar{X}$, $O\bar{Y}$, then it follows from (3.6) that $\mathfrak{M}_{\bar{x}\bar{x}}(\mathbf{r}_1,\mathbf{r}_2)=\mathfrak{M}_{\bar{y}\bar{y}}(\mathbf{r}_1,\mathbf{r}_2)$ and hence we have from (3.5) and (3.8) and (2.11),

$$\begin{aligned} I_{\bar{x}}(\mathbf{r}_1)&=I_{\bar{y}}(\mathbf{r}_1),\\ I_{\bar{x}}(\mathbf{r}_2)&=I_{\bar{y}}(\mathbf{r}_2). \end{aligned} \quad (4.18)$$

On substituting from (4.17) into (4.15) and using (4.18), we obtain the following expression for the correlation:

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =\frac{\alpha^2}{T}\sum_{i,j}I_i(\mathbf{r}_1)I_i(\mathbf{r}_2)|\gamma_{ij}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)|^2\xi_{ii}(T,\tau). \quad (4.19)$$

If we recall the relation $\xi_{\bar{x}\bar{x}}(T,\tau)=\xi_{\bar{y}\bar{y}}(T,\tau)$ which follows from (4.2) and (3.8), (4.19) reduces to

$$\langle \Delta S(\mathbf{r}_1, t+\tau)\Delta S(\mathbf{r}_2,t)\rangle \\ =[\xi_{\bar{x}\bar{x}}(T,\tau)/2T]\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle \\ \times[1+|\gamma_{\bar{x}\bar{y}}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)|^2]. \quad (4.20)$$

Consider now the special case when $\tau=0$ and the resolving time T of the detectors is large compared with the coherence time $1/\Delta\nu$. Then $\xi_{\bar{x}\bar{x}}(T,\tau)\approx\xi_{\bar{x}\bar{x}}(\infty,0)$ and

[26] I. Mandel and E. Wolf, J. Opt. Soc. Am. **51**, 815 (1961). Actually in this paper a coherent scalar field is considered, but the extension of the result to a coherent vector field is trivial.

(4.20) reduces to

$$\langle \Delta S(\mathbf{r}_1,t)\Delta S(\mathbf{r}_2,t)\rangle = [\xi_{\bar{x}\bar{x}}(\infty,0)/2T]\langle S(\mathbf{r}_1,t)\rangle\langle S(\mathbf{r}_2,t)\rangle \times[1+|\gamma_{\bar{x}\bar{y}}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)|^2]. \quad (4.21)$$

Now as mentioned earlier, $\xi_{\bar{x}\bar{x}}(\infty,0)$ may be identified with the coherence time of the light. Moreover, it has been shown by Wolf[21] that $|\gamma_{\bar{x}\bar{y}}(\boldsymbol{\varrho}_0,\boldsymbol{\varrho}_0,0)|$ is precisely the *degree of polarization* of the light as conventionally defined and (4.21) is then the classical analog of formula (8) derived by Wolf[11] for this special case.[27]

APPENDIX

Deviation of the Formula (2.9)

Let

$$W_i(t)=U_i(t)+iV_i(t), \quad (i=1, 2), \quad (A1)$$

be two analytic signals, i.e., functions such that their real and imaginary parts U_i, V_i form a conjugate pair. It will be assumed that the U's and V's are Gaussian random variables each with zero mean.

Consider the correlation function

$$\langle W_1(t+\tau)W_1^*(t+\tau)W_2(t)W_2^*(t)\rangle.$$

Then from (A1)

$$\langle W_1(t+\tau)W_1^*(t+\tau)W_2(t)W_2^*(t)\rangle = \langle U_1^2(t+\tau)U_2^2(t)\rangle + \langle V_1^2(t+\tau)V_2^2(t)\rangle + \langle U_1^2(t+\tau)V_2^2(t)\rangle + \langle V_1^2(t+\tau)U_2^2(t)\rangle. \quad (A2)$$

Now if F_1, F_2, F_3 and F_4 are real random variables with a Gaussian joint[3] probability density function and each with a zero mean, then the following identity is known to hold[28]:

$$\langle F_1F_2F_3F_4\rangle = \langle F_1F_2\rangle\langle F_3F_4\rangle + \langle F_1F_3\rangle\langle F_2F_4\rangle + \langle F_1F_4\rangle\langle F_2F_3\rangle. \quad (A3)$$

In particular, it follows from (A3) with $F_1=F_2=U_1(t+\tau)$, $F_3=F_4=U_2(t)$, that

$$\langle U_1^2(t+\tau)U_2^2(t)\rangle = \langle U_1^2(t)\rangle\langle U_2^2(t)\rangle + 2\langle U_1(t+\tau)U_2(t)\rangle^2, \quad (A4)$$

and there are similar expressions for the correlations $\langle V_1^2(t+\tau)U_2^2(t)\rangle$ etc. Now since U and V are conjugate functions, the following relations also hold [reference 21a, Eq. (A6)]:

$$\left.\begin{aligned}\langle U_1(t+\tau)U_2(t)\rangle &= \langle V_1(t+\tau)V_2(t)\rangle,\\ \langle U_1(t+\tau)V_2(t)\rangle &= -\langle V_1(t+\tau)U_2(t)\rangle.\end{aligned}\right\} \quad (A5)$$

It follows on substitution from (A4) and three similar formulas into (A2) and on using (A5) that

$$\langle W_1(t+\tau)W_1^*(t+\tau)W_2(t)W_2^*(t)\rangle = \langle U_1^2(t)+V_1^2(t)\rangle\langle U_2^2(t)+V_2^2(t)\rangle + 4\{\langle U_1(t+\tau)U_2(t)\rangle^2 + \langle U_1(t+\tau)V_2(t)\rangle^2\}. \quad (A6)$$

Now the first term on the right-hand side of (A6) is evidently equal to $\langle W_1(t)W_1^*(t)\rangle\langle W_2(t)W_2^*(t)\rangle$. To examine the significance of the second term consider the correlation $\langle W_1(t+\tau)W_2^*(t)\rangle$. One has from (A1) and (A5)

$$\langle W_1(t+\tau)W_2^*(t)\rangle = 2[\langle U_1(t+\tau)U_2(t)\rangle - i\langle U_1(t+\tau)V_2(t)\rangle].$$

Hence the second term on the right of (A6) is equal to $\langle W_1(t+\tau)W_2^*(t)\rangle\langle W_1^*(t+\tau)W_2(t)\rangle$ and it follows that

$$\langle W_1(t+\tau)W_1^*(t+\tau)W_2(t)W_2^*(t)\rangle = \langle W_1(t)W_1^*(t)\rangle\langle W_2(t)W_2^*(t)\rangle + \langle W_1(t+\tau)W_2^*(t)\rangle\langle W_1^*(t+\tau)W_2(t)\rangle. \quad (A7)$$

Finally, if $W_1(t)$ is identified with $E_i(\mathbf{r}_1,t)$ and $W_2(t)$ with $E_j(\mathbf{r}_2,t)$, then

$$\left.\begin{aligned}\langle W_1(t+\tau)W_1^*(t+\tau)W_2(t)W_2^*(t)\rangle &= R_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau),\\ \langle W_i(t+\tau)W_j^*(t)\rangle &= \mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau),\end{aligned}\right\} \quad (A8)$$

and (A7) reduces to Eq. (2.9) quoted in the text.

[27] In reference 11 the assumption of spectral purity was not explicitly made, but it is implicit in the derivation of formula (8) of that paper.

[28] Alternatively, formulae of the form (A4) may, of course, be derived by a direct calculation [cf. S. O. Rice, Bell Tech. J., **24**, 89 (1945), Eq. (3.9-7) or E. Wolf, Phil. Mag., **2**, 351 (1957), Eq. (1.10)].