Theory of the Electro- and Photoproduction of π Mesons

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The scattering amplitude for the electroproduction of pions from nucleons is derived using the Mandelstam representation. It is shown that the amplitude consists mainly of two parts which in the framework of the Cini-Fubini approximation are simply additive. One part describes the effects of a $\pi-\pi$ resonance, while the other part describes the $\frac{3}{2}-\frac{3}{2}$ resonant nucleon term. The theory is fully relativistic in the sense that no expansion in inverse powers of the nucleon mass is made. This means that even though the theory is applicable only in the low-energy region where the $\frac{3}{2}-\frac{3}{2}$ resonance dominates, the momentum transfer variable upon which the various form factors depend, may be very large. The meson-current Born term contains explicitly the form factor of the pion and can be isolated at high momentum transfers to the electrons, where the resonant nucleon term becomes small.

INTRODUCTION

THE method of dispersion relations has been applied by Fubini, Nambu, and Wataghin[1] to the electroproduction of pions in order to show how this process could reveal information on the form factors of the nucleons. More recently, it has been shown by Frazer[2] that it is also possible to obtain information about the pion form factor by carrying out a Chew-Low extrapolation on the electroproduction amplitude to a point in the unphysical region of the momentum transfer to the nucleon, corresponding to the single meson pole. In this paper we will show by the use of the Mandelstam representation that the electroproduction process is also particularly well suited to test further the nature of the pion-pion interaction, and that it allows for a direct determination of the form factor of the pion.

Starting from the Mandelstam representation means that we are taking into account not only the interaction between the outgoing pion and the nucleon core, but also the interaction between the outgoing pion and the pion cloud of the nucleon. We will assume that this latter interaction is mainly due to intermediate states of the nucleon-antinucleon channel containing 2 and 3 pions. It has been shown by Frazer and Fulco[3] that the existence of a $\pi-\pi$ interaction in the $J=I=1$ state, which has recently received some experimental support,[4] could explain rather well the main features of the isovector electromagnetic structure of the nucleons. It was also shown by Bowcock, Cottingham, and Lurié[5] that such an interaction could throw some light on our understanding of the small π-nucleon phase shifts. This interaction may also be necessary in order to explain the discrepancy between theoretical and experimental predictions in the photoproduction of pions.

Within the framework of the Cini-Fubini approximation,[6] the pion-core and pion-cloud interactions contribute additively to the S matrix. The pion-core interaction manifests itself in the S matrix by its singularities in the energy variable, whereas the pion-cloud interaction shows singularities in the variable corresponding to the momentum transfer to the nucleons.

The contribution to the S matrix from the pion-cloud interaction depends upon a function of the momentum transfer to the electrons, to be determined by experiments, and upon parameters that are directly obtainable from the experimental data on the form factors of the nucleons. The Born term due to the meson-current depends explicitly upon the electromagnetic form factor of the pion. It is important to note that, to the extent that the analytic properties of the scattering amplitude with respect to the momentum transfer to the nucleons are taken into account by the 2- and 3-pion exchanges, the meson-current Born term will depend upon a function of one variable only, and not of two as in Frazer. This function is identical to the form factor of the pion. Therefore, by using the Mandelstam representation, one circumvents the complicated extrapolation procedure that is needed to obtain the form factor of the pion, when a one-dimensional dispersion relation is used.

The contribution from the pion-core interaction is related, through the final-state theorem, to the scattering in the final π-nucleon state. We will consider total barycentric energies of the π-nucleon system that are in a region where only the $\frac{3}{2}-\frac{3}{2}$ resonance dominates. Experimentally, this condition can always be met by choosing an appropriate incident electron energy. On the other hand, the experiments of Hofstadter *et al.*,[7] on the form factors of the nucleons are carried out at a high value of the square of the momentum transfer from the electrons λ^2. The most recent experiments achieve a momentum transfer squared of the order of $40\mu^2$. The

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[2] W. R. Frazer, Phys. Rev. **115**, 1763 (1959).

[3] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

[4] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961); also D. Stonehill *et al.*, Phys. Rev. Letters **6**, 624 (1961).

[5] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960).

[6] M. Cini and S. Fubini, Ann. Phys. **3**, 352 (1960).

[7] R. Hofstadter, F. Bumiller, and M. Croissiaux, Phys. Rev. Letters, **5**, 263 (1960); S. Bergia, A. Stanghellini, S. Fubini, and C. Villi, Phys. Rev. Letters **6**, 367 (1961).

kinematics of the electroproduction process show that the combination of low total barycentric energy together with high momentum transfer from the electron favors higher values of the momentum transfer to the nucleons. It is therefore clear that the static theory of FNW, in which only the first term in an expansion of the amplitude in inverse powers of the nucleon mass is kept, cannot be expected to be very reliable when $|\lambda|$ is of the order of a nucleon mass. For this reason, we will give a complete relativistic treatment of the nucleon term. The separation of the terms into those generated by magnetic dipole, electric quadrupole, and longitudinal quadrupole radiation will be made in the hope that only some of these terms may be important. In the static limit, the magnetic dipole term dominates, but in the nonstatic case we have of course no way to judge beforehand the relative size of these terms.

In Secs. III and IV it will be shown that the magnetic dipole approximation improves as one attains higher values of λ^2. In Sec. I, after some kinematical preliminaries, we derive a representation for relativistic and gauge-invariant amplitudes. In Sec. II we consider the contribution to the isoscalar amplitude from the pion-pion interaction. We conclude with a discussion of our results.

I. KINEMATICS AND THE MANDELSTAM REPRESENTATION

The scattering amplitude T is related to the S matrix by the definition

$$S_{fi}=\delta_{fi}-i\delta^4(p_1+r_1-p_2-q_2-r_2)\left(\frac{m^2M^2}{\omega_2E_1E_2\epsilon_1\epsilon_2}\right)^{\frac{1}{2}}T_{fi},$$

where ϵ_1, E_1 are the energies of the incident electron and nucleon (of 4 momenta P_1 and r_1, and of masses m and M) and ϵ_2, E_2, ω_2 are the final energies of the electron, nucleon, and meson (of momenta r_2, p_2, q).

As shown by Dalitz and Yennie,[8] the electroproduction T-matrix element, to first order in the electromagnetic coupling constant, is given by

$$\langle p_2,q_2,r|T|p_1r_1\rangle=[eg/(2\pi)^{7/2}]\langle p_2,q|J_\mu|p_1\rangle\epsilon_\mu,$$

with

$$\epsilon_\mu=e\bar{u}(r_2)\gamma_\mu u(r_1)/(r_1-r_2)^2,$$

$\langle p_2,q,|J_\mu|p_1\rangle$ is the matrix element for the photoproduction of a pion by a virtual photon of (spacelike) 4 momentum:

$$k^2\equiv(r_1-r_2)^2=\lambda^2,$$

$(-\lambda^2)$ is the square of the "photon" mass. It is this matrix element which we propose to analyze in detail.

In spite of the fact that ϵ_μ has both longitudinal and timelike components, the Lorentz condition,

$$k^\mu\epsilon_\mu=0, \tag{1}$$

is still satisfied. The current J_μ also obeys the continuity equation

$$k^\mu J_\mu=0. \tag{2}$$

The T matrix may be written in terms of a linear combination of six Lorentz and gauge invariant quantities M_i:

$$T=\sum_{i=1}^{6}A_iM_i. \tag{3}$$

the M_i's will be given below. The amplitudes A_i are functions of three scalar quantities,

$$s_1=-(p_1+k)^2;\quad s_2=-(p_2-k)^2;\quad t=-(p_1-p_2)^2;$$

p_1, k have been chosen as incoming momenta, p_2, q as outgoing moments. We will take the "photon" mass to be constant. Then there are only two independent scalars and we have the relation

$$s_1+s_2+t=2M^2+\mu^2-\lambda^2. \tag{4}$$

The above amplitudes may be further decomposed into an isoscalar and into an isovector part, these designations referring to the character of the photon current. For each amplitude we may write

$$A_i=A_i^{(S)}+A_i^{(V)},$$
$$A_i^{(V)}=A_i^{(+)}\delta_{3\alpha}+A_i^{(-)}\tfrac{1}{2}[\tau_\alpha,\tau_3],$$
$$A_i^{(S)}=A_i^{(0)}\tau_\alpha.$$

These amplitudes are in turn related to amplitudes of given total isotopic spin

$$\begin{aligned}A_i^{(+)}&=\tfrac{1}{3}A_i^{\frac{1}{2}}+\tfrac{2}{3}A_i^{\frac{3}{2}},\\A_i^{(-)}&=\tfrac{1}{3}A_i^{\frac{1}{2}}-\tfrac{1}{3}A_i^{\frac{3}{2}}.\end{aligned} \tag{5}$$

Only the isotopic spin $\frac{1}{2}$ state contributes to $A_i^{(0)}$.

If one considers the process $\gamma+\pi\rightarrow N+\bar{N}$, then it can be shown from considerations of G invariance that intermediate states with an even number of pions contribute to $A_i^{(S)}$ only, whereas those with an odd number of pions contribute to $A_i^{(V)}$ only.

The decomposition of the total amplitude as in (3), into a linear combination of relativistic and gauge invariant quantities is not unique. However, the requirement that the associated amplitudes A_i should obey a representation with only those cuts and poles that have been conjectured by Mandelstam, places a severe limitation on this choice. In order to find the appropriate set of invariants we will follow a method due to Ball.[9]

We first write the T matrix in terms of 8 relativistic

[8] R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1959).

[9] J. S. Ball, University of California Radiation Laboratory Report UCRL-9172, 1960 (unpublished).

but not gauge-invariant quantities I_i:

$$T=\sum_{i=1}^{8} B_i I_i, \tag{6}$$

$$\begin{aligned}
&I_1=\gamma_5(\gamma k\gamma\epsilon-\gamma\epsilon\gamma k), && I_5=\gamma_5\gamma\epsilon,\\
&I_2=\gamma_5(2p_1\epsilon+k\epsilon), && I_6=\gamma_5\gamma k(2p_1\epsilon+k\epsilon),\\
&I_3=\gamma_5(2p_2\epsilon-k\epsilon), && I_7=\gamma_5\gamma k(2p_2\epsilon-k\epsilon),\\
&I_4=\gamma_5(2q\epsilon-k\epsilon), && I_8=\gamma_5\gamma k(2q\epsilon-k\epsilon).
\end{aligned}$$

The amplitudes B_i are taken to satisfy the Mandelstam representation. The above invariants are suggested by the form of the electric charge Born terms. One may note that the inhomogeneous electric charge terms are not gauge invariant when taken individually, whereas the magnetic moment terms are.

Gauge invariance means that when ϵ is replaced by k in (6), T must vanish. This requires that

$$\begin{aligned}
&-(s_1-M^2)B_2(s_1,t)+(s_2-M^2)B_3(s_1,t)\\
&\qquad\qquad+(t-\mu^2)B_4(s_1,t)=0.\\
&B_5(s_1,t)+(s_1-M^2)B_6(s_1,t)\\
&\qquad+(s_2-M^2)B_7(s_1,t)+(t-\mu^2)B_8(s_1,t)=0.
\end{aligned} \tag{7}$$

The kinematic factors which multiply the amplitudes B_i in (7) are just the denominators of the 3 poles in s_1, s_2, and t which appear in electroproduction. This means that no other denominators can appear in the gauge invariant amplitudes. We now write

$$T=\sum_{i=1}^{6} A_i M_i, \tag{8}$$

and define

$$\begin{aligned}
M_1&=\tfrac{1}{2}iI_1=\tfrac{1}{2}i\gamma_5F_{\mu\nu}\gamma_\mu\gamma_\nu,\\
M_2&=\tfrac{1}{2}i[(t-\mu^2)(I_2+I_3)+(s_1-s_2)I_4]\\
&=2i\gamma_5F_{\mu\nu}p_\mu(q-\tfrac{1}{2}h)_\nu,\\
M_3&=-\tfrac{1}{2}[I_6-I_7-(t-\mu^2+\lambda^2)I_5]=\gamma_5F_{\mu\nu}\gamma_\mu q_\nu,\\
M_4&=-\tfrac{1}{2}[I_6+I_7+(s_1-s_2)I_5]-MM_1\\
&=2\gamma_5F_{\mu\nu}\gamma_\mu P_\nu-MM_1,\\
M_5&=\tfrac{1}{2}i[(t-\mu^2)(I_2-I_3)-(t-\mu^2+\lambda^2)I_4]\\
&=i\gamma_5F_{\mu\nu}k_\mu q_\nu,\\
M_6&=[I_6-I_7-I_8-k^2I_5]=\gamma_5F_{\mu\nu}k_\mu\gamma_\nu,
\end{aligned} \tag{9}$$

where

$$F_{\mu\nu}=e_\mu k_\nu-e_\nu k_\mu,$$

and

$$P=(p_1+p_2)/2.$$

It is to be noted that the invariant M_2 differs from the corresponding one given by FNW.

The A_i are related to the B_i as follows:

$$\begin{aligned}
A_1&=-2iB_1-M(B_6+B_7),\\
A_2&=[-i/(t-\mu^2)](B_2+B_3),\\
A_3&=B_7-B_6-2B_8,\\
A_4&=-(B_6+B_7),\\
A_5&=[-i/(t-\mu^2)](B_2-B_3),\\
A_6&=-B_8.
\end{aligned} \tag{10}$$

Since A_1, A_3, A_4, and A_6 are related to the B's by numerical coefficients, they will also obey the Mandelstam representation:

$$\begin{aligned}
A_i={}&R_i^{(s)}\left(\frac{1}{M^2-s_1}\pm\frac{1}{M^2-s_2}\right)\\
&+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\rho_i(s')\left(\frac{1}{s'-s_1}\pm\frac{1}{s'-s^2}\right)ds'\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{4\mu^2}^{\infty}\frac{ds_1'dt'a_{13}^{(i)}(s'_1,t')}{(s_1'-s_1)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{(M+\mu)^2}^{\infty}\frac{ds_1'ds_2'a_{12}^{(i)}(s_1',s_2')}{(s_1'-s_1)(s_2'-s_2)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{4\mu^2}^{\infty}\frac{ds_2'dt'a_{23}^{(i)}(s_2',t')}{(s_2'-s_2)(t'-t)}
\end{aligned}$$

$$\text{(for } i=1,3,4,6). \tag{11}$$

To obtain the spectral representations for A_2 and A_5, we start from the Mandelstam representation and combine denominators according to (10). One obtains

$$\begin{aligned}
A_i={}&\frac{R_i^{(s)}}{(t-\mu^2)}\left(\frac{1}{M^2-s_1}\pm\frac{1}{M^2-s_2}\right)+\frac{R_i^{(t)}}{t-\mu^2}\\
&+\frac{1}{t-\mu^2}\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\rho_i(s')\left(\frac{1}{s'-s_1}\pm\frac{1}{s'-s_2}\right)ds'\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{4\mu^2}^{\infty}\frac{ds_1'dt'a_{13}^{(i)}(s_1',t')}{(s_1'-s_1)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{(M+\mu)^2}^{\infty}\frac{ds_1'ds_2'a_{12}^{(i)}(s_1',s_2')}{(s_1'-s_1)(s_2'-s_2)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{4\mu^2}^{\infty}\frac{ds_2'dt'a_{23}^{(i)}(s_2',t')}{(s_2'-s_2)(t'-t)}
\end{aligned}$$

$$\text{(for } i=2,5). \tag{12}$$

Because of the virtual character of the photon, the cuts are the same as in the usual Mandelstam representation.

The residues of the poles are as follows:

$$\begin{aligned}
&R^{(s)}(A_1^{(\pm,0)}) = -\tfrac{1}{2}gF_1^{(V,S)}(\lambda^2),\\
&R^{(s)}(A_2^{(\pm,0)}) = gF_1^{(V,S)}(\lambda^2),\\
&R^{(s)}(A_3^{(\pm,0)}) = R^{(S)}(A_4^{\pm,0}) = \tfrac{1}{2}gF_2^{(V,S)}(\lambda^2),\\
&R^{(s)}(A_5^{(\pm,0)}) = \pm\tfrac{1}{2}gF_1^{(V,S)}(\lambda^2),\\
&\qquad\qquad (+\text{ for } A_5^{(-)},\ -\text{ for } A_5^{(+,0)}),\\
&R^{(s)}(A_6^{(\pm,0)}) = 0,\\
&R^{(t)}(A_5^{(-)}) = (2g/\lambda^2)[eF_\pi(\lambda^2) - F_1^{(V)}(\lambda^2)],
\end{aligned} \tag{13}$$

$F_{1,2}^{(V,S)}$ are the usual Hofstadter form factors related to the proton and neutron form factors as follows[10]:

$$\begin{aligned}
&F_1^{(V)} = e[F_1^p(\lambda^2) - F_1^n(\lambda^2)],\\
&\qquad F_2^{(V)} = \mu_p' F_2^p(\lambda^2) - \mu_n F_2^n(\lambda^2),\\
&F_1^{(S)} = e[F_1^p(\lambda^2) + F_1^n(\lambda^2)],\\
&\qquad F_2^{(s)} = \mu_p' F_2^p(\lambda^2) + \mu_n F_2^n(\lambda^2),
\end{aligned}$$

$F_\pi(\lambda^2)$ is the form factor of the pion.

The Born terms have been made manifestly gauge invariant by adding to them the term

$$\tfrac{1}{2}[\tau_\alpha,\tau_3]ig\gamma_5\frac{k\epsilon}{\lambda^2}[F_1^{(V)}(\lambda^2) - eF_\pi(\lambda^2)],$$

which is identically zero on account of the Lorentz condition (1). As a consequence of crossing symmetry (interchange of incoming and outgoing nucleon lines) the amplitudes

$$A_1^{(+,0)},\ A_2^{(+,0)},\ A_3^{(-)},\ A_4^{(+,0)},\ A_5^{(-)},\ A_6^{(-)},$$

which are even under the interchange $s_1 \leftrightarrow s_2$ go with the positive sign in (11) and (12) whereas the amplitudes

$$A_1^{(-)},\ A_2^{(-)},\ A_3^{(+,0)},\ A_4^{(-)},\ A_5^{(+,0)},\ A_6^{(+,0)},$$

which are odd under $s_1 \leftrightarrow s_2$ go with the negative sign.

II. ISOSCALAR AMPLITUDE: THE CHANNEL $\gamma+\pi\rightarrow N+\bar{N}$

The lowest intermediate state that can contribute to the isoscalar amplitude in the process $\gamma+\pi\rightarrow N+\bar{N}$ is the 2-pion state. In this case s_1 and s_2 cannot simultaneously reach their lower limits in (11) and (12), and therefore one can apply the Cini-Fubini method[6] to reduce the Mandelstam representation to a one-dimensional form. One obtains

$$A_i^{(0)} = (A_i^{(0)})_{\text{Born}} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{a_i(t,t')}{t'-t}dt' + K_i^0, \tag{14}$$

K_i^0 is a constant that lumps together the effects of all the distant singularities. It vanishes for those amplitudes that are odd under crossing. We have omitted a term related to the electroproduction channel proper and dependent, through the final state theorem, upon the π-nucleon phase shifts. The reason is that for the isoscalar amplitude, the π-nucleon system is in an $I=\frac{1}{2}$ state, whereas we will be considering energy regions dominated by the $\frac{3}{2}-\frac{3}{2}$ resonance only.

The imaginary part $a_i(t,t')$ of $A_i^{(0)}$ in the approximation of keeping only 2-pion intermediate states, may be calculated using unitarity. Alternatively, if one assumes that there exists a sharp $J=I=1$ pion-pion resonance, then a completely equivalent method of procedure is to calculate the "bi-pion" graph in which the reaction $\gamma+\pi\rightarrow N+\bar{N}$ proceeds in lowest order via an intermediate particle of spin 1 and isotopic spin 1. The calculation for photoproduction is well known.[11,12] For electroproduction one obtains

$$\begin{aligned}
A_i^{(0)} = (A_i^{(0)})_{\text{Born}} &+ \frac{1}{8\sqrt{2}}\frac{e\Lambda(\lambda^2)}{\mu^3}t_R\phi(t_R)\\
&\times\Bigg[A_1^{(0)}M_1 + \frac{aM_4}{t_R-t} + \frac{g_V b}{M}\frac{1}{t_R-t}\\
&\times\left(-tM_1 + \frac{t-\mu^2+\lambda^2}{t-\mu^2}M_2 + \frac{2Pk}{t-\mu^2}M_5\right)\Bigg].
\end{aligned} \tag{15}$$

In the above $\phi(t)$ and $\Lambda(\lambda^2)$ are functions associated with the $\gamma\pi\rightarrow\pi\pi$ vertex. $\Lambda(\lambda^2)$ is to be determined from experiment. It reduces of course to a constant for photoproduction. We now wish to explain the meaning of the constant $A_1^{(0)}$. By going over to the center-of-mass frame of the $N\bar{N}$ system it is possible to see that, in relation to the center-of-mass amplitudes, A_1 has an extra energy factor as compared to the other amplitudes. It is to be expected therefore, that A_1 will have a dominant high-energy behavior. We have allowed for this by carrying out a subtraction in the dispersion relation for the amplitude A_1. This accounts for the constant $A_1^{(0)}$ in (15). The constants a, b and t_R are related to the parameters appearing in an empirical formula for the isovector form factors of the nucleons[7]:

$$\begin{aligned}
F_1^{(V)} &= e\left[(1-a) + \frac{a}{1-t/t_R}\right],\\
F_2^{(V)} &= \frac{g_V e}{2M}\left[(1-b) + \frac{b}{1-t/t_R}\right],
\end{aligned} \tag{16}$$

where $g_V = 3.69$.

Reference 7 gives $a=b=1.2$ when $t_R = 22\mu^2$. With the present value of t_R at about[4] $30\mu_-^2$, a and b will be somewhat modified.

[10] $\mu_p' = 1.78e/2M$, $\mu_n = -1.91e/2M$, $g^2/4\pi \approx 14$, $e^2/4\pi = 1/137$.

[11] M. Gourdin, D. Lurié, and A. Martin, Nuovo cimento 18, 933 (1960).

[12] B. de Tollis, E. Ferrari, and H. Munczek, Nuovo cimento 18, 198 (1961).

III. ISOVECTOR AMPLITUDE: THE NUCLEON TERM

The isovector amplitude, contrary to the isoscalar one, is not reducible to a one-dimensional form. We may, however, follow the analogy with the treatment of the isoscalar amplitude and replace the contribution from the intermediate 3π state by the contribution from an isobar with quantum numbers $J=1$, $I=0$ and neglect rescattering corrections. Such a particle was introduced by Nambu[13] and it was shown by Chew[14] that with the above choice for the quantum numbers, all pairs of pions in a 3π state would have $J=I=1$. We are then led to the following equation

$$A_i^{(\pm)}(s_1,t)=(A_i^{(\pm)})_{\text{Born}}+(A_i^{(\pm)})_{\text{tri-pion}}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\text{Im}\,A_i^{(\pm)}(s',t)\times\left\{\frac{1}{s'-s_1}\pm\frac{1}{s'-s_2}\right\}, \quad (17)$$

where

$$(A_i^{(-)})_{\text{tri-pion}}=0,$$

and where $(A_i^{(+)})_{\text{tri-pion}}$ is the same expression as (15) with Λ, a, b, and t_R replaced by Λ', a', b', and t_R'. The constants a', b', t_R' are related to the parameters appearing in the isoscalar form factors of the nucleons in the same way as a, b, t_R are related to those appearing in the expression for the isovector form factors. The reason that the tri-pion contribution is similar to the bi-pion contribution is of course that both particles have spin 1.

Due to the explicit separation of the tri-pion singularities in (17), $\text{Im}\,A_i^{(\pm)}(s,t)$ has only distant singularities in t and may therefore be expanded in powers of that variable, or equivalently, in states of given orbital and total angular momentum (multipole expansion). In such a representation the final state theorem[1] tells us that $\text{Im}\,A_i^{(\pm)}$ has the phase of π-nucleon scattering. This means that as long as the π-nucleon state is at sufficiently low energies, we may neglect all but the large $\frac{3}{2}-\frac{3}{2}$ resonant phase shift in $\text{Im}\,A_i^{(\pm)}$. Therefore we will proceed in two steps. First we express $\text{Im}\,A_i^{(\pm)}$ in terms of the imaginary parts of center-of-mass amplitudes $\mathfrak{F}_i$. Second, we expand the $\mathfrak{F}_i$'s in Legendre polynomials where the expansion coefficients are simply the magnetic dipole, electric quadrupole, and longitudinal quadrupole amplitudes. In this expression we shall retain only the $J=\frac{3}{2}$ state. The calculation is long but introduces no new physical concepts and is therefore relegated to Appendix I. The result is

$$\text{Im}\,A_i^{(\pm)}(W^2,t)=[\alpha_M^{(i)}(\lambda^2)+t\beta_M^{(i)}(\lambda^2)]M(W^2,\lambda^2)+[\alpha_E^{(i)}(\lambda^2)+t\beta_E^{(i)}(\lambda^2)]E(W^2,\lambda^2)+[\alpha_L^{(i)}(\lambda^2)+t\beta_L^{(i)}(\lambda^2)]L(W^2,\lambda^2), \quad (18)$$

[13] Y. Nambu, Phys. Rev. **106**, 1366 (1957).
[14] G. Chew, Phys. Rev. Letters **4**, 142 (1960).

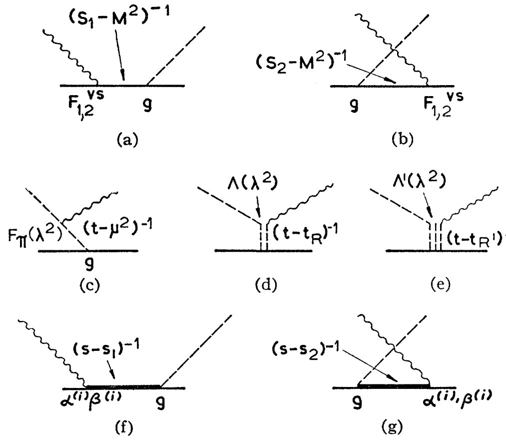

FIG. 1. Graphical representation of Eqs. (15), (16), and (18). The heavy lines in (1f) and (1g) represent particles of spin $\frac{3}{2}$.

where

$$\frac{M}{\text{Im}\,M_{1+}}=\frac{E}{\text{Im}\,E_{1+}}=\frac{L}{\text{Im}\,L_{1+}}=\frac{M[(W+M)^2+\lambda^2]^{\frac{1}{2}}}{qkW[(W+M)^2-\mu^2]^{\frac{1}{2}}},$$

M_{1+}, E_{1+}, L_{1+} are, respectively, the magnetic dipole, the electric quadrupole and the longitudinal quadrupole amplitudes. The coefficients $\alpha^{(i)}$ and $\beta^{(i)}$ are given in Appendix II.

Equations (15), (17), and (18) contain the essential results of the paper and it may be useful at this point to discuss briefly their physical content. Dispersion relations together with unitarity may be viewed as a set of of instructions which tells you that some graphs with their associated singularities are of greater importance than others. What we have actually done is merely to calculate the contributions to the total amplitude from these more important graphs. First there are the Born terms with the nucleon propagators in the variables s_1 and s_2 (Figs. 1a, 1b). These contain the Hofstadter form factors; then there are the peripheral graphs which include: (a) the meson current Born term with a propagator in t and the form factor of the pion (Fig. 1c), (b) the bi-pion and tri-pion terms (Figs. 1d, e) with spin 1 propagators in t at the square of their respective masses, and with "form factors" $\Lambda(\lambda^2)$ and $\Lambda'(\lambda^2)$. Finally, there are the direct and crossed resonant nucleon terms (Figs. 1f, g). On the resonance the quantities $\alpha^{(i)}$ and $\beta^{(i)}$ in (18) are pure numbers and may be regarded as a relativistic generalization of the Clebsch-Gordon coefficients which couple a photon and a nucleon to a particle of spin $\frac{3}{2}$. The functions M, E, and L refer to the fact that the transition to the $\frac{3}{2}$ state may be initiated by a magnetic, electric, or longitudinal excitation. These functions will also contain certain "form factors" which characterize the transition.

Therefore, the only unknown quantities which appear in the expression for the total amplitude are the various

form factors which are all functions of λ^2, and which must be determined from experiment.

It may be thought that the form factors associated with the nucleon term would be of a new kind, characteristic of the resonant state. It will be the task of the remaining part of this paper to show that in fact, because the resonant state is metastable, that these form factors are simply related to the Hofstadter ones.

IV. INTEGRAL EQUATIONS FOR THE 3-3 AMPLITUDES

By projection using (If), (17), and (18) we may obtain integral equations for the three multipoles. This will allow us to determine the functions M, E and L in (18). Let us first discuss the multipole projections of Born terms. These may be conveniently separated into those proportional to the linear combination of form factors:

$\mu^{(V)}(\lambda^2)=F_1^{(V)}(\lambda^2)/2M+F_2^{(V)}(\lambda^2)$ to $F_1^{(V)}(\lambda^2)$ and to $F_\pi(\lambda^2)$.

The work of FNW shows that in the static limit only the term proportional to $\mu^{(V)}(\lambda^2)$ is important. The results may be written as follows:

$$
\begin{aligned}
M_{1+}{}^{\mathrm{B}}(\mu^{(V)},F_1{}^{(V)},F_\pi)&=I_1(y)f_1(\mu^{(V)},F_1{}^{(V)},F_\pi)+I_2(y)f_2(\mu^{(V)},F_1{}^{(V)},F_\pi)\\
&\quad-I_3(y)f_3(\mu^{(V)},F_1{}^{(V)},F_\pi),\\
E_{1+}{}^{\mathrm{B}}(\mu^{(V)},F_1{}^{(V)},F_\pi)&=I_1(y)f_1(\mu^{(V)},F_1{}^{(V)},F_\pi)+I_2(y)f_2(\mu^{(V)},F_1{}^{(V)},F_\pi)\\
&\quad+I_3(y)f_3(\mu^{(V)},F_1{}^{(V)},F_\pi)\\
&\quad+(\tfrac{4}{3}-2yI_3(y)f_4(\mu^{(V)},F_1{}^{(V)},F_\pi),\\
L_{1+}{}^{\mathrm{B}}(\mu^{(V)},F_1{}^{(V)},F_\pi)&=I_1(y)(f_1-yf_3+f_4+f_5)-I_2(y)f_6\\
&\quad-(3yI_3(y)-2)f_4,
\end{aligned}
\tag{19}
$$

where

$$
I_1(y)=2-y\ln\frac{y+1}{y-1},\quad I_2(y)=3y-\frac{1-3y^2}{2}\ln\frac{y+1}{y-1},
$$

$$
I_3(y)=y+\frac{1-y^2}{2}\ln\frac{y+1}{y-1},
$$

and

$$
y=\begin{cases}\dfrac{2k_0E_2+\lambda^2}{2qk} & \text{for terms proportional to } \mu^{(V)} \text{ and } F_1{}^{(V)}\\[2ex] -\left(\dfrac{2k_0q_0+\lambda^2}{2qk}\right) & \text{for term proportional to } F_\pi,\end{cases}
$$

$$
f_1(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{-\mu^{(V)}Mf(W-M)[(E_1+M)(E_2+M)]^{\frac{1}{2}}}{4qk},0,0\right)
$$

$$
f_2(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{\mu^{(V)}fM(W+M)}{4[(E_1+M)(E_2+M)]^{\frac{1}{2}}},0,0\right),
$$

$$
f_3(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{-\mu^{(V)}f(E_2+M)^{\frac{1}{2}})(W+M)}{4(E_1+M)^{\frac{1}{2}}},\frac{F_1{}^{(V)}f(E_2+M)^{\frac{1}{2}}(W-M)}{8M(E_1+M)^{\frac{1}{2}}},\frac{eF_\pi f(E_2+M)^{\frac{1}{2}}}{4(E_1+M)^{\frac{1}{2}}}\right),
$$

$$
f_4(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{-\mu^{(V)}qf(W-M)(E_1+M)^{\frac{1}{2}}}{4k(E_2+M)^{\frac{1}{2}}},\frac{F_1{}^{(V)}f(E_1+M)^{\frac{1}{2}}(W+M)q}{8M(E_2+M)^{\frac{1}{2}}k},\frac{-eF_\pi f(E_1+M)^{\frac{1}{2}}q}{4(E_2+M)^{\frac{1}{2}}k}\right),
$$

$$
f_5(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{-\mu^{(V)}f(W-M)k(E_2+M)^{\frac{1}{2}}}{8q(E_1+M)^{\frac{1}{2}}},\frac{F_1{}^{(V)}f(W-M)k(E_2+M)^{\frac{1}{2}}}{16Mq(E_1+M)^{\frac{1}{2}}},\frac{-eF_\pi fk(E_2+M)^{\frac{1}{2}}}{8q(E_1+M)^{\frac{1}{2}}}\right),
$$

$$
f_6(\mu^{(V)},F_1{}^{(V)},F_\pi)=\left(\frac{-\mu^{(V)}f(W+M)(E_1-M)}{8[(E_1+M)(E_2+M)]^{\frac{1}{2}}},\frac{F_1{}^{(V)}f(W+M)(E_1+M)^{\frac{1}{2}}}{16M(E_2+M)^{\frac{1}{2}}},\frac{eF_\pi f(E_1+M)^{\frac{1}{2}}}{8(E_2+M)^{\frac{1}{2}}}\right).
$$

In these relations $f^2\approx 0.08$.

A numerical analysis of the Born terms has been carried out on a Mercury computer and the results show that of the nine Born terms, only $M_{1+}{}^{\mathrm{B}}(\mu^{(V)})$, $M_{1+}{}^{\mathrm{B}}(F_\pi)$ and $E_{1+}{}^{\mathrm{B}}(F_\pi)$ are not negligible. Among these terms, moreover, $M_{1+}{}^{\mathrm{B}}(F_\pi)$ and $E_{1+}{}^{\mathrm{B}}(F_\pi)$ decrease with λ^2 much more rapidly than $M_{1+}{}^{\mathrm{B}}(\mu^{(V)})$. Below $W=8.9$ and $\lambda^2=6$, $M_{1+}{}^{\mathrm{B}}(F_\pi)$ and $E_{1+}{}^{\mathrm{B}}(F_\pi)$ are at most of the order of 10% of $M_{1+}{}^{\mathrm{B}}(\mu^{(V)})$. Above these values of λ^2 and W, they are negligibly small.

A further result of the analysis shows that the 3 Born terms that survive may be very well approximated by simple poles in W, multiplied by a λ-dependent factor. We may write

$$
\begin{aligned}
\frac{M_{1+}{}^{\mathrm{B}}(\mu^{(V)})}{qk}&=\frac{f\mu^{(V)}(\lambda^2)P_1(\lambda^2)}{W-6.4},\\
\frac{M_{1+}{}^{\mathrm{B}}(F_\pi)}{qk}&=\frac{efF_\pi(\lambda^2)P_2(\lambda^2)}{W-7.3},\\
\frac{E_{1+}{}^{\mathrm{B}}(F_\pi)}{qk}&=\frac{efF_\pi(\lambda^2)P_3(\lambda^2)}{W-7.5},
\end{aligned}
\tag{20}
$$

where

$$P_1(\lambda^2)\approx[1-(0.275/40)\lambda^2],$$
$$P_2(\lambda^2)\approx 0.092(1+0.18\lambda^2)^{-1},$$
$$P_3(\lambda^2)\approx -0.05(1+0.38\lambda^2)^{-1}.$$

We have taken $M=6.8$. The inclusion of the factor $(qk)^{-1}$ in (20) gives the simple pole-like behavior. The representations are accurate to the first decimal place in the entire ranges $7.8<W<9.8$, $0\leqslant\lambda^2\leqslant 70$. As can be seen, $M_{1+}{}^{B}(F_\pi)$ and $E_{1+}{}^{B}(F_\pi)$ vanish when $W>8.9$ or when $\lambda^2\geqslant 6$. These results imply in particular that the magnetic dipole approximation of FNW becomes even better at high values of λ^2.

As a first approximation we will neglect the multipole projection of the tri-pion. The reasons are that we are uncertain that such a particle actually exists, and that we do not know its mass. The present treatment may of course be easily extended to include the tri-pion, if this turns out to be necessary.

We may now write the integral equations for the multipole amplitudes. If we evaluate the kernel of the integral at resonance, i.e., if in the kernel we set $W'=W$, we obtain

$$\mathrm{Re}\frac{M_{1+}(\mu^{(V)})}{qk}=\frac{f\mu^{(V)}(\lambda^2)P_1(\lambda^2)}{W-6.4}+\frac{1}{\pi}\int\frac{\mathrm{Im}\,M_{1+}(\mu^{(V)};W')}{W'-W}\frac{dW'}{q'k'}+\text{crossed term},\tag{21a}$$

$$\mathrm{Re}\frac{M_{1+}(F_\pi)}{qk}=\frac{efF_\pi(\lambda^2)P_2(\lambda^2)}{W-7.3}+\frac{1}{\pi}\int\frac{\mathrm{Im}\,M_{1+}(F_\pi;W')}{W'-W}\frac{dW'}{q'k'}+\text{crossed term},\tag{21b}$$

$$\mathrm{Re}\frac{E_{1+}(F_\pi)}{qk}=\frac{efF_\pi(\lambda^2)P_3(\lambda^2)}{W-7.5}+\frac{1}{\pi}\int\frac{\mathrm{Im}\,E_{1+}(F_\pi;W')}{W'-W}\frac{dW'}{q'k'}+\text{crossed term}.\tag{21c}$$

We give in Appendix III the crossed term of (21a).

In order to solve Eqs. (21), we will first neglect the crossed terms which are expected to be small compared to the uncrossed terms because of their much larger denominators; the consistency of this approximation will be checked later. We are then left with equations which resemble very much the integral equation of the static π-nucleon model[15] without crossing, which may be written

$$\mathrm{Re}\frac{h(W)}{q^2}=\frac{4}{3}\frac{f^2}{W-6.8}+\frac{1}{\pi}\int\frac{\mathrm{Im}\,h(W')}{W'-W}\frac{dW'}{q'^2},$$

and whose solution is given by

$$h(W)=e^{i\delta_{33}}\sin\delta_{33}/q.$$

To obtain the solution of (21a), e.g., we may proceed as follows:

Let

$$\frac{M_{1+}(\mu^{(V)})}{qk}=C_1\frac{(W-6.8)h(W)}{(W-6.4)q^2},$$

where C_1 is a real constant to be determined. Then

$$\lim_{W\to 6.4}(W-6.4)\,\mathrm{Re}\frac{M_{1+}(\mu^{(V)})}{qk}=\frac{-0.4C_1\,\mathrm{Re}\,h(6.4)}{q^2(6.4)}=f\mu^{(V)}(\lambda^2)P_1(\lambda^2),$$

which gives C_1. We may proceed in a similar way for the other two equations. The results are:

$$\begin{aligned}\frac{M_{1+}(\mu^{(V)})}{qk}&=C_1\frac{(W-6.8)h(W)}{(W-6.4)q^2},\\ \frac{M_{1+}(F_\pi)}{qk}&=C_2\frac{(W-6.8)h(W)}{(W-7.3)q^2},\\ \frac{E_{1+}(F_\pi)}{qk}&=C_3\frac{(W-6.8)h(W)}{(W-7.5)q^2},\end{aligned}\tag{22}$$

where

$$C_1=-f\mu^{(V)}(\lambda^2)P_1(\lambda^2)q^2(6.4)/0.4\,\mathrm{Re}\,h(6.4),$$
$$C_2=2efF_\pi(\lambda^2)P_2(\lambda^2)q^2(7.3)/\mathrm{Re}\,h(7.3),$$
$$C_3=efF_\pi(\lambda^2)P_3(\lambda^2)q^2(7.5)/0.7\,\mathrm{Re}\,h(7.5).$$

There is of course the possibility of adding to (22) arbitrary solutions of the homogeneous integral equations.[16] It seems reasonable to require, however, that the (unique) solutions to the problem be the same as the one obtained by iteration when the π-nucleon coupling constant is small. Such a requirement has led to satisfactory effective range formulas for π-nucleon scattering and to a correct behavior of the photoproduction amplitudes. In the $\pi-\pi$ problem, the method would not seem to be as trustworthy. But this may be due to the fact that in this problem, the inhomogeneous term is merely an arbitrary subtraction constant which may not have a particular physical meaning. The residue of the π-nucleon pole on the other hand is the physical π-nucleon coupling constant responsible for the existence of the composite resonant state. Analyticity

[15] G. Chew, M. L. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957) and *ibid.* **106**, 1345 (1957), quoted as CGLN.

[16] I thank Dr. M. Gourdin for reminding me of this possibility.

arguments with respect to this coupling constant are therefore likely to be valid.

An estimate of the error involved by neglecting the crossed term can be made by reinserting (22) into (21) and assuming a sharp resonance at $W=8.9$; it turns out that the crossed term is of the order of 7% of the uncrossed term at $W=9.5$. Nearer to resonance, the error is even less.

The total amplitude is now obtained by inserting (22) into (17) and (18).

DISCUSSION

We have obtained in this paper a theory of electro- and photo-pion production which avoids an expansion in powers of the inverse nucleon mass. The total amplitude as given in (17), (18), and (22) may be obtained by a simple numerical integration.

The results have shown that especially for photoproduction, there is in addition to the magnetic dipole contribution, a small but non-negligible contribution from the electric quadupole term. Aside from this, the essential differences with the static theory arise from the kinematical factors $\alpha^{(i)}$ and $\beta^{(i)}$ in Eq. (18), from the factors $P_i(\lambda^2)$ in the multipole amplitudes, and from the fact that the magnetic dipole amplitude has a pole in W that is somewhat displaced from the nucleon mass.

The inclusion of a $\pi-\pi$ interaction introduces directly the pion form factor in the meson current Born term. This circumstance is of particular significance because, as can be seen from the general form of the semiempirical formula for $\mu^{(V)}(\lambda^2)$ [Eq. (16)] and from the available experimental data on the nucleon form factors, the resonant nucleon term becomes very small at values of λ^2 of the order of $60\mu^2$. This in effect isolates the pion form factor and could allow for a rather direct determination of it.

It is not yet clear what roles the bi-pion and tri-pion will play in correlating theory with experiment. The ratio of positive- to negative-pion production cross sections is particularly sensitive to the bi-pion term. A change of 1% in the bi-pion term induces a change in this ratio approximately equivalent to a 10% change in the nucleon term. On the other hand, if the bi-pion mass is of the order of $(30)^{\frac{1}{2}}\mu$, it may not contribute to any appreciable extent in the resonance region of the electroproduction channel. In this respect, the correction to the static theory of FNW, which includes using the complete relativistic isoscalar Born terms, may be more significant.

The arbitrariness in the problem lies especially with the question of subtractions in the dispersion relations A preliminary calculation by Ferrari[17] for photoproduction based on the work of CGLN indicates that only one subtraction in the amplitude A_1 may be necessary. Whether this subtraction constant is of such a size as to mask the effect of the bi-pion is still an open question.

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APPENDIX I

In the center-of-mass frame of the $\pi-N$ state, the matrix may be written as

$$\mathfrak{F}=i\boldsymbol{\sigma}\cdot\mathbf{a}\mathfrak{F}_1+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\,\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{a}}{qk}\mathfrak{F}_2+\frac{i\boldsymbol{\sigma}\cdot\mathbf{k}\,\mathbf{q}\cdot\mathbf{a}}{qk}\mathfrak{F}_3+\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\,\mathbf{q}\cdot\mathbf{a}}{q^2}\mathfrak{F}_4+\frac{i\boldsymbol{\sigma}\cdot\mathbf{k}\,\mathbf{k}\cdot\mathbf{a}}{k^2}\mathfrak{F}_5+\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\,\mathbf{k}\cdot\mathbf{a}}{qh}\mathfrak{F}_6, \tag{Ia}$$

where

$$a=\epsilon-(\epsilon_0/k_0)k.$$

The coefficients $\mathfrak{F}_i$ are related to the A_i as follows:

$$\begin{aligned}
F_1&\equiv\frac{2M}{(W-M)[(E_1+M)(E_2+M)]^{\frac{1}{2}}}\mathfrak{F}_1=A_1-\frac{(t-\mu^2+\lambda^2)}{2(W-M)}(A_3-A_4)+(W-M)A_4+\frac{\lambda^2}{W-M}A_6,\\
F_2&\equiv\frac{2M[(E_1+M)(E_2+M)]^{\frac{1}{2}}}{qk(W+M)}\mathfrak{F}_2=-A_1-\frac{(t-\mu^2+\lambda^2)}{2(W+M)}(A_3-A_4)+(W+M)A_4+\frac{\lambda^2}{W+M}A_6,\\
F_3&\equiv\frac{2M(E_1+M)^{\frac{1}{2}}}{qk(E_2+M)^{\frac{1}{2}}(W+M)}\mathfrak{F}_3=\frac{\beta A_2}{W+M}+(A_3-A_4)-\frac{\lambda^2}{W+M}A_5,\\
F_4&\equiv\frac{2M(E_2+M)^{\frac{1}{2}}}{q^2(E_1+M)^{\frac{1}{2}}(W-M)}\mathfrak{F}_4=\frac{-\beta A_2}{W-M}+(A_3-A_4)+\frac{\lambda^2}{W-M}A_5,\\
F_5&=\frac{2M(E_1+M)^{\frac{1}{2}}}{k^2(E_2+M)^{\frac{1}{2}}}\mathfrak{F}_5=-A_1-\alpha A_2-(W-M)A_4+\frac{(t-\mu^2+\lambda^2)}{2}A_5-(W+M)A_6,\\
F_6&\equiv\frac{2M(E_2+M)^{\frac{1}{2}}}{qk(E_1+M)^{\frac{1}{2}}}\mathfrak{F}_6=\frac{-k_0}{E_1+M}A_1+\alpha A_2-\frac{(t-\mu^2+\lambda^2)}{2(E_1+M)}(A_3-A_4)+\frac{k_0(W+M)}{E_1+M}A_4-\frac{(t-\mu^2+\lambda^2)}{2}A_5-k_0\frac{W+M}{E_1+M}A_6,
\end{aligned} \tag{Ib}$$

[17] E. Ferrari (private communication).

where

$$\beta=\frac{2(W^2-M^2)+\lambda^2}{2};\quad \alpha=\frac{3(t-\mu^2+\lambda^2)}{4}+k_0W,\quad 2Wk_0=W^2-M^2-\lambda^2;\quad 2WE_1=W^2+M^2+\lambda^2;\quad 2WE_2=W^2+M^2-\mu^2.$$

Inverting these equations, one obtains

$$\begin{aligned}
2WA_1&=\frac{W-M}{Wk_0}\frac{[(W+M)^2-\lambda^2]}{2}F_1-\frac{(W+M)(W^2-M^2+\lambda^2)}{2W(E_1+M)}F_2+\frac{(t-\mu^2+\lambda^2)M(W+M)}{2Wk_0}\\
&\quad+\frac{(t-\mu^2+\lambda^2)M(W-M)}{2Wk_0}F_4+\frac{M\lambda^2}{Wk_0}(F_5+F_6),\\
2WA_2&=\frac{(W^2-M^2)}{\beta}(F_3-F_4)+\frac{2W\lambda^2}{\beta}A_5,\\
2WA_3&=2WA_4+(W+M)F_3+(W-M)F_4,\\
2WA_4&=\frac{W^2-M^2}{2Wk_0}F_1+\frac{(W+M)^2}{2W(E_1+M)}F_2+\frac{(W+M)(t-\mu^2+\lambda^2)}{4Wk_0}F_3+\frac{(t-\mu^2+\lambda^2)(W-M)}{4Wk_0}F_4+\frac{\lambda^2}{2Wk_0}(F_5+F_6),\\
2WA_5&=\frac{\beta}{Wk_0(t-\mu^2)}\left[(W-M)F_1+k_0\frac{W+M}{E_1+M}F_2+\frac{\alpha(W^2-M^2)}{\beta}(F_3-F_4)+(W-M)F_5-(W+M)F_6\right],\\
2WA_6&=\frac{W^2-M^2}{2Wk_0}\left[-(F_1+F_5+F_6)+\frac{k_0}{E_1+M}F_2-\frac{(t-\mu^2+\lambda^2)}{2(W-M)}F_3-\frac{(t-\mu^2+\lambda^2)}{2(W+M)}F_4\right].
\end{aligned}\tag{Ic}$$

The multipole expansion of the $\mathfrak{F}_i$'s may be obtained briefly as follows: we expand $\mathfrak{F}$ into a complete set of states of the orbital angular momentum l. Among these states, some will refer to states of total angular momentum $l+\frac{1}{2}$ or $l-\frac{1}{2}$. Therefore, we introduce the projection operators

$$P_{l+}=\frac{l+1+\boldsymbol{\sigma}\cdot\mathbf{l}_q}{2l+1};\quad P_{l-}=\frac{l-\boldsymbol{\sigma}\cdot\mathbf{l}_q}{2l+1},$$

where $\mathbf{l}_q=i^{-1}\mathbf{q}\times\nabla_q$ is the angular momentum of the meson. A further classification of states pertains to the photon variables. Some states will transform as transverse pseudovectors [magnetic radiation, parity $-(-1)^l$], others are transverse vectors [electric radiation, parity $(-1)^l$], still others as longitudinal vectors [longitudinal radiation, parity $(-1)^l$]. Because of (1), there is no scalar radiation. Finally, normalizing the various projection operators and noting that the total matrix must be a pseudoscalar, one obtains

$$\begin{aligned}
\mathfrak{F}=\sum_{l=0}^{\infty}[(M_{l+}\mathbf{a}\cdot\mathbf{l}_k+E_{l+1,-}\boldsymbol{\sigma}\cdot\mathbf{q}\;\mathbf{a}\cdot\mathbf{k}\times\mathbf{l}_k+lL_{l+1,-}\boldsymbol{\sigma}\cdot\mathbf{q}\;\mathbf{a}\cdot\mathbf{k})P_{l+}\\
+M_{l-}\mathbf{a}\cdot\mathbf{l}_k+E_{l-1,+}\boldsymbol{\sigma}\cdot\mathbf{q}\;\mathbf{a}\cdot\mathbf{k}\times\mathbf{l}_k+lL_{l-1,+}\boldsymbol{\sigma}\cdot\mathbf{q}\;\mathbf{a}\cdot\mathbf{k}](2_{l+1})P_l(\hat{q}\cdot\hat{k}),
\end{aligned}$$

$P_l(\hat{q}\cdot\hat{k})$ are the Legendre polynomials. (Id)

Carrying out the operations implied in (Id) and comparing with (Ia) yields

$$\begin{aligned}
\mathfrak{F}_1&=\sum_{l=0}^{\infty}[(lM_{l+}+E_{l+})P_{l+1}'(x)+((l+1)M_{l-}+E_{l-})P_{l-1}'(x)].\\
\mathfrak{F}_2&=\sum_{l=1}^{\infty}[(l+1)M_{l+}+lM_{l-}]P_l'(x),\\
\mathfrak{F}_3&=\sum_{l=1}^{\infty}[(E_{l+}-M_{l+})P_{l+1}''(x)+(E_{l-}+M_{l-})P_{l-1}''(x)],\\
\mathfrak{F}_4&=\sum_{l=1}^{\infty}(M_{l+}-E_{l+}-M_{l-}-E_{l-})P_l''(x),\\
\mathfrak{F}_5&=-\mathfrak{F}_1-x\mathfrak{F}_3+\sum_{l=0}^{\infty}[(l+1)L_{l+}P_{l+1}'(x)-lL_{l-}P_{l-1}'(x)],\\
\mathfrak{F}_6&=-x\mathfrak{F}_4+\sum_{l=0}^{\infty}(lL_{l-}-(l+1)L_{l+})P_l'(x).
\end{aligned}\tag{Ie}$$

The $\mathfrak{F}_1\cdots\mathfrak{F}_4$ expansions are the same as for photoproduction and have been given by CGLN.

The above formulas may be inverted by using the orthogonality properties of the Legendre polynomials. We will only need those amplitudes referring to states with total angular momentum $l+\frac{1}{2}$.

$$M_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx\left[\mathfrak{F}_1P_l(x)-\mathfrak{F}_2P_{l+1}(x)-\frac{\mathfrak{F}_3P_l'(x)(1-x^2)}{l(l+1)}\right],$$

$$E_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx\left[\mathfrak{F}_1P_l(x)-\mathfrak{F}_2P_{l+1}(x)+\frac{\mathfrak{F}_3P_l'(x)(1-x^2)}{(l+1)}+\frac{\mathfrak{F}_4P_{l+1}'(x)(1-x^2)}{l+2}\right], \qquad \text{(If)}$$

$$L_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx[(\mathfrak{F}_1+x\mathfrak{F}_3+\mathfrak{F}_5)P_l(x)+(x\mathfrak{F}_4+\mathfrak{F}_6)P_{l+1}(x)],$$

Now we express Im A_i in terms of the $\mathfrak{F}_i$'s using (Ib) and (Ic). Then, using (Ie) and keeping only the $J=\frac{3}{2}$ state, we obtain an expression for Im A_i in terms of Im M_{1+}, Im E_{1+}, Im L_{1+}. The result is

$$(A_i)=(A_i)_{\text{Born}}+(A_i)_{\text{tri-pion}}+\frac{1}{\pi}\int dW'\left[\frac{1}{W'^2-s_1}\pm\frac{1}{W'^2-s_2}\right]\{[\alpha_M{}^{(i)}(\lambda^2)+t\beta_M{}^{(i)}(\lambda^2)]M(W',\lambda^2)$$
$$+[\alpha_E{}^{(i)}(\lambda^2)+t\beta_E{}^{i}(\lambda^2)]E(W',\lambda^2)+[\alpha_L{}^{(i)}(\lambda^2)+t\beta_L{}^{(i)}(\lambda^2)]L(W',\lambda^2)\}, \qquad \text{(Ig)}$$

where

$$\frac{M}{\text{Im }M_{1+}}=\frac{E}{\text{Im }E_{1+}}=\frac{L}{\text{Im }L_{1+}}=\frac{M[(W+M)^2+\lambda^2]^{\frac{1}{2}}}{qkW[(W+M)^2-\mu^2]^{\frac{1}{2}}}.$$

The coefficients $\alpha^{(i)}$ and $\beta^{(i)}$ are given in Appendix II. In (Ig) these coefficients may be evaluated at resonance, and then they are only functions of λ^2.

APPENDIX II

We list here the coefficients $\alpha^{(i)}$ which appear in Eq. (18). The $\beta^{(i)}$ may be read off directly as follows; they are simply the coefficient of $(\lambda^2-\mu^2)$ in the corresponding $\alpha^{(i)}$.

$$\alpha_M{}^{(1)}=\frac{1}{[(W+M)^2+\lambda^2]}\{3(\lambda^2-\mu^2)(W+M)+3q_0[(W+M)^2-\lambda^2]-(W^2-M^2+\lambda^2)[(W+M)^2-\mu^2]W^{-1}\},$$

$$\alpha_M{}^{(2)}=\frac{2}{[2(W^2-M^2)+\lambda^2]}[\lambda^2\alpha_M{}^{(5)}-3(W-M)],$$

$$\alpha_M{}^{(3)}=\alpha_M{}^{(4)}-3,$$

$$\alpha_M{}^{(4)}=\frac{1}{[(W+M)^2+\lambda^2]}\left\{\frac{3(\lambda^2-\mu^2)}{2}+3q_0(W+M)+\frac{(W+M)[(W+M)^2-\mu^2]}{W}\right\},$$

$$\alpha_M{}^{(5)}=\frac{1}{(t-\mu^2)[(W+M)^2+\lambda^2]}\left\{3q_0[2(W^2-M^2)+\lambda^2]+\frac{[(W+M)^2-\mu^2][2(W^2-M^2)+\lambda^2]}{W}\right.$$
$$\left.+\frac{3(\lambda^2-\mu^2)(W-3M)}{2}-3(W-M)[(W+M)^2+\lambda^2]\right\},$$

$$\alpha_M{}^{(6)}=\frac{1}{[(W+M)^2+\lambda^2]}\left\{\frac{-3(\lambda^2-\mu^2)}{2}-3q_0(W+M)+\frac{(W-M)[(W+M)^2-\mu^2]}{W}\right\},$$

$$\alpha_L{}^{(1)}=\frac{-3(\lambda^2-\mu^2+2q_0k_0)Mk_0}{W(k_0{}^2+\lambda^2)}+\frac{2k_0M}{W}\frac{[(W+M)^2-\mu^2]}{[(W+M)^2+\lambda^2]},$$

$$\alpha_L{}^{(2)}=\frac{2\lambda^2}{[2(W^2-M^2)+\lambda^2]}\alpha_L{}^{(5)},$$

$$\alpha_L{}^{(3)}=\alpha_L{}^{(4)},$$

$$\alpha_L{}^{(4)}=\frac{-3k_0(\lambda^2-\mu^2+2q_0k_0)}{2W(k_0{}^2+\lambda^2)}+\frac{k_0[(W+M)^2-\mu^2]}{W[(W+M)^2+\lambda^2]},$$

$$\alpha_L{}^{(5)}=\frac{-k_0[2(W^2-M^2)+\lambda^2]}{2W\lambda^2(t-\mu^2)}\left\{\frac{3(\lambda^2-\mu^2+2q_0k_0)(W-M)}{(k_0{}^2+\lambda^2)}+\frac{2(W+M)[(W+M)^2-\mu^2]}{[(W+M)^2+\lambda^2]}\right\},$$

$$\alpha_L{}^{(6)}=\frac{3k_0(W^2-M^2)(\lambda^2-\mu^2+2q_0k_0)}{2W\lambda^2(k_0{}^2+\lambda^2)}-\frac{k_0(W^2-M^2)[(W+M)^2-\mu^2]}{W\lambda^2[(W+M)^2+\lambda^2]},$$

$$\alpha_Q{}^{(1)}=\frac{3}{[(W+M)^2+\lambda^2]}\left\{q_0[(W+M)^2-\lambda^2]-\frac{2M\lambda^2q_0[(W+M)^2+\lambda^2]}{W(k_0{}^2+\lambda^2)}\right.$$
$$\left.+\frac{(\lambda^2-\mu^2)}{2W(k_0{}^2+\lambda^2)}\Big[2W\lambda^2(W+M)+k_0W[(W+M)^2-\lambda^2]+k_0M[(W+M)^2+\lambda^2]\Big]\right\},$$

$$\alpha_Q{}^{(2)}=\frac{2\lambda^2}{[2(W^2-M^2)+\lambda^2]}\alpha_Q{}^{(5)}+\frac{6(W-M)}{[2(W^2-M^2)+\lambda^2]},$$

$$\alpha_Q{}^{(3)}=\alpha_Q{}^{(4)}+3,$$

$$\alpha_Q{}^{(4)}=\frac{3q_0(W+M)}{[(W+M)^2+\lambda^2]}-\frac{3\lambda^2q_0}{W(k_0{}^2+\lambda^2)}+\frac{3(\lambda^2-\mu^2)}{4W}\left[\frac{2W\lambda^2+k_0(3W^2+4WM+M^2+\lambda^2)}{[(W+M)^2+\lambda^2](k_0{}^2+\lambda^2)}\right],$$

$$\alpha_Q{}^{(5)}=\frac{[2(W^2-M^2)+\lambda^2]}{2W(t-\mu^2)}\left\{\frac{6Wq_0}{[(W+M)^2+\lambda^2]}+\frac{6W(W-M)}{[2(W^2-M^2)+\lambda^2]}-\frac{6q_0(W-M)}{(k_0{}^2+\lambda^2)}\right.$$
$$+\frac{3(\lambda^2-\mu^2)}{2(k_0{}^2+\lambda^2)[(W+M)^2+\lambda^2][2(W^2-M^2)+\lambda^2]}[k_0(7W^3-7WM^2-3M^3+3W^2M+5W\lambda^2-3M\lambda^2)$$
$$\left.+2W(4M^3-4W^3+M\lambda^2-4W^2M+4M^2W-3W\lambda^2)]\right\},$$

$$\alpha_Q{}^{(6)}=\frac{3(W^2-M^2)q_0}{W(k_0{}^2+\lambda^2)}-\frac{3(W+M)q_0}{[(W+M)^2+\lambda^2]}+\frac{3(\lambda^2-\mu^2)}{4W(k_0{}^2+\lambda^2)[(W+M)^2+\lambda^2]}$$
$$\times[4W(W+M)^2+2W\lambda^2-k_0(3W^2+4WM+M^2+\lambda^2)].$$

As an example:

$$\beta_L{}^{(5)}=-\frac{3k_0[2(W^2-M^2)+\lambda^2](W-M)}{2W\lambda^2(t-\mu^2)(k_0{}^2+\lambda^2)}.$$

APPENDIX III

The crossed term of Eq. (21a) for the 3–3 amplitude is:

$$\frac{1}{12}\int_{-1}^{1}dx\int\frac{dW'^2}{W'^2-S_2}\bigg[(O_1x+P_2(x)O_2)A_1'-\frac{(1-x^2)}{2}\frac{O_3}{W'+M}(\beta A_2'+\lambda^2A_5')+\bigg(\frac{O_1x(t-\mu^2+\lambda^2)}{2(W'-M)}-\frac{O_2P_2(x)(t-\mu^2+\lambda^2)}{2(W'+M)}$$
$$+\frac{(1-x^2)O_3}{2}\bigg)(A_3'+A_4')+(O_1x(W'-M)-O_2P_2(x)(W'+M))A_4'-\bigg(\frac{O_1x}{W'-M}-\frac{O_2P_2(x)}{W'+M}\bigg)\lambda^2A_6'\bigg],$$

where $P_2(x)=\frac{1}{2}(3x^2-1)$, $O_i=(\mathfrak{F}_i/F_i)$ [Eq. (Ib)] and $A_i'=A_i{}^{\pm}-(A_i{}^{\pm})_{\text{tri-pion}}$, and are given in (17), (18).