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Coupling Constant for Small Binding Energy in Dispersion Theory

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The renormalized coupling constant associated with a bound state and its two composite particles is investigated as a function of binding energy using the N/D method of Chew and Mandelstam.

INTRODUCTION

It has often been suggested that some of the so-called elementary particles may be considered as bound states of a pair of elementary particles. In particular, it has been proposed recently that the renormalized coupling constant associated with the vertex of such a bound state and its composite particles should be related to the binding energy according to the zero effective range approximation.[1] As is well known, this is the case with the deuteron-neutron-proton vertex, where the coupling constant is the square of the asymptotic normalization of the deuteron wave function and is approximately determined by the deuteron binding energy and the nucleon mass. It has been proposed, for instance, that this relation should hold in a similar way for the Σ-Λ-π vertex, where the Σ hyperon is considered as an S-wave bound state of the Λ hyperon and the pion.[2,3]

Although the effective range theory was first obtained in the framework of nonrelativistic quantum mechanics,[4] it was subsequently shown that it follows also from unitarity and the analytic properties of the relativistic scattering matrix near physical thresholds,[5] and has been widely applied to the analysis of the interactions of elementary particles. Using the N/D method of Chew and Mandelstam,[5] we investigate the relation between coupling constant and binding energy of the zero effective range approximation by considering a relativistic S-wave elastic scattering amplitude of two particles which have a weakly bound state, satisfying unitarity and dispersion relations. We assume that the pole in the scattering amplitude due to the bound state corresponds to a zero in the D function, and study the behavior of the residue of the pole as a function of the binding energy. In nonrelativistic theory the magnitude of this residue is proportional to the square of the asymptotic bound-state wave function normalization, while in relativistic theory it is proportional to the square of the renormalized coupling constant of the vertex consisting of the bound state and the two scattering particles. We restrict ourselves first to scattering without inelastic thresholds, but later we take these into account.

THEORY

Let $A(s)$ be the relativistic S-wave elastic scattering amplitude of two particles of mass M and μ, respectively, which have a bound state with binding energy $B \ll M+\mu$. Assume that $A(s)$ satisfies dispersion relations and the elastic unitarity condition,

$$\operatorname{Im}A(s)=A^*(s)[q(s)/s^{\frac{1}{2}}]A(s) \quad \text{for} \quad s_1 \equiv (M+\mu)^2 \leq s, \tag{1}$$

where s is the square of the total center-of-mass energy and $q(s)=\{[s-(M+\mu)^2][s-(M-\mu)^2]/4s\}^{\frac{1}{2}}$ is the momentum of each particle. Then, according to Chew and Mandelstam,[5] $A(s)$ can be written in the form

$$A(s)=N(s)D^{-1}(s), \tag{2}$$

where

$$N(s)=\frac{1}{2\pi i}\int_\Gamma \frac{[A(s')]D(s')ds'}{(s'-s)}, \tag{3}$$

and

$$D(s)=1-\frac{1}{\pi}\int_{s_1}^{\infty}\frac{q(s')N(s')ds'}{s'^{\frac{1}{2}}(s'-s)}. \tag{4}$$

The function $[A(s)]$ is the discontinuity of $A(s)$ across the cut denoted by Γ, which we assume lies entirely in the unphysical region.

If $[A(s)]$ were known, $N(s)$ and $D(s)$ could be determined by solving the nonsingular integral equation obtained by substituting Eq. (4) in Eq. (3) or vice versa.[5] We shall assume here only that a solution exists, and that $D(s)$ has a zero at $s_B=(M+\mu-B)^2<s_1$ giving rise to a pole in $A(s)$ corresponding to the bound state. The residue Γ_B of this pole is given by

$$\frac{1}{\Gamma_B}\equiv\frac{1}{N(s_B)}\frac{dD(s_B)}{ds_B}=-\frac{1}{N(s_B)\pi}\int_{s_1}^{\infty}\frac{q(s')N(s')ds'}{s'^{\frac{1}{2}}(s'-s_B)^2}. \tag{5}$$

Since $N(s)$ is analytic and $q(s)\sim(s-s_1)^{\frac{1}{2}}$ near the threshold s_1, the integral in Eq. (5) diverges as the binding energy B approaches zero. For $B\ll(M+\mu)$ we obtain

$$\frac{2(M+\mu)^2}{\Gamma_B}=-\left(\frac{m}{2B}\right)^{\frac{1}{2}}+mr+O(B^{\frac{1}{2}}), \tag{6}$$

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[1] L. D. Landau, J. Exptl. Theor. Phys. (U.S.S.R.) **12**, 1294 (1961).
[2] Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **6**, 377 (1961).
[3] J. Bernstein and R. Oehme, Phys. Rev. Letters **6**, 639 (1961).
[4] R. G. Sachs, *Nuclear Theory* (Addison Publishing Company, Inc., Reading, Massachusetts, 1953), p. 49.
[5] G. F. Chew, *Dispersion Relations*, edited by G. R. Screaton (Interscience Publishers Inc., New York), 1961, p. 167.

where $m=M\mu/(M+\mu)$ is the reduced mass and[6]

$$mr=\frac{2}{\pi}\left\{1+\frac{1}{2}\frac{M-\mu}{M+\mu}\ln\frac{M}{\mu}+\frac{(M+\mu)^2}{N(s_1)\pi}\int_{s_1}^{\infty}\frac{q(s')[N(s_1)-N(s')]ds'}{s'^{\frac{1}{2}}(s'-s_1)^2}\right\}. \quad (7)$$

Equation (6) is the familiar expression for the pole residue in the effective range theory; one can verify by direct calculation that r, Eq. (7), is indeed the effective range r_e up to terms of order $B^{\frac{1}{2}}$, with r_e defined conventionally by

$$\tfrac{1}{2}r_e\equiv(d/dq^2)[s^{\frac{1}{2}}/A(s)+iq(s)]_{s=s_1}. \quad (8)$$

In nonrelativistic theory, $-\Gamma_B$ is proportional to the square of the asymptotic bound-state wave function normalization, while in relativistic theory it is proportional to the square of the renormalized coupling constant of the vertex consisting of the two scattering particles and the bound state. In either case $\Gamma_B<0$, and therefore we must have $mr<(m/2B)^{\frac{1}{2}}$ for a consistent interpretation of this pole as a bound state. We cannot evaluate the integral in Eq. (7) and hence determine r without specifying the discontinuity function $[A(s)]$ and obtaining $N(s)$ on the physical cut by solving the coupled equations (3) and (4). However, it seems plausible on the basis of simple models that if $[A(s)]$ gives rise to only a single bound state, this integral is positive.[7] In addition to this integral which depends on the dynamics, there are also kinematical contributions to r given by the constants in Eq. (7),[8] which do not depend on the discontinuity function $[A(s)]$ and in general will be of a different order of magnitude than the integral. If the integral is small we have $mr\cong 1$, while if it gives the main contribution to r, we have $1\ll mr$. In either case, the approximation of neglecting the range requires that $2B\ll m$. This condition is certainly fulfilled by the deuteron, but not, for instance, by the Σ hyperon, considered as a bound state of the Λ and the π, where $2B\approx m$.

For completeness we also include here the effective-range expansion for the scattering length a, which can be readily obtained by evaluating the scattering amplitude $A(s)$ at threshold, with one subtraction in the D function, Eq. (4), at the position of the bound state $s=s_B$.

$$\frac{1}{a}\equiv\frac{s_1^{\frac{1}{2}}D(s_1)}{N(s_1)}=-(2mB)^{\frac{1}{2}}+mrB+O(B^{\frac{3}{2}}). \quad (9)$$

We now drop the unphysical requirement of elastic unitarity, Eq. (7), at all energies, and assume for simplicity, that there are n two-particle channels with thresholds at $s_i=(M_i+\mu_i)^2$, $i=1\cdots n$, where M_i and μ_i are the masses of the two particles in the ith channel and $s_1<s_2\cdots<s_n$. The channel $i=1$ with the lowest threshold corresponds to the channel under consideration. Let $A(s)$ denote the $n\times n$ scattering matrix

$$A(s)=\begin{bmatrix}A_{11}(s) & A_{12}(s) & \cdots & A_{1n}(s)\\ A_{21}(s) & & & \vdots\\ \vdots & & & \\ A_{n1}(s) & & \cdots & A_{nn}(s)\end{bmatrix}, \quad (10)$$

where $A_{ij}(s)$ is the S-wave scattering amplitude of the particles in channel i into the particles in channel j, and satisfies dispersion relations. Equations (1)–(4) are now valid as matrix equations[9] with $N(s)$ and $D(s)$ as $n\times n$ matrix functions, and $q(s)$ as a diagonal matrix with elements $q_i(s)\theta(s-s_i)$, where

$$q_i(s)=\left(\frac{[s-(M_i+\mu_i)^2][s-(M_i-\mu_i)^2]}{4s}\right)^{\frac{1}{2}}$$

is the momentum of the particles in channel i, and $\theta(x)$ is the step function

$$\theta(x)=1,\quad 0<x$$
$$=0,\quad x<0.$$

The bound state now corresponds to a zero in the determinant of $D(s)$ at $s=s_B<s_1$. The determinant is given by

$$\det D(s)=\sum_{i_1\cdots i_n}(-)^P D_{1,i_1}(s)D_{2,i_2}(s)\cdots D_{n,i_n}(s), \quad (11)$$

where the summation is carried over the $n!$ permutations of the indices, with $P=0$ (1) for even (odd) permutations. Since each function $D_{i,j}(s)$ is analytic in the cut s plane and $D_{i,i}(s)\to 1$ asymptotically, we have

$$\det D(s)=1+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Im}\,\det D(s')ds'}{(s'-s)}. \quad (12)$$

From Eqs. (4) and (11) we obtain for the imaginary part of $\det D(s)$:

$$\mathrm{Im}\,\det D(s)=\sum_{l=1}^{n}-\frac{q_l(s)}{s^{\frac{1}{2}}}G_l(s)\theta(s-s_l), \quad (13)$$

[6] A similar expression was obtained in the case of a nonrelativistic scattering amplitude by J. D. Bjorken (private communication).

[7] For instance, if $[A(s)]=-\Gamma_0\delta(s-s_0)$, where $\Gamma_0>0$ and $s_0<s_1$, $N(s)$ is monotonically decreasing on the physical cut $s>s_1$, which is a sufficient condition for this integral to be positive. Replacing the left-hand cut by a pole with positive residue is equivalent to the Lee model, [T. D. Lee, Phys. Rev. **95**, 1329 (1954)], where this pole is interpreted as a ghost state.

[8] These constants are due to relativistic kinematics and are not present for a nonrelativistic scattering amplitude.

[9] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960); J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961); R. Blankenbecler, *ibid.* **122**, 983 (1961).

where

$$G_l(s)=\sum_{i_1\cdots i_n}(-)^P D^-{}_{1,i_1}(s)D^-{}_{2,i_2}(s)\cdots D^-{}_{l-1,i_{l-1}}(s)\times N_{l,i_l}(s)D^+{}_{l+1,i_{l+1}}(s)\cdots D^+{}_{n,i_n}, \quad (14)$$

and

$$D^{\pm}{}_{ij}(s)=D_{ij}(s\pm i\epsilon), \quad s_i\leq s.$$

The residue of the pole in the elastic scattering amplitude $A_{11}(s)$ is obtained by evaluating

$$\frac{1}{\Gamma_{11}}=\Big[\sum_{l=1}^{n}N_{1l}(s)D_{1l}^{-1}(s)\,\det D(s)\Big]^{-1}\frac{d}{ds}\times\det D(s)\Big|_{s=s_B}. \quad (15)$$

Substituting Eqs. (12)–(14), we obtain

$$\frac{1}{\Gamma_{11}}=\sum_{l=1}^{n}-\frac{1}{G_1(s_B)\pi}\int_{s_l}^{\infty}\frac{q_l(s')G_l(s')ds'}{s'^{\frac{1}{2}}(s'-s_B)^2}. \quad (16)$$

The first term in this sum is of the same form as Eq. (5) where $G_1(s)$ replaces $N(s)$, and is also analytic at the elastic threshold $s_1=(M+\mu)^2$, as can be readily verified from Eq. (14). The remaining terms approach a constant as $s_B\to s_1$. Therefore, Eq. (6) remains valid, but now the effective range r has a contribution from the inelastic thresholds given by

$$\frac{2(M_1+\mu_1)^3}{M_1\mu_1}\sum_{l=2}^{n}-\frac{1}{G_1(s_1)\pi}\int_{s_l}^{\infty}\frac{q_l(s')G_l(s')ds'}{s'^{\frac{1}{2}}(s'-s_B)^2}, \quad (17)$$

in addition to the contributions of the unphysical singularities given in Eq. (7), with $N(s)$ replaced by $G_1(s)$. The same is true in the expression for the scattering length Eq. (9).

If the inelastic thresholds $s_i=(M_i+\mu_i)^2$ are very far above the elastic threshold, we can expect the contribution of Eq. (17) to the effective range to be small, and since $G_1(s)\cong N_{11}(s)$ for $s\ll s_2$, Eq. (7) should remain approximately valid. However, if this is not the case as, for instance, in π-Λ scattering, where the π-Σ and $\bar{K}$-N inelastic thresholds are located at a distance comparable to that of the nearest unphysical singularity, the contribution to the effective range of the inelastic threshold, Eq. (17), may not be neglected.

CONCLUSION

The main conclusion that we can derive from these considerations is that the effective range in a relativistic S-wave scattering amplitude of two particles of mass M and μ, respectively, which have a weakly bound state, i.e., the binding energy $B\ll M+\mu$, is at least of order $1/m$ where $m=M\mu/(M+\mu)$ is the reduced mass. Therefore, according to Eqs. (6) and (9), the effective range cannot be neglected in estimating the pole residue or renormalized coupling constant and the scattering length, unless $2B\ll m$, a condition which is not fulfilled by any of the elementary particle vertices.

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