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Application of the Mandelstam Representation to Photoproduction of Pions from Nucleons

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The Mandelstam representation is applied to the invariant amplitudes for photoproduction. By treating gauge invariance as a subsidiary condition, it is shown that the fixed-momentum-transfer dispersion relations of Chew, Goldberger, Low, and Nambu (CGLN) are probably valid without subtractions for the $(-)$ amplitudes while a three-pion resonance would perhaps require a subtraction in the $(+)$ amplitudes. The two-pion resonance will certainly require a subtraction for the (0) amplitudes, but to a good approximation the contribution of the two-pion intermediate state is found to produce a simple additive correction to the CGLN (0) formula. The strength of this new term is determined by a parameter Λ, which has been introduced elsewhere in treating the photon, three-pion problem. Otherwise, the form of the new term can be expressed in terms of nucleon electromagnetic form factors. Finally, the photoproduction amplitudes are calculated in the threshold region, and an estimate of the size of Λ is made.

I. INTRODUCTION

RECENTLY Chew and Mandelstam have proposed a method for calculating the behavior of systems of strongly interacting particles and have applied it to the problem of pion-pion scattering.[1] This method is based on Mandelstam's generalization of dispersion relations,[2] which provides a means of extending scattering amplitudes into the complex plane for both the energy- and momentum-transfer variables. The new method has already been applied to the process $\gamma+\pi \leftrightarrow \pi+\pi$,[3] $\pi+\pi \leftrightarrow N+\bar{N}$,[4] and $N+N \leftrightarrow N+N$,[5] in addition to π-π scattering. Our purpose here is to extend the new approach to pion photoproduction from nucleons and in particular to investigate the effect of the pion-pion interaction on photoproduction.

In the case of photoproduction, the invariant amplitudes satisfying the Mandelstam representation will be the scattering amplitudes for the three processes shown in Fig. 1(a)–(c) when the variables are in the appropriate physical region for each process. The Mandelstam singularities appear in the energy variables for each of these processes, $N+\gamma \leftrightarrow N+\pi$, $\gamma+\pi \leftrightarrow \bar{N}+N$, and $\gamma+\bar{N} \leftrightarrow \pi+\bar{N}$. Their location is determined by the masses of intermediate states that have the same quantum numbers as the initial and final states for the reaction in question.

In the following section, the invariant amplitudes are defined in terms of invariant spin and isotopic-spin matrices. The angular-momentum decomposition of the invariant amplitudes in terms of multipoles is given for photoproduction in Sec. III, expressing the connection between the invariant amplitudes and the eigenamplitudes for this channel. Section IV deals with the angular-momentum decomposition for $\gamma+\pi \leftrightarrow N+\bar{N}$.

In Sec. V, the invariant amplitudes are expressed in the Mandelstam form. A general procedure to obtain a complete solution of the photoproduction problem is discussed in Secs. VI and VII. Finally Secs. VIII and IX deal with a low-energy approximation for the photoproduction amplitudes, based on the assumption that pion-pion and pion-nucleon interactions are both dominated by P-wave resonances.

II. THE INVARIANT AMPLITUDE

Let P_1 and P_2 denote the initial and final nucleon four-vector momenta and Q and K represent those of the pion and the photon, respectively. Since we will consider all three processes in Fig. 1, it is convenient to define the variables:

$$s=-(P_1+K)^2,\quad t=-(Q-K)^2,\quad \bar{s}=-(P_2-K)^2, \tag{2.1}$$

which are the squares of the total energy in the barycentric system for the three processes in Fig. 1. The amplitudes satisfying the Mandelstam representation will have singularities in s, t, and $\bar{s}$, corresponding to the possible intermediate states for each channel.

Conservation of energy-momentum,

$$P_1+K=P_2+Q, \tag{2.2}$$

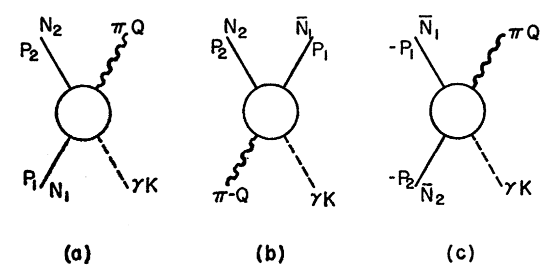

FIG. 1. The three channels of the pion, photon, two-nucleon problem.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 (1959); **115**, 1752 (1959).

[3] H. S. Wong, Phys. Rev. Letters **5**, 70 (1960) and private communication; M. Gourdin and A. Martin, Nuovo cimento **16**, 78 (1960).

[4] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

[5] D. Y. Wong, Phys. Rev. Letters **2**, 406 (1959); H. P. Noyes and D. Y. Wong, *ibid.* **3**, 191 (1959); and M. L. Goldberger, M. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

and the mass-shell restrictions, $P_1^2=P_2^2=-M^2$, $Q^2=-1$,[5a] and $K^2=0$, lead to

$$s+t+\bar{s}=2M^2+1. \tag{2.3}$$

The most general form of the photomeson transition matrix element has been shown by Chew, Goldberger, Low, and Nambu[6] (hereafter CGLN) to be

$$T=\sum_{i=1}^{4}[A_i^{+}g_\beta^{+}+A_i^{-}g_\beta^{-}+A_i^{0}g_\beta^{0}]M_i, \tag{2.4}$$

where the A's are scalar functions of s, $\bar{s}$, and t. The four gauge-invariant spin matrices introduced by CGLN are

$$M_1=i\gamma_5\gamma\cdot\epsilon\gamma\cdot K, \tag{2.5a}$$

$$M_2=2i\gamma_5(P\cdot\epsilon Q\cdot K-P\cdot KQ\cdot\epsilon), \tag{2.5b}$$

$$M_3=\gamma_5(\gamma\cdot\epsilon Q\cdot K-\gamma\cdot KQ\cdot\epsilon), \tag{2.5c}$$

$$M_4=2\gamma_5(\gamma\cdot\epsilon P\cdot K-\gamma\cdot KP\cdot\epsilon-iM\gamma\cdot\epsilon\gamma\cdot K), \tag{2.5d}$$

where ϵ is the photon polarization and $P=\frac{1}{2}(P_1+P_2)$. The isotopic-spin matrices have the following form:

$$g_\beta^{+}=\delta_{\beta3}, \tag{2.6a}$$

$$g_\beta^{-}=\tfrac{1}{2}[\tau_\beta,\tau_3], \tag{2.6b}$$

$$g_\beta^{0}=\tau_\beta, \tag{2.6c}$$

β being the isotopic-spin index of the pion.

The crossing relations obtained by CGLN give the symmetry of the A's under exchange of s and $\bar{s}$. The symmetric functions are $A_1^{(+,0)}$, $A_2^{(+,0)}$, $A_3^{(-)}$, and $A_4^{(+,0)}$, while $A_1^{(-)}$, $A_2^{(-)}$, $A_3^{(+,0)}$, and $A_4^{(-)}$ are antisymmetric.

III. KINEMATICS FOR THE PHOTOPRODUCTION CHANNEL

For the photoproduction channel we have

$$s=(E_1+k)^2=(E_2+\omega)^2, \tag{3.1a}$$

$$t=1-2\omega k+2qk\cos\theta, \tag{3.1b}$$

and

$$s=M^2-2E_2k-2qk\cos\theta. \tag{3.1c}$$

With reference to the barycentric system, q and k are the magnitudes of the meson and photon momenta, $E_1=(k^2+M^2)^{\frac{1}{2}}$ and $E_2=(q^2+M^2)^{\frac{1}{2}}$ are the initial and final nucleon energies, $\omega=(q^2+1)^{\frac{1}{2}}$ is the meson energy, and $\cos\theta=(\mathbf{Q}\cdot\mathbf{K})/qk$ defines the production angle.

The differential cross section for meson production in the barycentric system was written by CGLN in the form

$$d\sigma/d\Omega=(q/k)|x_f\mathfrak{F}x_i|^2, \tag{3.2}$$

where

$$q=\{[s-(M+1)^2][s-(M-1)^2]/4s\}^{\frac{1}{2}},$$
$$k=(s-M^2)/2\sqrt{s},$$

and x_f, x_i are the final, initial Pauli spinor for the nucleon. The amplitude $\mathfrak{F}$ is given by

$$\mathfrak{F}=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathfrak{F}_1+(\boldsymbol{\sigma}\cdot\mathbf{Q}\boldsymbol{\sigma}\cdot\mathbf{K}\times\boldsymbol{\varepsilon}/qk)\mathfrak{F}_2+i(\boldsymbol{\sigma}\cdot\mathbf{K}\mathbf{Q}\cdot\boldsymbol{\varepsilon}/qk)\mathfrak{F}_3+i(\boldsymbol{\sigma}\cdot\mathbf{Q}\mathbf{Q}\cdot\boldsymbol{\varepsilon}/q^2)\mathfrak{F}_4, \tag{3.3}$$

where the $\mathfrak{F}$'s are functions of A_1, A_2, A_3, and A_4. Still following CGLN we define F_1, F_2, F_3, and F_4 as follows:

$$F_1=4\pi\frac{2W}{W-M}\frac{\mathfrak{F}_1}{[(E_2+M)(E_1+M)]^{\frac{1}{2}}}$$
$$=A_1+(W-M)A_4-\frac{t-1}{2(W-M)}(A_3-A_4), \tag{3.4}$$

$$F_2=4\pi\frac{2W}{W-M}\left(\frac{E_2+M}{E_1+M}\right)^{\frac{1}{2}}\frac{\mathfrak{F}_2}{q}$$
$$=-A_1+(W+M)A_4-\frac{t-1}{2(W+M)}(A_3-A_4), \tag{3.5}$$

$$F_3=4\pi\frac{2W}{\mathrm{W}-M}\frac{\mathfrak{F}_3}{[(E_2+M)(E_1+M)]^{\frac{1}{2}}q}$$
$$=(W-M)A_2+(A_3-A_4), \tag{3.6}$$

$$F_4=4\pi\frac{2W}{W-M}\left(\frac{E_2+M}{E_1+M}\right)^{\frac{1}{2}}\frac{\mathfrak{F}_4}{q^2}$$
$$=-(W+M)A_2+(A_3-A_4). \tag{3.7}$$

It is well known that, for photoproduction, the unitarity condition requires the phase of an amplitude leading to an outgoing pion-nucleon state of definite angular-momentum, isotopic spin, and parity to be the same as the phase of the pion-nucleon scattering amplitude leading to the same final state.[7] To use the unitarity condition we must decompose the $\mathfrak{F}$'s into definite parity eigenamplitudes. This angular-momentum decomposition has been carried out by CGLN; they obtain

$$\mathfrak{F}_1=\sum_{l=0}[lM_{l+}+E_{l+}]P_{l+1}'(x)+[(l+1)M_{l-}+E_{l-}]P_{l-1}'(x), \tag{3.8}$$

$$\mathfrak{F}_2=\sum_{l=1}[(l+1)M_{l+}+lM_{l-}]P_l'(x), \tag{3.9}$$

$$\mathfrak{F}_3=\sum_{l=1}[E_{l+}-M_{l+}]P_{l+1}''(x)+[E_{l-}+M_{l-}]P_{l-1}''(x), \tag{3.10}$$

$$\mathfrak{F}_4=\sum[M_{l+}-E_{l+}-M_{l-}-E_{l-}]P_l''(x), \tag{3.11}$$

where $x=\cos\theta$. The energy-dependent amplitudes $M_{l\pm}$ and $E_{l\pm}$ refer to transitions initiated by magnetic and

[5a] The units employed throughout are $\hbar=c=$pion mass$=1$ and the metric used is $g_i=1$ for $i=1$, 2, or 3 and $g_0=-1$.

[6] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[7] K. M. Watson, Phys. Rev. **95**, 228 (1954).

electric radiation, respectively, leading to final states of orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$.

These expressions can be inverted, yielding

$$M_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx\left[\mathfrak{F}_1P_l(x)-\mathfrak{F}_2P_{l+1}(x)-\mathfrak{F}_3\frac{P_{l-1}(x)-P_{l+1}(x)}{2l+1}\right] \quad (3.12)$$

for $l>0$,

$$E_{l+}=\frac{1}{2(l+1)}\int_{-1}^{1}dx\left[\mathfrak{F}_1P_l(x)-\mathfrak{F}_2P_{l+1}(x)+\mathfrak{F}_3l\frac{P_{l-1}(x)-P_{l+1}(x)}{2l+1}+\mathfrak{F}_4(l+1)\frac{P_l(x)-P_{l+2}(x)}{2l+3}\right], \quad (3.13)$$

$$M_{l-}=\frac{1}{2l}\int_{-1}^{1}dx\left[-\mathfrak{F}_1P_l(x)+\mathfrak{F}_2P_{l-1}(x)+\mathfrak{F}_3\frac{P_{l-1}(x)-P_{l+1}(x)}{2l+1}\right] \quad (3.14)$$

for $l>0$, and

$$E_{l-}=\frac{1}{2l}\int_{-1}^{1}dx\left[\mathfrak{F}_1P_l(x)-\mathfrak{F}_2P_{l-1}(x)-\mathfrak{F}_3(l+1)\frac{P_{l-1}(x)-P_{l+1}(x)}{2l+1}-\mathfrak{F}_4l\frac{P_{l-2}(x)-P_l(x)}{2l-1}\right] \quad (3.15)$$

for $l>1$.

Superscripts $(+,-,0)$ may be added to each quantity in Eqs. (3.2) to (3.15) to designate its isotopic spin dependence.

IV. ANGULAR-MOMENTUM DECOMPOSITION OF THE SCATTERING AMPLITUDE FOR THE PROCESS $\gamma+\pi\rightarrow N+N$

In a discussion of the kinematics of the process $\gamma+\pi\rightarrow N+\bar{N}$ depicted in Fig. 1(b), it is useful to introduce the four vectors P_1' and Q' representing the energy-momentum of the antinucleon and pion. We can write

$$P_1'=-P_1, \quad (4.1a)$$

and

$$Q'=-Q. \quad (4.1b)$$

Then Eq. (2.2) becomes

$$Q'+K=P_2+P_1'. \quad (4.2)$$

Again, expressing s, t, and $\bar{s}$ in terms of P_1' and Q', we have

$$s=-(K-P_1')^2=M^2-2Ek'-2pk'\cos\theta', \quad (4.3a)$$

$$t=-(Q'+K)^2=(2E)^2, \quad (4.3b)$$

and

$$\bar{s}=-(P_2-K)^2=M^2-2Ek'+2pk'\cos\theta', \quad (4.3c)$$

where p and k' are the magnitudes of the nucleon and photon momenta, E is the nucleon energy, and $\cos\theta'=\mathbf{P}_2\cdot\mathbf{K}/pk'$, all in the barycentric system. We can write p and k' as

$$k'=(t-1)/2\sqrt{t}, \quad (4.4a)$$

$$p=\tfrac{1}{2}(t-4M^2)^{\frac{1}{2}}. \quad (4.4b)$$

The S matrix for $\pi+\gamma\rightarrow N+\bar{N}$ is

$$S_{fi}=\frac{i}{(2\pi)^2}M\frac{\delta^4(P_2+P_1'-K-Q')\bar{u}(P_2)T(-P_1',P_2,-Q',K)v(P_1')}{(4E_1E_2\omega k')^{\frac{1}{2}}}, \quad (4.5)$$

where E_1, E_2, ω, and k' are the energies of the four particles in whatever reference system is employed.

The differential cross section for $\gamma+\pi\rightarrow N+\bar{N}$ in the barycentric system is

$$(d\sigma/d\Omega)=(p/k')|x_NGy_{\bar{N}}|^2, \quad (4.6)$$

where

$$G=(\mathbf{p}_2\cdot\boldsymbol{\varepsilon}/p)G_1+(i\boldsymbol{\sigma}\cdot\mathbf{p}_2\times\boldsymbol{\varepsilon}/p)G_2+(i\boldsymbol{\sigma}\cdot\mathbf{p}_2\mathbf{p}_2\cdot\mathbf{K}\times\boldsymbol{\varepsilon}/p^2k')G_3+(i\boldsymbol{\sigma}\cdot\mathbf{K}\times\boldsymbol{\varepsilon}/k')G_4, \quad (4.7)$$

and x_N, $y_{\bar{N}}$ are the nucleon and antinucleon Pauli spinors. The G's are the following functions of A_1, A_2, A_3, and A_4:

$$G_1=(k'p/16\pi E)[A_1+tA_2], \quad (4.8)$$

$$G_2=-(k'p/4\pi)A_3, \quad (4.9)$$

$$G_3=[(M-E)k'/8\pi E][A_1+(t)^{\frac{1}{2}}A_4], \quad (4.10)$$

$$G_4=(k'/16\pi E)[2MA_1-tA_4]. \quad (4.11)$$

The angular-momentum decomposition of the G's is now obtained by using the helicity amplitudes treated by Jacob and Wick.[8] The resulting expansion is

$$G_1=-\textstyle\sum_J(J+\frac{1}{2})\beta_J^-P_J'(x'), \quad (4.12)$$

$$G_2=-\tfrac{1}{2}\textstyle\sum_J\{\alpha_J^-[JP_{J+1}''(x')+(J+1)P_{J-1}''(x')]+(2J+1)\alpha_J^+P_J''(x')\}, \quad (4.13)$$

$$G_3=+\tfrac{1}{2}\textstyle\sum_J\{\alpha_J^+[JP_{J+1}''(x')+(J+1)P_{J-1}''(x')]-(2J+1)\alpha_J^-P_J''(x')-(2J+1)\beta_J^+P_J'(x')\}, \quad (4.14)$$

$$G_4=-\tfrac{1}{2}\textstyle\sum_J\{\alpha_J^+[JP_{J+1}''(x')+(J+1)P_{J-1}''(x')]-(2J+1)\alpha_J^-P_J''(x'), \quad (4.15)$$

[8] M. Jacob and G. C. Wick, Ann. phys. 7, 404 (1959).

where

$$\alpha_J^{(\pm)}=\frac{T_J(+,-,1)\pm T_J(-,+,1)}{J(J+1)(2pk')^{\frac{1}{2}}}, \quad (4.16a)$$

$$\beta_J^{(\pm)}=\frac{T_J(+,+,1)\pm T_J(-,-,1)}{[J(J+1)]^{\frac{1}{2}}(2pk')^{\frac{1}{2}}}, \quad (4.16b)$$

and $T_J(\pm,\pm,1)$ are the T-matrix elements for transitions initiated by a photon of helicity $+1$ producing a nucleon of helicity $\pm\frac{1}{2}$ and an antinucleon of helicity $\pm\frac{1}{2}$ with total angular momentum J. The first argument refers to nucleon and the second to the antinucleon. The photon phase has been adjusted to make the A_i's real when the T_J's are real.

The energy-dependent amplitudes α_J^+ and β_J^+ represent transitions initiated by magnetic radiation leading to triplet nucleon-antinucleon final states of parity $(-1)^J$ and total angular momentum J. Electric transitions leading to a triplet final state of parity $(-1)^{J+1}$ are represented by α_J^-, while β_J^- represents an electric transition to a singlet final state of angular momentum J.

Again one may add superscripts to each term of Eqs. (4.7) to (4.16) to denote the term's isotopic spin character. Some physical meaning of g^0, and g^+, and g^- for this channel is obtained by noticing that the isotopic-spin projection operators for the process $\pi+\pi^0\rightarrow N+\bar{N}$ given by Frazer and Fulco,[4]

$$P_0(\beta,3)=\delta_{\beta 3}/\sqrt{6}, \quad (4.17)$$

and

$$P_1(\beta,3)=\tfrac{1}{4}[\tau_\beta,\tau_3] \quad (4.18)$$

are just proportional to g^+ and g^- while g^0 leads only to the $I=1$ amplitude.

V. THE MANDELSTAM REPRESENTATION

According to Mandelstam's postulate, each element of the transition matrix is an analytic function of the momenta except for the dynamical singularities corresponding to the three physical processes which this matrix represents. By taking appropriate traces over the nucleon and photon spin indices, scalars can be constructed which are analytic functions of the momenta, and thus by the Hall-Wightman theorem[9] are analytic functions of the scalar products of the momenta. These scalars are then assumed to satisfy the Mandelstam representation.

The analytic scalars will be linear combinations of the scalar amplitudes defined in terms of the spin matrices. If the transformation between these two sets of amplitudes introduces no kinematic singularities, both sets will satisfy the Mandelstam representation. Since the Dirac equation has been used to reduce the number of independent amplitudes we must remove that portion of the general T matrix which vanishes between nucleon spinors. This is easily done by use of projection operators. The fact that gauge invariance has also been used in constructing the A_i's is more serious as the nongauge invariant terms in the general T matrix cannot be removed by projection in the case of physical photons. To avoid this difficulty and also to see the effect of the gauge condition on the analyticity, gauge invariance will be considered as a subsidiary condition to be imposed after the analyticity properties of the non-gauge invariant amplitudes have been investigated. In this case the most general form of T is

$$T=\sum_{i=1}^{8}B_i(s,t,\bar{s})N_i(P_1,P_2,K,\epsilon,\gamma), \quad (5.1)$$

where the N_i's are all of the independent Lorentz-invariant matrices that can be formed containing γ and ϵ, and the B_i's are scalar functions of s, t, and $\bar{s}$. A suitable set of N's is

$$\begin{aligned}
&N_1=i\gamma_5\gamma\cdot\epsilon\gamma\cdot K, && N_5=\gamma_5\gamma\cdot\epsilon,\\
&N_2=2i\gamma_5 P\cdot\epsilon, && N_6=\gamma_5\gamma\cdot KP\cdot\epsilon,\\
&N_3=2i\gamma_5 Q\cdot\epsilon, && N_7=\gamma_5\gamma\cdot KK\cdot\epsilon,\\
&N_4=2i\gamma_5 K\cdot\epsilon, && N_8=\gamma_5\gamma\cdot KQ\cdot\epsilon,
\end{aligned} \quad (5.2)$$

by which we will define our B's. Gauge invariance requires that the B's satisfy the following equations:

$$(s-\bar{s})B_2=2(t-1)B_3, \quad (5.3)$$

$$B_5+\tfrac{1}{4}(\bar{s}-s)B_6+\tfrac{1}{2}(t-1)B_8=0. \quad (5.3)$$

If these conditions are imposed on T, then the A's and B's have the following connection:

$$\begin{aligned}
&A_1=B_1-MB_6, && A_3=-B_8,\\
&A_2=2B_2/(t-1), && A_4=-\tfrac{1}{2}B_6.
\end{aligned} \quad (5.4)$$

In the Appendix it is shown that B's contain no kinematical singularities due to the choice of invariant spin matrices, and therefore will satisfy the spectral representation proposed by Mandelstam.

We may now express the B's as

$$\begin{aligned}
B_i=&\frac{R_s{}^i}{s-M^2}+\frac{R_t{}^i}{t-1}+\frac{R_{\bar{s}}{}^i}{\bar{s}-M^2}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{\rho_s{}^i(s')}{s'-s}\\
&+\frac{1}{\pi}\int_4^{\infty}dt'\frac{\rho_t{}^i(t')}{t'-t}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{\rho_{\bar{s}}{}^i(\bar{s}')}{\bar{s}'-\bar{s}}\\
&+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_4^{\infty}dt'\frac{b_{12}{}^i(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{b_{13}{}^i(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\
&+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}d\bar{s}'\int_4^{\infty}dt'\frac{b_{23}{}^i(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)}.
\end{aligned} \quad (5.5)$$

[9] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 31, No. 5 (1957).

The one-dimensional spectral functions in Eq. (5.5) represent reducible Feynman diagrams which have the same form as the diagrams that produce the poles. Thus on the basis of perturbation theory, an amplitude that has a pole in a particular variable will in general also have a one-dimensional spectral function in that variable. The possibility of an over-all subtraction constant (independent of s, t, and $\bar{s}$) or of polynomials multiplying the one-dimensional terms is removed by the unitary requirements on the asymptotic behavior of the eigenamplitudes in each channel.

Since A_1, A_3, and A_4 contain no kinematic factors in their relation to the B's, they have the same representation as the B's [Eq. (5.5)]; furthermore, since they have no pole in the t variable, they will have no one-dimensional spectral function in that variable.

The gauge condition, Eq. (5.3), is now applied to B_2 and B_3. We see that if B_3 is to remain finite as s approaches infinity then $R_t^{(2)}$ and ρ_t^2 must be zero. If Eq. (5.3) is evaluated at $t=1$, we obtain

$$(s-M^2)B_2(s,1)=R_t^{(3)}, \tag{5.6}$$

where $R_t^{(3)}$ is a known constant. We can take advantage of this relation by making a subtraction at $t=1$, obtaining a spectral representation for

$$B_2/(t-1)=-\tfrac{1}{2}A_2.$$

The resulting form for A_2 is

$$\begin{aligned} A_2^{(\pm\,0)}=&-\frac{e_rg_r}{t-1}\left(\frac{1}{s-M^2}\pm\frac{1}{\bar{s}-M^2}\right)\\ &+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_4^{\infty}dt'\frac{a_{12}^{(2)(+\,-\,0)}(s',t')}{(s'-s)(t'-t)}\\ &+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{a_{13}^{(2)(+\,-\,0)}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\ &+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}d\bar{s}'\int_4^{\infty}dt'\frac{a_{23}^{(2)(+\,-\,0)}(s',t')}{(\bar{s}'-\bar{s})(t'-t)}. \end{aligned} \tag{5.7}$$

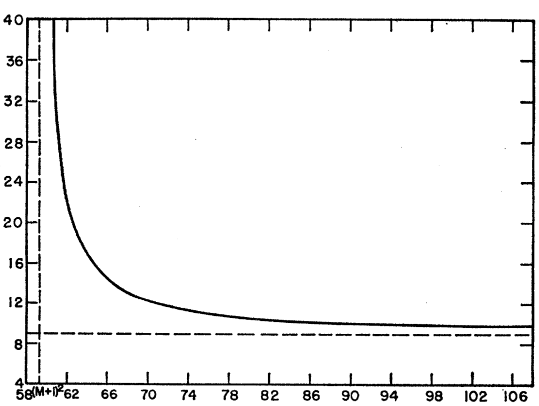

FIG. 2. The boundary curve for $a_{12}^{(i)(+,-)}$. The dashed lines are the asymptotes of the curve.

It should be noted that this subtraction procedure has removed all one-dimensional spectral terms.

The crossing condition now requires

$$a_{12}^{(i)(\pm,0)}(s',t')=\pm a_{23}^{(i)(\pm,0)}(s',t') \tag{5.8}$$

and

$$a_{13}^{(i)(\pm,0)}(s',\bar{s}')=\pm a_{13}^{(i)(\pm,0)}(\bar{s}',s'), \tag{5.9}$$

for $i=1$, 2, 3, 4. The upper sign in Eqs. (5.7) through (5.9) is to be used with the even A's, while the lower goes with the odd functions.

The double spectral functions $a_{12}^{(i)}$, $a_{13}^{(i)}$, and $a_{23}^{(i)}$ are real and nonzero within the regions discussed below.

The Γ_i's, the residues of the poles in the A_i's, have been given by CGLN:

$$\Gamma_1^{(\pm,0)}=e_rg_r/2, \tag{5.10a}$$

$$\Gamma_3^{(\pm)}=\Gamma_4^{(\pm)}=-\tfrac{1}{2}g_r(\mu_{p_r}'-\mu_{n_r}), \tag{5.10b}$$

$$\Gamma_3^{(0)}=\Gamma_4^{(0)}=-\tfrac{1}{2}g_r(\mu_{p_r}'+\mu_{n_r}), \tag{5.10c}$$

where μ_{p_r}' and μ_{n_r} are the rationalized anomalous static nucleon moments and e_r and g_r are the rationalized and renormalized electronic charge and pion-nucleon coupling constant, respectively. These have the following values:

$$e_r^2/4\pi=1/137,\quad g_r^2/4\pi\simeq 14.$$

Mandelstam has given a general method for determining the regions in which the spectral functions a_{12}, a_{13}, and a_{23} are nonzero.[2] These boundaries result from considering the lowest mass intermediate states possible in any pair of the variables s, t, and $\bar{s}$.

Examining first the t spectrum, we must consider the processes $\gamma+\pi\rightarrow n\pi$. Conservation of G parity requires n to be odd for the isotopic-scalar part. Thus the $(+,-)$ spectra will have intermediate states containing odd numbers of pions, starting with the three-pion state, while the 0 spectrum will contain only intermediate states with even numbers of pions, starting with the two pion state. The s and $\bar{s}$ spectra both start with the pion and nucleon intermediate state; with no differences for the various isotopic combinations.

Following Mandelstam's method, we find that the region in which the functions $a_{12}^{(i)(+,-)}$ are nonzero is bounded by the following curve (see Fig. 2),

$$[s-(M+1)^2][s-(M-1)^2](t-9)-8(3s-M^2+1)=0. \tag{5.11}$$

The spectral functions $a_{12}^{(i)0}$ are bounded by the two curves (see Fig. 3),

$$[s-(M+2)^2][s-(M-2)^2]t(t-4)-2t(9s+31M^2-28)-(4M^2-1)=0, \tag{5.12a}$$

and

$$[s-(M+1)^2][s-(M-1)^2]t(t-16)-8t(9s+M^2-1)-16(M^2-1)=0. \tag{5.12b}$$

The curves bounding $a_{23}^{(i)(+,-)}$ and $a_{23}^{(i)0}$ can be obtained from Eqs. (5.11), (5.12a), and (5.12b) by

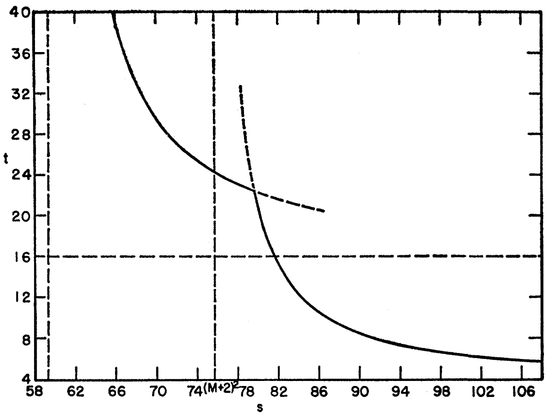

FIG. 3. The boundary curve for $a_{12}^{(i)(0)}$. The dashed lines are the asymptotes of the curve.

changing s to $\bar{s}$. The spectral functions $a_{13}^{(i)(+,-,0)}$ are bounded by the curve (see Fig. 4)

$$[\bar{s}-(M+1)^2][\bar{s}-(M-1)^2][s-(M+1)^2]\times[s-(M-1)^2]-(4M^2-1)[2s\bar{s}-2(M^2-1)(s+\bar{s})+2M^4-1]=0. \quad (5.13)$$

VI. ONE-DIMENSIONAL DISPERSION RELATIONS

It is now possible to obtain one-dimensional dispersion relations with either s, t, or $\bar{s}$ held fixed. We define the following functions:

$$\mathrm{Im}_{\mathrm{I}}A_i(s,t)=\rho_i(s)+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{a_{13}^{(i)}(s,\bar{s}')}{\bar{s}'+t+s-2M^2-1}+\frac{1}{\pi}\int_4^{\infty}dt'\frac{a_{12}^{(i)}(s,t')}{t'-t}, \quad (6.1\mathrm{a})$$

$$\mathrm{Im}_{\mathrm{II}}A_i(s,t)=\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{a_{12}^{(i)}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{a_{23}^{(i)}(\bar{s}',t)}{\bar{s}'+s+t-2M^2-1}, \quad (6.1\mathrm{b})$$

$$\mathrm{Im}_{\mathrm{III}}A_i(s,\bar{s})=\pm\rho_i(\bar{s})+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{a_{13}^{(i)}(s',\bar{s})}{s'-s}+\frac{1}{\pi}\int_4^{\infty}dt'\frac{a_{23}^{(i)}(\bar{s},t')}{t'+\bar{s}+s-2M^2-1}. \quad (6.1\mathrm{c})$$

It can be seen from Eqs. (5.5) and (5.7) that $\mathrm{Im}_{\mathrm{I}}\,A_i$ is the imaginary part of A_i when the variables are in the physical region for process I, $\gamma+N\rightarrow\pi+N$, and represents the analytic continuation of Im A_i outside of this region. Functions $\mathrm{Im}_{\mathrm{II}}\,A_i$ and $\mathrm{Im}_{\mathrm{III}}\,A_i$ have the same meaning for process II, $\gamma+\pi\rightarrow N+\bar{N}$, and process III, $\gamma+\bar{N}\rightarrow\pi+\bar{N}$, respectively.

The dispersion relations with one variable held fixed are now obtained from Eqs. (5.5), (5.7), and (6.1):

Fixed s:

$$A_i(s,t,\bar{s})=\mathrm{Poles}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{\rho_i(s')}{s'-s}+\frac{1}{\pi}\int_4^{\infty}dt'\frac{\mathrm{Im}_{\mathrm{II}}A_i(s,t')}{t'-t}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{\mathrm{Im}_{\mathrm{III}}A_i(s,\bar{s}')}{\bar{s}'-\bar{s}}, \quad (6.2)$$

Fixed t:

$$A_i(s,t,\bar{s})=\mathrm{Poles}+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\times\mathrm{Im}_{\mathrm{I}}A_i(s',t)\left(\frac{1}{s'-s}\pm\frac{1}{s'-\bar{s}}\right), \quad (6.3)$$

Fixed $\bar{s}$:

$$A_i(s,t,\bar{s})=\mathrm{Poles}\pm\frac{1}{\pi}\int_{(M+1)^2}^{\infty}d\bar{s}'\frac{\rho_i(\bar{s}')}{\bar{s}'-\bar{s}}\pm\frac{1}{\pi}\int_4^{\infty}dt'\frac{\mathrm{Im}_{\mathrm{II}}A_i(\bar{s},t')}{t'-t}\pm\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{\mathrm{Im}_{\mathrm{III}}A_i(\bar{s},s')}{s'-s}. \quad (6.4)$$

The crossing symmetry has been employed to produce Eqs. (6.3) and (6.4). These are, of course, simply different ways to representing the same functions.

In previous work on pion-nucleon scattering and on photoproduction, only the fixed momentum-transfer dispersion relation, Eq. (6.3), has been employed. It is noteworthy that, according to the above considerations, this is the only one of the three for which a subtraction is not required by elementary perturbation-theory arguments. In practice, however, a strongly interacting

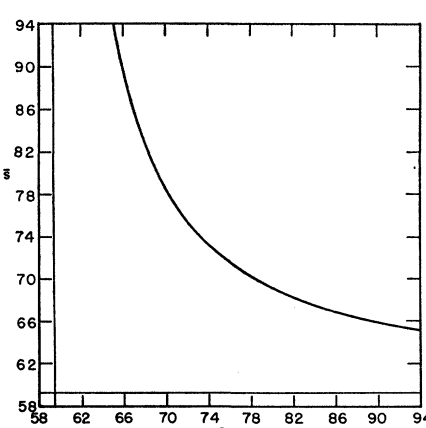

FIG. 4. The boundary curve for $a_{13}^{(i)}$. The boundary lines are the asymptotes of the curve.

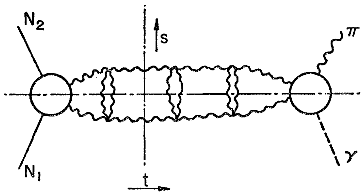

FIG. 5. A typical diagram representing a two-pion state connecting $\gamma+\pi$ to $N\bar{N}$.

intermediate state connecting $(\gamma\pi)$ to $N\bar{N}$ may necessitate a subtraction. To illustrate this point, we consider for example a resonant 2π intermediate state. This can be represented schematically by the diagram in Fig. 5. As the π-π interaction becomes stronger, the lifetime of the 2π state becomes longer, and more pion pairs are exchanged between the resonating pions. This can be represented in Fig. 5 by adding more pairs to ladder of pions representing the intermediate state. If we now look at the singularities produced in s by this diagram, we see that as more pairs are added, the contribution of this diagram comes from higher values of s'. Finally as the interaction becomes strong enough to produce a bound state, the contribution to the s spectrum moves to infinity, requiring a subtraction. Another way to understand this effect is to recall that if there were a 2π bound state, we should certainly have to add a new pole in t together with the associated ρ_t. Thus if one wants to treat only the lower intermediate states in the s spectrum, approximating $\mathrm{Im_I}\, A_i$ in Eq. (6.3) by the first few terms of a polynomial expansion, a resonance in an intermediate state of the t spectrum may necessitate a subtraction. The approximation of replacing $\mathrm{Im_I}\, A_i$ in Eq. (6.3) by its polynomial expansion in the physical region, will be discussed in a later section.

The fixed-s dispersion relation, Eq. (6.2), is useful for the photoproduction channel in that the $\cos\theta$ dependence of A_i is given explicitly. This allows the use of projection operators to obtain an integral representation for the eigenamplitudes.

VII. THE APPROACH THROUGH EIGENAMPLITUDES

Before we use the basic machinery developed in the preceding sections, it is useful to examine the type of information provided by the unitarity conditions for photoproduction and $\gamma+\pi\rightarrow N+\bar{N}$. For photoproduction below the threshold for production of two pions, unitarity gives the phase of each eigenamplitude in terms of the corresponding pion-nucleon phase. Of course, the pion-nucleon phase must be supplied as starting information.

The unitarity condition for a particular angular-momentum state of the process $\gamma+\pi\rightarrow N+\bar{N}$ is

$$2\,\mathrm{Im}\langle\bar{N}N|\pi\gamma\rangle_J=\textstyle\sum_n\langle\bar{N}N|n\rangle_J\langle n|\pi\gamma\rangle_J, \qquad (7.1)$$

where the sum n runs over all physical intermediate states having the same quantum numbers as the $N\bar{N}$ and $\pi\gamma$ states. Mandelstam has shown that Eq. (7.1) is valid in the nonphysical region $4M^2>t>4$.[10] We see that only for $t>4M^2$ does the right of Eq. (7.1) contain $\langle\bar{N}N|\pi\gamma\rangle$. Thus, for $4M^2>t>4$ the imaginary part of the A_i's will be a function that must be supplied by solutions of other scattering problems. At present, the only information available is for $\gamma+\pi\rightarrow\pi+\pi$ and $\pi+\pi\rightarrow N+\bar{N}$, therefore we are restricted to treating only the 2π intermediate state. It should be noted, however, that in the treatment of $\gamma+\pi\rightarrow\pi+\pi$ by Wong, it has been necessary to introduce a new coupling constant, which if large enough would make the 2π intermediate state an important singularity.[3] A further enhancement of this state would arise from the pion-pion p-wave resonance proposed by Chew and Mandelstam.

In order to make use of all the information obtained from the unitarity conditions, it is convenient to develop dispersions relations for the individual eigenamplitudes. The first step toward this end is to insert Eqs. (3.4) to (3.7) into Eqs. (3.12) to (3.15), obtaining projection operators to be applied to the A_i's. These projection operators are then applied to the fixed-s representation of the A_i's, Eq. (6.2), yielding an integral representation for each eigenamplitude. By examining these expressions we can obtain the analyticity properties of the eigenamplitudes.

In general, the eigenamplitudes will have the singularities present in the A_i's plus kinematical singularities arising from the relation between the $\mathfrak{F}$'s and the A_i's, and from expressing $\bar{s}$ and t as functions of s and $\cos\theta$. It is possible, however, to construct a function from each eigenamplitude that is free from kinematic singularities in the $\sqrt{s}$ or W plane.

A simple reflection property in W for the eigenamplitudes can be obtained by noting that $\mathfrak{F}_1(-W)=-\mathfrak{F}_2(W)$ and $\mathfrak{F}_3(-W)=\mathfrak{F}_4(W)$, and using these relations in Eqs. (3.12) to (3.15). The resulting relation for the eigenamplitudes is

$$M_{l+}(-W)=\frac{1}{l+1}[(l+2)M_{(l+1)-}(W)+E_{(l+1)-}(W)], \qquad (7.2a)$$

and

$$E_{l+}(-W)=\frac{1}{l+1}[M_{(l+1)-}-lE_{(l+1)-}(W)]. \qquad (7.2b)$$

If Eqs. (7.2a) and (7.2b) are used together with the analyticity properties of the eigenamplitudes, the following dispersion relation results:

$$F_l^i(W)=G_l^i(W)+\frac{1}{\pi}\int_{-\infty}^{-(M+1)}dW'\frac{\mathrm{Im}\,F_l^i(W')}{W'-W}+\frac{1}{\pi}\int_{(M+1)}^{\infty}dW'\frac{\mathrm{Im}\,F_l^i(W')}{W'-W}, \qquad (7.3)$$

[10] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

where the eigenamplitudes $F_l^1(W)$ and $F_l^2(W)$ differ from M_{l+} and E_{l+}, respectively, by factors that remove the kinematical singularities. The function $G_l^i(W)$ contains all the singularities arising from the t and $\bar{s}$ spectra and may be obtained from Eq. (6.2) if $\mathrm{Im}_{\mathrm{II}}\, A_i$ and $\mathrm{Im}_{\mathrm{III}}\, A_i$ are replaced by their appropriate polynomial expansions in the physical regions for channels II and III, respectively. When the projection operation is applied to Eq. (6.2) after the above replacement has been made, $G_l^i(W)$ will now result.

It is now possible to write a solution to Eq. (7.3) imposing the unitarity requirement on F_l^i in the physical region for photoproduction. The reflection law in the W plane for the pion-nucleon eigenamplitudes has been given by MacDowell to be[11] $f_{l+}(W) = f_{(l+1)-}(-W)$, where $f_{l\pm} = \sin\delta_{l\pm} \exp(i\delta_{l\pm})/q$. Thus it is seen that the function $f_{l+}(W)$ that has the same phase as $F_l(W)$ for $W>0$ will also have the correct phase for $W<0$. We will now assume that the pion-nucleon problem has been solved by use of the N/D technique employed by Chew and Mandelstam in the pion-pion problem.

The function N_l will be a real analytic function of W for $W>M+1$ and $W<-(M+1)$, while D_l will have two branch cuts running from $(M+1)$ to ∞ and from $-(M+1)$ to $-\infty$ and will be analytic elsewhere. The function N_l/D_l will be the eigenamplitude for pion-nucleon scattering that is free of kinematical singularities. The relation of N_l/D_l to the pion-nucleon phase shifts will be

$$\begin{aligned} N_l/D_l &= R_l(W) \sin\delta_{l+} \exp(i\delta_{l+}) \quad \text{for} \quad W>M+1 \\ &= R_l(W) \sin\delta_{(l+1)-} \exp(i\delta_{(l+1)-}) \\ &\qquad \text{for} \quad W<-(M+1). \end{aligned} \tag{7.4}$$

where $R_l(W)$ is the factor needed to remove the kinematical singularities. The phase requirements on F_l will now be satisfied by a solution to Eq. (7.3) of the type employed by Chew and Low[12]:

$$\begin{aligned} F_l^i(W) &= G_l^i(W) \\ &+ \frac{1}{D_l(W)} \left[\frac{1}{\pi} \int_{-\infty}^{-(M+1)} dW' \frac{G_l^i(W') N_l(W')}{(W'-W) R_l(W')} \right. \\ &\left. + \frac{1}{\pi} \int_{M+1}^{\infty} dW' \frac{G_l^i(W') N_l(W')}{(W'-W) R_l(W')} \right]. \end{aligned} \tag{7.5}$$

This approach to the photoproduction problem is based on the assumption, discussed by Chew, that nearby singularities in the complex plane dominate the behavior of amplitudes in the low-energy region.[13] The solution given by Eq. (7.5) includes the lowest singularities and must be considered the first step in a series of approximations that will include successively higher singularities.

[11] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).
[12] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).
[13] G. F. Chew, Ann. Rev. Nuclear Sci. **9**, 29 (1959).

The main obstacle preventing the use of Eq. (7.5) at present is the lack of information about the pion-nucleon problem. The only pion-nucleon eigenamplitude that has been studied by the N/D method so far is the resonant $I=\frac{3}{2}$, $J=\frac{3}{2}$, p wave.[14] If only the 3–3 amplitude and the pole terms are to be treated, Eq. (7.5) represents little improvement over the fixed-t dispersion-relation approach which has been used by CGLN. Since they included the 3–3 amplitude, we must conclude that, with the pion-nucleon information presently available, no significant improvement can be made in the treatment of the s or $\bar{s}$ spectra. However, the 2π branch cut in the t spectrum can now be included, which will be the first modification to the pole terms for the (0) amplitudes.

VIII. PHOTOPRODUCTION NEAR THRESHOLD

The threshold region provides us with a situation in which the contribution of the 2π intermediate state is maximized, first by virtue of a small denominator in Eq. (6.2) and second because the measurable cross sections will not be dominated by the 3–3 resonance of the pion-nucleon system.

The approximation now employed is to assume that the $I=\frac{1}{2}$ phases for pion-nucleon scattering are negligible, meaning that the s and $\bar{s}$ branch cuts may be ignored for the (0) amplitudes. The amplitude F_l^i will then just be given by its projection from the poles and from the t spectrum. For this reason we can do the sum over the F_l's, undoing the projection and allowing us to work directly with $A_i^{(0)}$. The resulting functions $A_i^{(0)}$ will just be given by Eq. (7.2) in which the polynomial expansion is used for $\mathrm{Im}_{\mathrm{II}}\, A_i^0$, with $\mathrm{Im}_{\mathrm{III}}\, A_i^0 = 0$.

We will now employ unitarity for process II to obtain $\mathrm{Im}_{\mathrm{II}}\, A_i^0$. The unitarity condition resulting from the 2π intermediate state is

$$2\,\mathrm{Im}\, T_J^{\lambda(N),\lambda(\bar{N}),\lambda(\gamma)}(t) = \tau_J^{\lambda(N),\lambda(\bar{N})}(t) t_J^{\lambda(\gamma)}(t), \tag{8.1}$$

where $\tau_J^{\lambda(N),\lambda(\bar{N})}(t)$ is the helicity eigenamplitudes for $\pi+\pi \rightarrow N+\bar{N}$ and $t_J^{\lambda(\gamma)}(t)$ is the helicity eigenamplitudes for $\gamma+\pi \rightarrow \pi+\pi$. This relation is exact for $4<t<16$ and will be assumed to be approximately true for larger t.

As the process $\gamma+\pi \rightarrow \pi+\pi$ contains only odd angular-momentum states, we will neglect F-wave and higher states, keeping only the p-wave state. We are concerned only with photon helicity $\lambda(\gamma)=+1$, because Eqs. (4.12) to (4.15) are expressed for this photon state. Equation (8.1) becomes

$$\mathrm{Im}\, T_1^{++1} = \tfrac{1}{2}\tau_1^{++} t_1^{*1} = \mathrm{Im}\, T_1^{--1}, \tag{8.2}$$

and

$$\mathrm{Im}\, T_1^{+-1} = \tfrac{1}{2}\tau_1^{+-} t_1^{*1} = \mathrm{Im}\, T_1^{-+1}, \tag{8.3}$$

[14] W. R. Frazer and J. R. Fulco, Lawrence Radiation Laboratory (private communication); and S. Frautschi and D. Walecka, Department of Physics, University of California (private communication).

where we denote $+\frac{1}{2}$ helicity as $+$ and $-\frac{1}{2}$ as $-$. The relation between $T_1{}^{++1}$ and $T_1{}^{--1}$, and between $T_1{}^{+-1}$ and $T_1{}^{-+1}$ arises from the fact that only the odd-parity part of the $J=1$ nucleon-antinucleon system contributes.

The amplitudes $\tau_1{}^{++}$ and $\tau_1{}^{+-}$ have been treated by Frazer and Fulco.[4] They defined

$$T_+{}^1=(2q'/p)^{\frac{1}{2}}\tau_1{}^{++}, \tag{8.4}$$

and

$$T_-{}^1=(2q'/p)^{\frac{1}{2}}\tau_1{}^{+-}, \tag{8.5}$$

where q' and p are the magnitude of the initial meson and final nucleon momenta in the barycentric system. If we now define

$$t_1{}^1/k'=(2q'/k')^{\frac{1}{2}}M^1, \tag{8.6}$$

where q' and k' are the magnitudes of initial photon and final meson momenta in the barycentric system for the process $\gamma+\pi\rightarrow\pi+\pi$, the following expressions for the imaginary parts of the A's are obtained:

$$\operatorname{Im} A_1=-\frac{6\pi E}{p^2k'}\left[ET_+{}^1-\frac{MT_-{}^1}{\sqrt{2}}\right]M^{1*}, \tag{8.7}$$

$$\operatorname{Im} A_2=+\frac{3\pi}{2p^2k'E}\left[ET_+{}^1-\frac{MT_-{}^1}{\sqrt{2}}\right]M^{1*}, \tag{8.8}$$

$$\operatorname{Im} A_3=0, \tag{8.9}$$

$$\operatorname{Im} A_4=-\frac{3\pi}{p^2k'}\left[MT_+{}^1-E\frac{T_-{}^1}{\sqrt{2}}\right]M^{1*}. \tag{8.10}$$

It is now convenient to introduce the notation of Frazer and Fulco (hereafter denoted FF)[15]:

$$g_1{}^V(t)=+\frac{eF_\pi{}^*(t)(t-4)^{\frac{1}{2}}}{4p^2}\left[\frac{ET_-{}^1(t)}{\sqrt{2}}-MT_+{}^1(t)\right], \tag{8.11}$$

$$g_2{}^V(t)=\frac{eF_\pi{}^*(t)(t-4)^{\frac{1}{2}}}{8p^2E}\left[ET_+{}^1-\frac{M}{\sqrt{2}}T_-{}^1\right], \tag{8.12}$$

where $F_\pi(t)$ is the pion form factor. If we now use the factor that the P-wave $\gamma+\pi\rightarrow\pi+\pi$ amplitude will have the phase of $\pi-\pi$ scattering, then $M^{1*}(t)$ may be written

$$M^{1*}(t)=ek'(t)(t-4)^{\frac{1}{2}}h(t)F_\pi{}^*(t)/12\pi, \tag{8.13}$$

where $h(t)$ is a real function and the factors $k'(t)$, $(t-4)^{\frac{1}{2}}$, e, and 12π are included for convenience in subsequent calculations.

Expressing the Im $A_i{}^0$'s in this notation, we obtain

$$\operatorname{Im} A_1{}^0(t)=-th(t)g_2{}^V(t), \tag{8.14}$$

$$\operatorname{Im} A_2{}^0(t)=+h(t)g_2{}^V(t), \tag{8.15}$$

$$\operatorname{Im} A_4{}^0(t)=+h(t)g_1{}^V(t). \tag{8.16}$$

[15] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

The final expressions for the $A_i{}^0$'s are then:

$$A_1{}^0=\frac{e_rg_r}{2}\left(\frac{1}{s-M^2}+\frac{1}{s-M^2}\right)+\frac{1}{\pi}\int_4^\infty dt'\frac{t'h(t')g_2{}^V(t')}{t'-t}, \tag{8.17}$$

$$A_2{}^0=+\frac{e_rq_r}{(s-M^2)(\bar{s}-M^2)}-\frac{1}{\pi}\int_4^\infty dt'\frac{h(t')g_2{}^V(t')}{t'-t}, \tag{8.18}$$

$$A_3{}^0=-\tfrac{1}{2}g_r(\mu_{p_r}{}'+\mu_{n_r})\left(\frac{1}{s-M^2}-\frac{1}{\bar{s}-M^2}\right)$$

$$A_4{}^0=-\tfrac{1}{2}g_r(\mu_{p_r}{}'+\mu_{n_r})\left(\frac{1}{s-M^2}+\frac{1}{\bar{s}-M^2}\right)$$

$$-\frac{1}{\pi}\int_4^\infty dt'\frac{h(t')g_1{}^V(t')}{t'-t}, \tag{8.20}$$

where the factor (-1) arises from the sign of $i\epsilon$ that appears in the t denominators when s is given a small imaginary part and $\bar{s}$ is held fixed.

It is now possible to take advantage of the appearance of the functions $g_2{}^V$ and $g_1{}^V$ in these integrals. From FF we observe that

$$G_1{}^V(t)=\frac{1}{\pi}\int_4^\infty dt'\frac{g_1{}^V(t')}{t'-t}, \tag{8.21a}$$

and

$$G_2{}^V(t)=\frac{1}{\pi}\int_4^\infty dt'\frac{g_2{}^V(t')}{t'-t}, \tag{8.21b}$$

where $G_1{}^V$ and $G_2{}^V$ are the nucleon form factors. The function $h(t)$ has been shown by Wong to be well represented by the form[3]

$$h(t)=\frac{3\Lambda}{8\sqrt{2}F_\pi(1)e}\left(\frac{1+a}{t+a}\right), \tag{8.22}$$

where, if FF's solution of the F_π is used, we have $a=5$, $F_\pi(1)=1.08$, and Λ is the arbitrary constant previously mentioned. Knowing the form of $h(t)$, we can by forming subtracted forms of Eqs. (8.21a) and (8.21b) express the integrals in Eqs. (8.17) to (8.20) directly in terms of $G_1{}^V(t)$ and $G_2{}^V(t)$.

The resulting expressions for $A_1{}^0$, $A_2{}^0$, $A_3{}^0$, and $A_4{}^0$ are

$$A_1{}^0=\frac{e_rg_r}{2}\left(\frac{1}{s-M^2}+\frac{1}{\bar{s}-M^2}\right)$$

$$+\lambda'\left[G_2{}^V(t)-\frac{a}{t+a}[G_2{}^V(t)-G_2{}^V(-a)]\right], \tag{8.23}$$

$$A_2{}^0=\frac{e_rg_r}{(s-M^2)(s-M^2)}-\lambda'\frac{G_2{}^V(t)-G_2{}^V(-a)}{t+a}, \tag{8.24}$$

$$A_3{}^0=-\tfrac{1}{2}g_r(\mu_{p_r}{}'+\mu_{n_r})\left(\frac{1}{s-M^2}-\frac{1}{\bar{s}-M^2}\right), \tag{8.25}$$

and

$$A_4{}^0=-\tfrac{1}{2}g_r(\mu_{p_r}'+\mu_{n_r})\left(\frac{1}{s-M^2}+\frac{1}{\bar{s}-M^2}\right)\times\lambda'\frac{G_1{}^V(t)-G_1{}^V(-a)}{t+a},\quad(8.26)$$

where

$$\lambda'=\frac{3\Lambda}{8\sqrt{2}e}\left(\frac{1+a}{F_\pi(1)}\right).$$

In order to compare the above results with experimental data, we must know the $(+)$ and $(-)$ amplitudes in the threshold region. These amplitudes have been given by CGLN, but because of a $1/M$ expansion used within their dispersion integrals, they were forced to introduce undetermined correction terms $N^{(+)}$ and $N^{(-)}$ into each of these amplitudes. Since the effect of our new Λ-dependent terms will be fairly sensitive to the values of N^+ and N^-, we will recalculate the $(+,-)$ amplitudes avoiding any expansions in $1/M$. The fixed $-t$ dispersion relation without subtractions as given in Eq. (6.3) will be used. It should be noted that a strongly interacting 3π state as has been proposed by Chew[16] would necessitate the use of a subtracted form of Eq. (6.3), but only for the $(+)$ amplitude, as this 3π state would have $I=0$. However, the $(+)$ amplitude does not contribute to charged-pion photoproduction, which will prove to be most sensitive to the value of Λ.

A polynomial expansion in $\cos\theta$ for $\mathrm{Im}_{\mathrm{I}}\,A_i{}^{(+,-)}$ in Eq. (6.3) will now converge for $\cos\theta$ within an ellipse with foci at $+1$ and -1 and semimajor axis given by the value of $\cos\theta$ at the nearest singularity in t. Thus for a maximum value of t allowed for convergence, there is also a minimum corresponding to the negative limit of the ellipse. The relation between these limits is

$$t_{\max}+t_{\min}=2-4\omega k.\quad(8.27)$$

If we now express $t_{\max}$ as given by the boundary of $a_{12}{}^{(i)(+,-)}$, we obtain

$$t_{\min}=-4\omega(s)k(s)-7-\frac{8(3s-M^2+1)}{[s-(M+1)^2][s-(M-1)^2]}.\quad(8.28)$$

Since $\mathrm{Im}\,A_i{}^{(+,-)}$ will be used in the integral in Eq. (6.3), the region of convergence is determined by the maximum value of $t_{\min}$ and is found to be $9>t>-19.3$. For comparison, we state the result rigorously proved by Oehme and Taylor that the polynomial expansion converges, at least for t, in the range $0>t>t_0\simeq-12$.[17] The smallest value of t corresponding to physical $\cos\theta$ will be larger than -10 in the energy region we are considering; therefore the expansion for $\mathrm{Im}\,A_i{}^{(+,-)}$ should converge rapidly, making it plausible to neglect the high-angular-momentum contributions to $\mathrm{Im}\,A_i{}^{(+,-)}(s',t)$. A general feature of pion-nucleon scattering below 400 Mev is that the only large phase shift is in the $J=\frac{3}{2}$, $I=\frac{3}{2}$, state. Thus a reasonable first approximation is obtained by including in $\mathrm{Im}\,A_i{}^{(+,-)}(s't)$ only the parts containing this large pion-nucleon phase.

We must now calculate the absorptive parts of the A_i's for photoproduction $[s>(M+1)^2]$ including only the 33 part. Since, in the previous treatment by CGLN, the M_{1+} amplitude was found to be much more important than the E_{1+}, the imaginary part of E_{1+} is neglected in the following.

The resulting expressions for the A_i's are

$$A_1{}^{\pm}=\frac{e_rg_r}{2}\left(\frac{1}{s-M^2}\pm\frac{1}{\bar{s}-M^2}\right)\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}ds'C(s')[\omega(s')(s'^{\frac{1}{2}}+M)+t+1]\,\mathrm{Im}\,M_{1+}{}^{\frac{3}{2}}(s)\left(\frac{1}{s'-s}\pm\frac{1}{s'-\bar{s}}\right),\quad(8.29)$$

$$A_2{}^{\pm}=-\frac{e_rg_r}{t-1}\left(\frac{1}{s-M^2}\pm\frac{1}{\bar{s}-M^2}\right)\mp\binom{2}{1}\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\times C(s')\,\mathrm{Im}\,M_{1+}{}^{\frac{3}{2}}(s')\left(\frac{1}{s'-s}\pm\frac{1}{s'-\bar{s}}\right),\quad(8.30)$$

$$A_3{}^{\pm}=-\tfrac{1}{2}g_r(\mu_{p_r}'-\mu_{n_r})\left(\frac{1}{s-M^2}\mp\frac{1}{\bar{s}-M^2}\right)\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}ds'C(s')\left[\frac{3}{2}\left(\frac{t-1}{s'^{\frac{1}{2}}+M}\right)+\omega(s')-(s'^{\frac{1}{2}}+M)\right]\mathrm{Im}\,M_{1+}{}^{\frac{3}{2}}(s')\left(\frac{1}{s'-s}\mp\frac{1}{s'-\bar{s}}\right),\quad(8.31)$$

and

$$A_4{}^{\pm}=-\tfrac{1}{2}g_r(\mu_{p_r}'-\mu_{n_r})\left(\frac{1}{s-M^2}\pm\frac{1}{\bar{s}-M^2}\right)\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}ds'\,C(s')\left[\frac{3}{2}\left(\frac{t-1}{s'^{\frac{1}{2}}+M}\right)+\omega(s')+2(s'^{\frac{1}{2}}+M)\right]\mathrm{Im}\,M_{1+}{}^{\frac{3}{2}}(s')\left(\frac{1}{s'-s}\pm\frac{1}{s'-s}\right),\quad(8.32)$$

where

$$C(s)=\frac{4\pi}{q(s)k(s)}\{[s^{\frac{1}{2}}+M]^2-1\}^{-\frac{1}{2}}.$$

These integrals can be carried out numerically if the CGLN expression for $M_{1+}{}^{\frac{3}{2}}$ is used with an effective range formula to represent the 33 amplitude for pion-nucleon scattering. It should be noted that expressions

[16] G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).
[17] R. Oehme and J. G. Taylor, Phys. Rev. **113**, 371 (1959).

TABLE I. Matrix elements of $g^{(\pm,0)}$ for the four possible charge configurations.

	$\gamma+p\to\pi^0+p$	$\gamma+n\to\pi^0+n$	$\gamma+p\to\pi^++n$I	$\gamma+n\to\pi^-+p$
g^+	1	1	0	0
g^-	0	0	$\sqrt{2}$	$-\sqrt{2}$
g^0	1	-1	$\sqrt{2}$	$+\sqrt{2}$

(8.29) to (8.32) are identical to those of CGLN except that no $1/M$ expansion has been made and only M_{1+} has been kept in the imaginary parts of the amplitudes.

X. EVALUATION OF THE PHOTOPRODUCTION AMPLITUDES

Evaluation of the integrals in Eq. (9.53) to Eq. (9.56) is accomplished by using the CGLN solution for $M_{1+}{}^{\frac{3}{2}}$:

$$\frac{M_{1+}{}^{\frac{3}{2}}(s)}{q(s)k(s)}=\frac{\mu_p-\mu_n}{2f}\frac{f_{33}}{q^2(s)}. \tag{9.1}$$

A relativistic effective-range formula suggested by Chew and Wong,[18]

$$\operatorname{Im} f_{33}=\frac{q^5}{q^6+\Gamma(s-s_r)^2(s-M^2)^2}, \tag{9.2}$$

is used to represent the 33 amplitude, where Γ and s_r are parameters which have been adjusted to fit the Chiu and Lomon[19] δ_{33} at 150 and 220 Mev and to the low-energy behavior of δ_{33} as given by Barnes *et al.*[20] The resulting parameters are $\Gamma=3.5\times10^{-4}$ and $s_r=76.6$. In performing the integrations, we expanded the denominators in powers of $\cos\theta$, keeping only the first two terms because the expansion converges quite rapidly since $\cos\theta$ is always multiplied by the nucleon velocity.

The $\operatorname{Re} M_{1+}(s)$ produced by the integrals in Eqs. (8.29) to (8.32) must be considered an iterative solution for $M_{1+}(s)$. As there seems to be no guarantee that such a procedure will converge, we projected this contribution from the $\mathfrak{F}$'s by means of Eq. (3.12) and replaced it by the value given by Eq. (9.1). It was noted, however, that this correction was not large, indicating that the solution given by Eq. (9.1) is reasonably good.

To form the scattering amplitude for any of the charge states of interest, we must know the matrix element of $g_\beta{}^+$, $g_\beta{}^-$, and $g_\beta{}^0$ for each of these states. These matrix elements as evaluated by CGLN are given in Table I. The scattering amplitudes for $\gamma+p\to\pi^0+p$, denoted $\mathfrak{F}(\pi^0)$, and $\gamma+p\to\pi^++n$, denoted $\mathfrak{F}(\pi^+)$, are formed as

$$\mathfrak{F}(\pi^0)=\mathfrak{F}^++\mathfrak{F}^0, \tag{9.3a}$$

and

$$\mathfrak{F}(\pi^+)/\sqrt{2}=\mathfrak{F}^-+\mathfrak{F}^0. \tag{9.3b}$$

In Table II the calculated values of $\mathfrak{F}(\pi^0)$ for $\Lambda=0$ are given, again with only the first two terms in $x=\cos\theta$ retained. (The values $M=6.7$ and $f^2=0.08$ have been employed throughout.) Only the pole terms for the (0) amplitude have been included. We define $\omega^*=s^{\frac{1}{2}}-M$. The photon laboratory energy can be obtained from

$$K_L=(s-M^2)/2M=\omega^*+\omega^{*2}/2M. \tag{9.4}$$

In the case of charged-pion production, an expansion of the meson-current pole term is not possible. We separate these terms as follows:

$$\mathfrak{F}(\pi^+)=\mathfrak{F}'+\mathfrak{F}^R/(1-vx), \tag{9.5}$$

where v is the velocity of the final meson.

It is now possible to expand $\mathfrak{F}'(\pi^+)$, and the resulting expressions for $\Lambda=0$ are given in Table III. Values of $\mathfrak{F}_3{}^R$, $\mathfrak{F}_4{}^R$, v, $\operatorname{Im} M_{1+}{}^+$, and $\operatorname{Im} M_{1+}{}^-$ are given in Table IV, and $\mathfrak{F}_1{}^R$ and $\mathfrak{F}_2{}^R$ are zero.

The imaginary part of any of the $\mathfrak{F}$'s may be obtained with the aid of Eqs. (3.8) to (3.11). The pole terms for $\mathfrak{F}^0$ are given in Table V.

The differential cross section for unpolarized photons and nucleons is

$$\begin{aligned}d\sigma/d\Omega=(q/k)|M|^2=(q/k)\{&|\mathfrak{F}_1|^2+|\mathfrak{F}_2|^2\\&+\tfrac{1}{2}|\mathfrak{F}_3|^2+\tfrac{1}{2}|\mathfrak{F}_4|^2+\operatorname{Re}\mathfrak{F}_1{}^*\mathfrak{F}_4+\operatorname{Re}\mathfrak{F}_2{}^*\mathfrak{F}_3\\&+\cos\theta[\operatorname{Re}\mathfrak{F}_3{}^*\mathfrak{F}_4-2\operatorname{Re}\mathfrak{F}_1{}^*\mathfrak{F}_2]\\&-\cos^2\theta[\tfrac{1}{2}|\mathfrak{F}_3|^2+\tfrac{1}{2}|\mathfrak{F}_4|^2+\operatorname{Re}\mathfrak{F}_1{}^*\mathfrak{F}_4\\&\qquad+\operatorname{Re}\mathfrak{F}_2{}^*\mathfrak{F}_3]-\cos^3\theta\operatorname{Re}\mathfrak{F}_3{}^*\mathfrak{F}_4\}.\end{aligned} \tag{9.6}$$

TABLE II. Values of the scattering amplitudes, $\mathfrak{F}$, for photoproduction of π^0 with $\Lambda=0$.

ω^*	$\mathfrak{F}_1\times10^3$	$\mathfrak{F}_2\times10^3$	$\mathfrak{F}_3\times10^3$	$\mathfrak{F}_4\times10^3$
1.00	-2.780	0	0	0
1.05	$-2.682+6.865x$	$3.424-0.064x$	$-7.104+0.123x$	$-0.360+0.016x$
1.10	$-2.556+10.159x$	$5.199-0.128x$	$-10.563+0.250x$	$-0.704+0.045x$
1.15	$-2.398+13.046x$	$6.844-0.193x$	$-13.631+0.380x$	$-1.033+0.081x$
1.20	$-2.204+15.828x$	$8.504-0.257x$	$-16.619+0.513x$	$-1.350+0.124x$
1.25	$-1.969+18.632x$	$10.243-0.321x$	$-19.659+0.648x$	$-1.654+0.170x$
1.30	$-1.690+21.531x$	$12.099-0.384x$	$-22.831+0.787x$	$-1.949+0.223x$
1.40	$-0.986+27.796x$	$16.263-0.503x$	$-29.765+1.073x$	$-2.510+0.334x$
1.50	$-0.078+34.784x$	$21.077-0.609x$	$-37.602+1.368x$	$-3.040+0.458x$

[18] G. F. Chew and D. Y. Wong, Lawrence Radiation Laboratory (private communication).
[19] H. Y. Chiu and E. L. Lomon, Ann. phys. **6**, 50 (1959).
[20] S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, Phys. Rev. **117**, 238 (1960).

TABLE III. Scattering amplitudes, $\mathfrak{F}$, excluding the meson-current term for photoproduction of π^+ with $\Lambda=0$.

ω^*	$\mathfrak{F}_1'\times10^3/\sqrt{2}$	$\mathfrak{F}_2'\times10^3/\sqrt{2}$	$\mathfrak{F}_3'\times10^3/\sqrt{2}$	$\mathfrak{F}_4'\times10^3/\sqrt{2}$
1.00	19.679	0	0	0
1.05	$19.403-2.824x$	$-2.773+0.036x$	$2.899-0.083x$	$0.049-0.002x$
1.10	$19.116-4.219x$	$-4.082+0.072x$	$4.353-0.168x$	$0.099-0.006x$
1.15	$18.818-5.471x$	$-5.217+0.108x$	$5.670-0.254x$	$0.151-0.011x$
1.20	$18.507-6.702x$	$-6.300+0.142x$	$6.979-0.343x$	$0.205-0.017x$
1.25	$18.180-7.964x$	$-7.383+0.175x$	$8.334-0.433x$	$0.262-0.024x$
1.30	$17.836-9.288x$	$-8.496+0.207x$	$9.767-0.525x$	$0.319-0.032x$
1.40	$17.090-12.203x$	$-10.886+0.264x$	$12.959-0.713x$	$0.439-0.052x$
1.50	$16.260-15.512x$	$-13.539+0.310x$	$16.628-0.907x$	$0.566-0.076x$

The differential cross section for $\gamma+p\rightarrow p+\pi^0$ in the threshold region may be expressed as

$$d\sigma/d\Omega=(q/k)[A+B\cos\theta+C\cos^2\theta+D\cos^3\theta]. \quad (9.7)$$

In Figs. 6–9 the values of A, B, C, and D calculated from the $\mathfrak{F}$'s in Table II are given together with experimental data.[21] The fact that D has been set equal to zero in the analysis of the experimental data, while the calculated value of D is comparable to B, makes a quantitative comparison between our values of B and C and those from experiment unreliable.

TABLE IV. Values of $\mathfrak{F}^R$, the meson velocity, Im M_{1+}^+, and Im M_{1+}^-.

ω^*	$\mathfrak{F}_3^R\times10^3/\sqrt{2}$	$\mathfrak{F}_4^R\times10^3/\sqrt{2}$	Meson velocity, V	Im $M_{1+}^+\times10^3$	Im $M_{1+}^-\times10^3$
1.00	0	0	0	0	0
1.05	5.947	−1.818	0.2854	0.012	−0.006
1.10	8.098	−3.382	0.3911	0.053	−0.027
1.15	9.565	−4.735	0.4649	0.130	−0.065
1.20	10.667	−5.910	0.5217	0.252	−0.126
1.25	11.532	−6.936	0.5676	0.433	−0.217
1.30	12.232	−7.834	0.6058	0.691	−0.346
1.40	13.284	−9.318	0.6661	1.537	−0.769
1.50	14.023	−10.475	0.7118	3.090	−1.5451

In Fig. 10, $|M|^2$ at 90 deg for $\gamma+p\rightarrow n+\pi^+$ as calculated from Tables III and IV is given, together with experimental data.[22] Also included in Fig. 10 are the results of a theoretical calculation by Robinson based on the results of CGLN with $N^+=N^-=0$.[23]

In obtaining these cross sections a correction has been made for the mass difference between π^+ and π^0 by using as a unit the mass of the pion in question. [The conversion factors used are $\mu_{\pi^0}=135$ Mev, $\mu_{\pi^+}=140$ Mev, $(1/\mu_{\pi^0})^2=18.66$ mb, and $(1/\mu_{\pi^+})^2=19.96$ mb.] This means that, in effect, the value of the nucleon mass used in the calculation for the π^0 amplitude was too small, being 6.7 instead of 6.9. Since all energies are expressed relative to the nucleon mass, no serious error will be introduced by this procedure.

It is now possible to estimate how large a Λ would be allowed on the basis of present experimental information. First we will take G_1^V and G_2^V to be linear functions of t for $0>t>-5$, and will use

$$G_1^{V\prime}(0)/G_1^V(0)\simeq0.08\simeq G_2^{V\prime}(0)/G_2^V(0)=\alpha, \quad (9.8)$$

as given by FF. We may express $G_1^V(t)$ and $G_2^V(t)$ as

$$G_1^V(t)=G_1^V(0)[1+\alpha t]=\tfrac{1}{2}e[1+\alpha t], \quad (9.9)$$

and

$$G_2^V(t)=G_2^V(0)[1+\alpha t]=\tfrac{1}{2}(\mu_p'-\mu_n)[1+\alpha t]. \quad (9.10)$$

The quantities most sensitive to Λ are the threshold values of $(k/q)(d\sigma/d\Omega)$ for π^+ and $d\sigma(\pi^-)/d\sigma(\pi^+)$. This can be seen by noticing that the correction to A_1 is several times the correction to the other amplitudes

TABLE V. The pole terms for the (0) amplitude.

ω^*	$\mathfrak{F}_1^0$	$\mathfrak{F}_2^0$	$\mathfrak{F}_3^0$	$\mathfrak{F}_4^0$
1.00	-1.290			
1.05	$-1.330+0.439x$	$0.030-0.010x$	$-0.403+0.018x$	$-0.162+0.007x$
1.10	$-1.371+0.623x$	$0.043-0.020x$	$-0.570+0.036x$	$-0.316+0.020x$
1.15	$-1.410+0.766x$	$0.055-0.030x$	$-0.699+0.055x$	$-0.462+0.036x$
1.20	$-1.448+0.887x$	$0.066-0.040x$	$-0.808+0.073x$	$-0.602+0.055x$
1.25	$-1.485+0.995x$	$0.076-0.051x$	$-0.903+0.092x$	$-0.735+0.075x$
1.30	$-1.521+1.093x$	$0.086-0.062x$	$-0.990+0.112x$	$-0.863+0.097x$
1.40	$-1.590+1.269x$	$0.106-0.084x$	$-1.143+0.151x$	$-1.106+0.146x$
1.50	$-1.655+1.425x$	$0.124-0.107x$	$-1.277+0.191x$	$-1.332+0.199x$

[21] V. I. Goldansky, B. B. Govorkov, and R. G. Vassikov, Soviet Phys.—JETP 7, 37 (1960).
[22] A. Barbaro, E. L. Goldwasser, and D. Carlson-Lee, Bull. Am. Phys. Soc. 4, 273 (1959); M. Beneventano, G. Bernardini, D. Carlson-Lee, G. Stoppini, and L. Tau, Nuovo cimento 4, 323 (1956).
[23] C. S. Robinson, "Tables of cross sections for π^+ production from hydrogen, according to the theory of Chew, Goldberger, Low, and Nambu," University of Illinois Report, May 22, 1959 (unpublished).

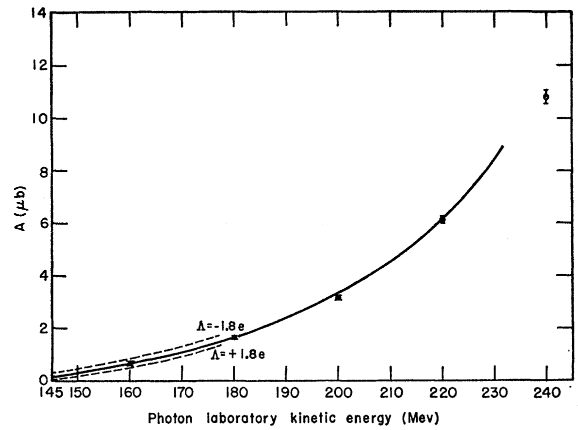

FIG. 6. The coefficient A for π^0 photoproduction. Solid line: prediction with $\Lambda=0$; dashed lines: predictions with $\Lambda=1.8e$ and $\Lambda=-1.8e$. The experimental points are those of Goldansky *et al.*[21]

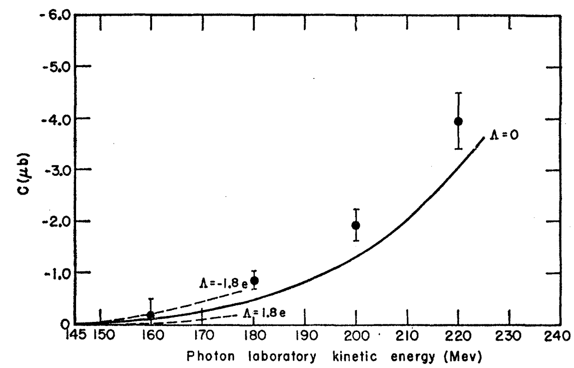

FIG. 7. The coefficient B for π^0 photoproduction. Solid line: prediction with $\Lambda=0$; dashed lines: predictions with $\Lambda=1.8e$ and $\Lambda=-1.8e$. The experimental points are those of Goldansky *et al.*[21]

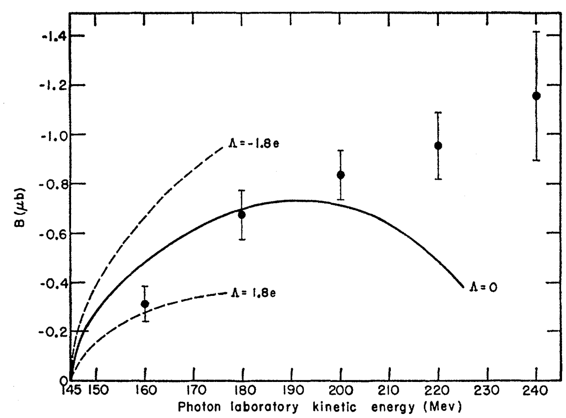

FIG. 8. The coefficient C for π^0 photoproduction. Solid line: prediction with $\Lambda=0$; dashed lines: predictions with $\Lambda=1.8e$ and $\Lambda=-1.8e$. The experimental points are those of Goldansky *et al.*[21]

causing a large correction to $\mathfrak{F}_1$. Also, since $\mathfrak{F}_1$ is larger for charged-pion production, $|\mathfrak{F}_1|^2$ will be sensitive to small changes in $\mathfrak{F}_1^0$.

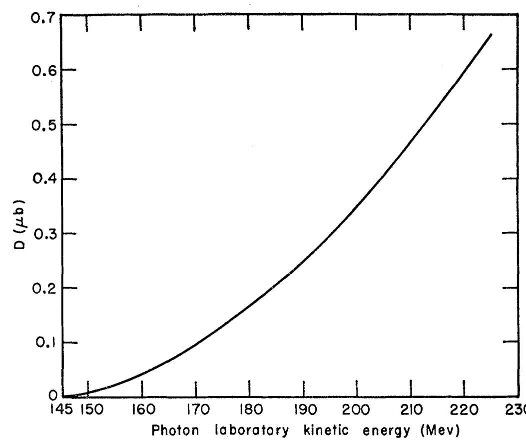

FIG. 9. The coefficient D for π^0 photoproduction as predicted for $\Lambda=0$.

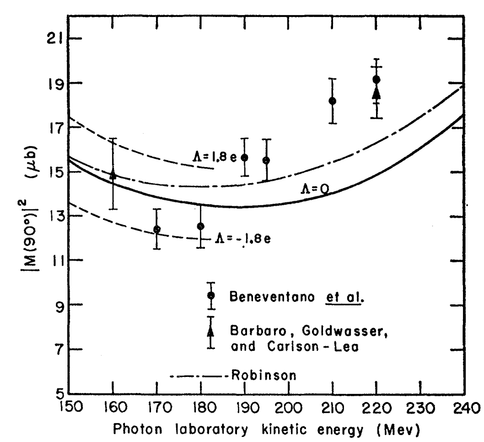

FIG. 10. The matrix element squared at $\theta=90$ deg for π^+ photoproduction. Solid line: prediction for $\Lambda=0$; dashed lines: predictions for $\Lambda=1.8e$ and $\Lambda=-1.8e$; dot-dash line: prediction of CGLN as calculated by Rosinson.[23] The experimental points are those of Barbaro *et al.* and Beneventano *et al.*[22]

At threshold, the correction to be added to $\mathfrak{F}_1^0$ is

$$\Delta\mathfrak{F}_1^0=6.8\times10^{-4}\Lambda/e, \tag{9.11}$$

which produces a fractional change of $1+(0.074\Lambda/e)$ in $|M|^2$ for π^+ production. Since $\mathfrak{F}^0$ enters with opposite signs into π^- production, $R=d\sigma(\pi^-)/d\sigma(\pi^+)$ at threshold will be even more sensitive to Λ. Including the correction given by Eq. (9.11) in R, we obtain

$$R=1.28\left\{\frac{1-(0.031\Lambda/e)}{1+(0.037\Lambda/e)}\right\}^2\simeq1.28[1-(0.14\Lambda/e)]. \tag{9.12}$$

The quantities A, B, and C for π^0 production and $|M|^2$ at $\theta=90$ deg for π^+ production have been calculated for $\Lambda=\pm1.8e$ (see Figs. 6–10).

XI. CONCLUSIONS

The results we have obtained in this work may be summarized by saying that after a more careful analysis

of photoproduction based on the Mandelstam representation, the work of CGLN survives almost unchanged from a practical point of view. The only modification is an additive term to correct the (0) amplitude in terms of the parameter Λ. However, it should be remembered that a change in the treatment of CGLN may also be required in the (+) amplitude if there is a resonant three-pion intermediate state in the t spectrum.

The evaluation of the dispersion integrals in their relativistic form for the (+) and (−) amplitudes did not produce any significant change from CGLN. The values $N^{+}=-0.062$ and $N^{-}=4.5\times 10^{-3}$ were obtained, indicating that the often used procedure of setting $N^{+}=N^{-}=0$ does not cause an important error.

To investigate what limit the various experimental data place on the size of Λ, consider first the coefficients A, B, and C giving the angular distribution for π^0 production. The value of A, which is the most accurately determined by experiment, proves to be quite insensitive to the values of Λ. The calculated values of A for $|\Lambda|<1.8e$ are all in good agreement with the experimental data. While the coefficients B and C are more sensitive to Λ, the difficulty encountered in comparing the theoretical values of these coefficients with those from experiment make these data a poor test for Λ. A further uncertainty in the theoretical values of A, B, and C arises from the possibility of a strongly interacting three-pion intermediate state, which would require a subtraction in the (+) amplitude.

The threshold π^+ data provide a better test of the magnitude of Λ. As can be seen from Fig. 10, with $f^2=0.08$ the experimental data constrain Λ to lie between $1.8e$ and $-1.8e$. Changing f^2 by ± 0.01, which is perhaps the maximum allowed by other considerations, can be compensated in the threshold π^+ cross section by giving Λ a value $\mp 1.75e$. (This compensating effect will not remain at higher energies, as the energy dependence of the Λ term is different from that of the other terms.)

The value of $R=d\sigma(\pi^-)/d\sigma(\pi^+)$ at threshold given by formula (9.12) provides a measure of Λ which is insensitive to small variations in f^2. However, the corrections and extrapolation necessary to obtain R cast some doubt as to its exact value. The range $-1.8e<\Lambda<1.8e$ corresponds to $1.0<R<1.6$, which is roughly the current uncertainty in the $(-/+)$ ratio.

An estimate of Λ based on the π^0 lifetime has been made by Wong.[3] His results are $|\Lambda|\gtrsim e$; however, the possibility of a resonant three-pion state produces some uncertainty in this estimate.

As the theoretical understanding of pion-nucleon scattering improves, an approach to photoproduction through multipole amplitudes as outlined in Sec. VIII should be carried through. Such a procedure could extend the description of photoproduction to the region in which phases other than the 3–3 become important. It could also improve the crude CGLN formula for the magnetic-dipole amplitude which has been accepted here as the basis for many of our calculations.

ACKNOWLEDGMENTS

The author is greatly indebted to Professor Geoffrey F. Chew for suggesting this problem and for his constant advice and encouragement throughout the course of this work. The author would like to thank Professor David Wong and Dr. How-sen Wong for many helpful discussions. The author would also like to thank Professor Adam Bincer for several helpful discussions on the problem of proving the analyticity of the scalar amplitudes for photoproduction.

APPENDIX

The first step in showing that the B's are free from kinematical singularities is the construction of a set of analytic scalar amplitudes (analytic here means analytic except for the appropriate Mandelstam singularities). After multiplying the general T matrix by the positive energy nucleon projection operators, we then write

$$(-i\gamma\cdot P_2+M)T\cdot\epsilon(-i\gamma\cdot P_1+M)$$
$$=(-i\gamma\cdot P_2+M)\Big[\sum_{i=1}^{8}B_iN_i\Big](-i\gamma\cdot P_1+M). \quad \text{(A1)}$$

Since this equation holds for all photon polarization and nucleon spins, we may multiply each side of this equation by the N_i's which are functions of nucleon and photon spin indices. If traces over the spin indices are now taken, each of these products will yield a scalar equation, and since the quantity on the left is still an analytic function of the components, the Hall-Wightman theorem can be used to show that these eight quantities are analytic function of the scalars. We will denote these scalars as follows:

$$T_i=\mathrm{Tr}[N_i^{\mu}(-i\gamma\cdot P_2+M)T^{\mu}\cdot(-i\gamma\cdot P_1+M)]. \quad \text{(A2)}$$

By taking the same traces on the right-hand side of Eq. (A1) the T_i's can be related to the B_i's. These T_I's are related to the B_I's by a linear transformation as follows:

$$T_i=R_{ij}B_j. \quad \text{(A3)}$$

The elements of R are polynomials in s, t, and $\bar{s}$. Thus the only singularities in the B's will be poles at the positions of the zeros of the determinant of R. However, since the determinant of R will be a very high order polynomial in s, $\bar{s}$, and t, it is somewhat simpler to consider a series of transformation between sets of amplitudes and then show at each step that the zeros in the determinants produce no singularities. The total transformation will then be just a product of these individual transformations.

Let us first consider the transformation from the T's

to a set of amplitudes X_i where these are defined as follows:

$$\begin{aligned}
X_1&=\tfrac{1}{2}(M^2-s)B_1+\tfrac{1}{2}(t-4M^2)B_2+\tfrac{1}{2}(\bar{s}-s)(B_3+B_4),\\
X_2&=MB_1+\tfrac{1}{4}(t-4M^2)B_6+\tfrac{1}{4}(\bar{s}-s)(B_7+B_8),\\
X_3&=(M^2-s)B_1+\tfrac{1}{2}(\bar{s}-s)B_2-2B_3+(t-1)B_4+2MB_5,\\
X_4&=B_5+\tfrac{1}{4}(\bar{s}-s)B_6+\tfrac{1}{2}(t-1)B_7-B_8,\\
X_5&=\tfrac{1}{2}(\bar{s}-s)B_2+(t-1)B_3,\\
X_6&=B_5+\tfrac{1}{4}(\bar{s}-s)B_6+\tfrac{1}{2}(t-1)B_8,\\
X_7&=T_1,\\
X_8&=T_5.
\end{aligned}\tag{A4}$$

Six of the T_i's are then given by a linear combination of two of the X_i's and the coefficients of this matrix are again polynomials in s and t. These six form three pairs of equations and each of these pairs contains only two X's and two T's and thus can be solved independently. Since the coefficients of each pair of coupled equations are identical, the first six X_i's are of the following two forms:

$$\begin{aligned}
X_1&=\frac{M(t-1)T_6-(s-M^2)(\bar{s}-M^2)T_2}{2D},\\
X_2&=\frac{M(t-1)T_2-tT_6}{2D},
\end{aligned}\tag{A5}$$

where $4D=M^2(t-1)^2-t(s-M^2)(\bar{s}-M^2)$ and is of course just the determinant of each pair of these equations. If D is evaluated in the physical photoproduction channel, it can be written $-16k^2q^2s(\sin^2\theta)$, and its two zeros for s as a function of t are in the forward and backward directions. We now use the fact that K for forward (backward) scattering can be expressed in terms of P_1 and P_2, and therefore $\gamma\cdot K$ can be expressed in terms of scalars, by use of the Dirac equation. The results of this are that for the forward and backward directions the T's satisfy the following 3 equations:

$$T_6=M\left(\frac{t-1}{t}\right)T_2,\quad T_8=M\left(\frac{t-1}{t}\right)T_3,$$

and

$$T_7=M\left(\frac{t-1}{t}\right)T_4.\tag{A6}$$

These sets of equations cause the numerators in Eq. (A5) to vanish and since they are analytic they must vanish as a power of s just cancelling the zero in D for the forward (backward) direction, proving that the X's have the same analyticity as the T's.

The next step is to express the B's in terms of the X's, by inverting Eq. (A4). In this case the determinant of the transformation matrix is just D^3. It can now be seen that this result is a natural one and implies that the B's are free from kinematical singularities. The three zeros in the determinant for each the forward and backward direction simply reflect the fact that there are three equations relating the various X's in each of these directions. The fact that in either of these directions P, Q, and K are not independent relates T_2, T_3, T_4, and T_6, T_7, T_8 yielding relations between X_1, X_3, X_5 and X_2, X_4, X_6. The fact that K can be expressed in terms of P_1 and P_2 relates X_7 and X_8. Since each of these equations would cause a first-order zero to appear in the determinant, we see that the third-order zero is produced by the set of three equations. This means that the set of equations relating the B's to the X's is consistent and that the numerator in each equation giving B_i in terms of the X_i's will vanish with the proper power in the forward (backward) direction to keep the B's regular.

We have now shown that no singularities have been introduced into the B amplitudes by the particular choice of spin functions used to define them.