state.[26] At these lower energies the $t=0$ interaction might be expected to dominate as pointed out by Carruthers and Bethe[27] and be observed first as the photon energy increases. In this connection, it should be noted that the produced pair of pions, as calculated here, is a mixture of $t=0$, 1, and 2 isotopic spin states and is not restricted on the basis of charge parity arguments since all orders of interaction with the Coulomb field of the nucleus were taken into consideration.

[26] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[27] P. Carruthers and H. A. Bethe, Phys. Rev. Letters **4**, 536 (1960).

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. T. A. Welton who suggested this problem and whose advice contributed markedly to every phase of this work. The continued encouragement of E. P. Blizard contributed to carrying this problem to completion and is gratefully acknowledged.

PHYSICAL REVIEW VOLUME 124, NUMBER 6 DECEMBER 15, 1961

Radiative Modes of K-Meson Leptonic Decay*

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(Received August 11, 1961)

The gamma-ray spectra for the decays $K\rightarrow\mu\nu\gamma$ and $K\rightarrow e\nu\gamma$ as well as $\pi\rightarrow\mu\nu\gamma$ and $\pi\rightarrow e\nu\gamma$ are calculated in full: Terms in the spectra proportional to lepton mass are retained so that the results are applicable to the muon decay mode, and the calculations take into account both inner bremsstrahlung radiation and radiation arising from structure in the meson vertex. The latter contributions are expressed in terms of two form factors.

The modes $K\rightarrow\mu\nu\gamma$ and $K\rightarrow e\nu\gamma$ can be used to test the validity of Wigner time-reversal invariance. Two ways of doing this are given, one based on measurements of the gamma spectra and the other on measurements of the transverse polarization of the muons from the $K\rightarrow\mu\nu\gamma$ decay.

Calculations have been carried out on the effect of a possible intermediate vector boson on the decay $K\rightarrow e\nu\gamma$. The calculations are in substantial agreement with those of Kanazawa, Sugawara, and Tanaka (KST). Contributions from internal bremsstrahlung radiation, not calculated in KST, are given in the present paper. The strongly interacting intermediate states which give rise to structure-dependent radiation are listed, and a discussion of possible ambiguities in the KST test, arising from these states, is given.

I. INTRODUCTION

In this paper, differential rates are calculated for the decay modes

$$K^{\pm}\rightarrow\mu^{\pm}+\nu+\gamma,$$
$$K^{\pm}\rightarrow e^{\pm}+\nu+\gamma, \tag{1.1}$$

as well as for the decays in which the K meson is replaced by a pion. These results extend the calculations of earlier workers[1]; in particular, the present calculations are applicable to the muon mode in (1.1) because all terms proportional to lepton mass are retained. The rates depend on photon energy, on the angle between photon and charged lepton (alternatively, on the kinetic energy of the charged lepton), on the known nonradiative decay lifetimes, and on two unknown functions of photon energy. Rates integrated over the angle are also given.

Electromagnetic and weak couplings are treated to first order in perturbation theory, while the effects of strong interactions are given without approximation in terms of form factors h_1 and h_2.

Besides Lorentz and gauge invariance, it is assumed that the leptons couple to K mesons via the V and A variants (i.e., vector coupling with the two-component neutrino). This assumption seems reasonable because in the related K_{l2} processes (l = muon or electron),

$$K^{\pm}\rightarrow\mu^{\pm}+\nu,$$
$$K^{\pm}\rightarrow e^{\pm}+\nu, \tag{1.2}$$

the muon mode of decay predominates. Just as in π_{l2} decay, such a result points to vector coupling.

The leptons are taken to couple "locally," with no particles mediating between the emission of the ν and charged lepton. As a consequence the form factors h_1 and h_2 for muon and electron modes in (1.1) are the same; therefore, such an assumption can be checked by measuring the $h_i(\mu)$ and $h_i(e)$ in turn and comparing results. In this manner several authors[2] have verified a local coupling hypothesis for the decays $\pi\rightarrow$(2 leptons).

The complete expressions for the photon spectra of (1.1) are given in Sec. II, Eqs. (2.7) through (2.15), and the Appendix. Magnitudes of the various terms of the decay rates are discussed in Sec. III in connec-

* Work done under the auspices of the U. S. Atomic Energy Commission.
† National Science Foundation Predoctoral Fellow.
[1] S. A. Bludman and J. A. Young, Phys. Rev. **118**, 602 (1960); V. G. Vaks and B. L. Ioffe, Nuovo cimento **10**, 342 (1958). These papers contain further references.

[2] T. Fazzini, G. Fidecaro, A. W. Merrison, H. Paul, and A. V. Tollestrup, Phys. Rev. Letters **1**, 247 (1958); G. Impeduglia, R. Plano, A. Prodell, N. Samios, M. Schwartz, and J. Steinberger, *ibid.* **1**, 249 (1958).

tion with the first test of Wigner time-reversal (T) invariance.

If T invariance holds for the processes (1.1), they are restricted in two ways. Firstly, h_1 and h_2 must be real; their phases are restricted to be 0 or π. Secondly, the rate cannot depend on scalar triple products of the momentum, spin, and polarization vectors characterizing the initial and final states since such scalars change sign under time reversal. Because these scalars occur only in interference terms, this second test measures phases indirectly. Both types of restriction are considered in this paper.

The first test of T invariance, discussed in Sec. III, involves a determination of the quantities $\mathrm{Im}|h_1 \pm h_2|$, where Im denotes the imaginary part. The possibility of a measurement of $\mathrm{Im}|h_1 \pm h_2|$ depends upon an interesting feature of the decays (1.1). The terms which predominate in the expressions for the decay to muons are different from those which predominate in the expressions for the decay to electrons, due to the great difference in electron and muon masses. This circumstance helps in obtaining a sufficient number of independent equations to solve for $\mathrm{Im}h_1$ and $\mathrm{Im}h_2$, since measurement of the same rate using a different mode can result in two equations for h_1 and h_2 instead of one.

The second test, discussed in Sec. IV, involves a measurement of the three vectors $\boldsymbol{\sigma}$, $\mathbf{k}$, and $\mathbf{l}$. σ is the muon polarization, and $\mathbf{k}$ and $\mathbf{l}$ are the gamma and muon 3-momenta. The scalar triple product is $\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{l}$.[3] Thus if the muons possess any net polarization perpendicular to the production plane (i.e., if the distribution of electrons from the subsequent decays of the muons is asymmetric with respect to that plane), T invariance does not apply to the process.

As an aid to understanding the physical significance of h_1 and h_2, it is helpful to list the intermediate states (structure) which contribute to the weight functions of a dispersion relation for h_1 and h_2. In Sec. V it is shown that such states must have total angular momentum $J=1$, and that the lowest mass states contributing to h_1 and h_2 are $K+2\pi$ and $K+\pi$, respectively. This last result is interesting in view of the report by Alston[4] of a resonance, called the K^*, at 885 Mev in the $K\pi$ system. If the K^* were $J=0$, it would have no effect on either form factor; if the K^* were $J=1$, it would have no effect on h_1 and would contribute to h_2, the term given in Eq. (5.2).

Recently[5] the decay $K\to e\nu\gamma$ has been considered as a possible test of the intermediate vector boson hypothesis.[6] We verify the formulas (6) through (12) of KST and in addition calculate the corrections to them arising from terms of higher order in the electron mass [see Eqs. (2.16) and (2.17)].

To facilitate comparison of the present results with those in KST, we note that in this paper a Feynman-Dyson approach is used, rather than the Low technique which is used in KST. The present approach leads immediately to a separation of the total amplitude into two parts, an inner bremsstrahlung amplitude and a structure-dependent amplitude. The separation exhibits the gauge invariance of the theory explicitly. It is essential to a full understanding of the structure terms, since each part represents a contribution from a different class of intermediate states (cf. Sec. V). In KST the separation is obtained by means of their definitions (5) and (7), although its significance is not explained.

In the present paper the structure-dependent amplitude is further separated into the contribution from the intermediate vector boson [Eq. (2.5)] and the contribution from other intermediate states.

Even if there were no weak interaction boson, an energy dependence of h_1 and h_2 arising from the strongly interacting intermediate states could conceivably counterfeit the effects of one. Accordingly in KST two tests are proposed to check that no such energy dependences are present. In Sec. VI of the present paper the sensitivity of these tests is examined, and the possibility of ambiguities, arising from the strongly interacting intermediate states, is discussed.

Section VII is an application of the formulas of this paper to the pion decay modes, $\pi\to\mu\nu\gamma$ and $\pi\to e\nu\gamma$.

II. CALCULATION OF THE PHOTON SPECTRUM

The amplitude for $K\to l\nu\gamma$ decay ($l=$ charged lepton) is the sum of two terms M_B and M_S. M_B is due to inner bremsstrahlung processes, while M_S is the structure amplitude. Their forms, which are known,[1] are given here for completeness. The form of M_S follows from Lorentz and gauge invariance and the (V,A) law:

$$M_S=\left[\frac{iegg'}{M^2}\frac{m_K}{(4q_0k_0)^{\frac{1}{2}}}\right]\bar{U}_l\left\{\frac{h_1}{m_K{}^2}[(q\cdot k)\epsilon\cdot\gamma-(\epsilon\cdot q)k\cdot\gamma]+\frac{h_2}{m_K{}^2}i\epsilon_{\lambda\mu\rho\sigma}\gamma_\lambda q_\mu\epsilon_\rho k_\sigma\right\}(1+\gamma_5)U_\nu. \quad (2.1)$$

The spatial γ matrices are anti-Hermitian; γ_0 and γ_5 are Hermitian. m_K is the mass of the K meson; the dot refers to the four-dimensional scalar product (e.g., $q\cdot\epsilon=q_0\epsilon_0-q_1\epsilon_1\cdots$); q_μ, k_μ, and ϵ_μ are the four-momentum of the K meson and the four-momentum and polarization of the gamma; $\epsilon_{\lambda\mu\rho\sigma}$ is a 4-dimensional antisymmetric unit tensor; U_l and U_ν are the charged lepton and neutrino spinors; h_1 and h_2 are two arbitrary functions of gamma energy k_0; the coupling constants $e=(4\pi/137)^{\frac{1}{2}}$ and gg'/M^2 are explicitly factored out of

[3] Since $\boldsymbol{\epsilon}$, the photon polarization, is harder to measure than $\boldsymbol{\sigma}$, we ignore triple products involving the former.

[4] M. Alston and M. Ferro-Luzzi, *Proceedings of the Conference on Strong Interactions, Berkeley, 1960* [Revs. Modern Phys. **33**, 416 (1961)].

[5] A. Kanazawa, M. Sugawara, and K. Tanaka, Phys. Rev. **122**, 341 (1961). This paper will be referred to as KST. The form factors $M^2h/[M^2-(k-q)^2]$ and $M^2h_2/[M^2-(k-q)^2]$ of these authors should be identified with the h_1 and h_2 of this report.

[6] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960).

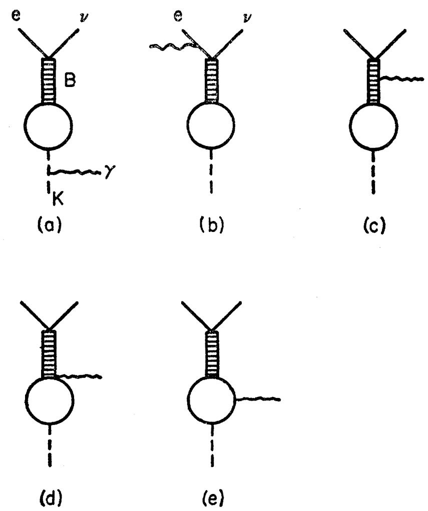

FIG. 1. Diagrams contributing to $K\to e\nu\gamma$ decay, assuming an intermediate vector boson B. The circle indicates possible structure due to strong interactions. Diagrams (a)–(d) give the inner bremsstrahlung terms and the contribution of the vector boson to h_1. Diagram (e) indicates contributions to h_1 and h_2 from structure other than the intermediate boson.

h_1 and h_2 for reasons of symmetry between M_S and M_B [cf. expression (2.2) and the discussion following Eq. (2.3)].

The amplitude M_B can be obtained from a perturbation approach. The calculation is made so that it also applies when an intermediate vector boson is present. One first writes down the Lagrangian for the free fields (electron, neutrino, intermediate boson,[7] and K meson) as well as the weak interaction Lagrangian for a point interaction [Eq. (2.2)] and then obtains the electromagnetic interaction Lagrangian by means of the replacement $\partial/\partial x_\mu \to \partial/\partial x_\mu + ieA_\mu$. The Feynman diagrams contributing to M_B are indicated in Fig. 1. Diagram (d) is obtained from the derivative in the point interaction[8] that would be assumed for $K\to e+\nu$ decay if there were no "structure" other than the vector boson:

$$gm_K i(\partial\phi^\dagger/\partial x_\mu)W_\mu + g'\bar{\psi}_e\gamma_\sigma(1+\gamma_5)\psi_\nu W_\sigma^\dagger + \text{Hermitian conjugate}; \quad (2.2)$$

ϕ is the K-meson field, W_μ is the intermediate boson field, and ψ_e, ψ_ν are the lepton fields.

Diagrams (a), (b), and that part of (d) which does not vanish with $M\to\infty$ for the vector boson mass M, form a gauge-invariant sum, the "internal bremsstrahlung" term:

$$M_B = \left[ie\frac{gg'}{M^2}\frac{m_K}{(4q_0k_0)^{\frac{1}{2}}}\right] m_l\bar{U}_l \times\left[\frac{l\cdot\epsilon}{l\cdot k}-\frac{q\cdot\epsilon}{q\cdot k}+\frac{(\epsilon\cdot\gamma)k\cdot\gamma}{2l\cdot k}\right](1+\gamma_5)U_\nu. \quad (2.3)$$

The quotient gg'/M^2 is known in terms of the inverse lifetime $W_{l\nu}$, calculated from expression (2.2) for the process $K\to l+\nu$:

$$W_{l\nu} = (gg'/M^2)^2(m_K{}^3 m_l{}^2/4\pi)(1-m_l{}^2/m_K{}^2)^2. \quad (2.4)$$

In the limit of no vector boson, $M^2\to\infty$, the product gg' is to increase so that gg'/M^2 remains constant.

The rest of diagram (d) plus diagram (c) is separately gauge-invariant and vanished as $M\to\infty$. This sum is the vector boson term. Since it is merely one kind of structure term, it will have the form given in formula (2.1). δh_1 and δh_2, the contributions to h_1 and h_2 from the intermediate vector boson, are

$$\begin{aligned}\delta h_1 &= m_K{}^2/[M^2-(k-q)^2]\\ &= m_K{}^2/(M^2-m_K{}^2+2k_0m_K);\\ \delta h_2 &= 0.\end{aligned} \quad (2.5)$$

In addition, all other contributions to h_1 and h_2 will contain a factor $M^2/[M^2-(k-q)^2]$, which arises from the propagator corresponding to the vector boson line in diagram 1(e). This line is present because of the assumption that the boson is an intermediary in all weak interactions. In contrast, the term (2.5) is due to electromagnetic, rather than weak interactions of the boson. No factor $M^2/[M^2-(k-q)^2]$ is present in the inner bremsstrahlung terms, since the denominator of the vector boson propagator is always canceled by terms arising from the numerator.

From M_B+M_S one may calculate the differential rates $k_{\max}d^2W/dk_0d\Omega$ and $k_{\max}dW/dk_0$. Ω refers to the angle θ between gamma and charged lepton, while $k_{\max}=(m_K{}^2-m_l{}^2)/2m_K$, the maximum allowed gamma energy, is a convenient unit of energy for this process. m_l is the lepton mass. Integrating over the four-dimensional δ function of energy-momentum conservation, $\delta^{(4)}$, in the formula

$$dW=(2\pi)^4\sum_{\sigma,\epsilon}|M_B+M_S|^2\delta^{(4)}\frac{d^3l\,d^3k\,d^3\nu}{(2\pi)^9}, \quad (2.6)$$

one obtains, for a stationary K meson, the differential rate of decay,

$$k_{\max}\frac{d^2W}{d\Omega dk_0}(\theta,k_0)=\frac{dL}{d\Omega}\frac{W_{Kl}\alpha}{4\pi}\frac{1}{k_{\max}}[S_0+S_1+S_2], \quad (2.7)$$

where

$$S_0=\frac{1}{l\cdot k}[l^2\sin^2\theta(2m_Kk_{\max}/l\cdot k)+2\nu\cdot k], \quad (2.8)$$

[7] The spin-one electromagnetic interaction Lagrangian used is that given in G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), p. 90. The vector boson was assumed to have no anomalous magnetic moment.

[8] It is possible to derive the form of the inner bremsstrahlung terms without writing down any specific kaon-lepton coupling. For such a derivation, which uses a generalization of Ward's identity, see H. Chew, Phys. Rev. **123**, 377 (1961). A further interesting result of the derivation is that, in the limit of small k_0, Eq. (2.3) gives the complete amplitude correct to all orders of the coupling constant e.

$$S_1=\frac{-4}{l\cdot k}[l^2\sin^2\theta\,\mathrm{Re}h_1+\nu\cdot k\,\mathrm{Re}(h_1-h_2)]\frac{k_0}{m_K},\tag{2.9}$$

$$S_2=\frac{4}{m_l^2}\bigg\{[l\cdot\nu-l^2\sin^2\theta](|h_1|^2+|h_2|^2)+l_0D_0\left[\frac{l\cdot k}{l_0k_0}-\frac{\nu\cdot k}{\nu_0k_0}\right]\times[2|h_1|\,|h_2|\cos(\phi_1-\phi_2)]\bigg\}\frac{k_0^2}{m_K^2},\tag{2.10}$$

$$\frac{dL}{d\Omega}=\frac{1}{2\pi}\frac{l^2k_0}{l(m_K-k_0)+l_0k_0\cos\theta},\tag{2.11}$$

l is the magnitude of the electron (muon) 3-momentum; l_μ and ν_μ are the electron (muon) and neutrino 4-momentum; W_{Kl} is the rate for $K\to l+\nu$, where l may be e or μ; θ is the angle between electron (muon) and gamma; ϕ_1 and ϕ_2 are the phases of h_1 and h_2.

If the differential rate $k_{\max}^2d^2W/dk_0dL$ is needed ($L=l_0-m_l=$ lepton kinetic energy), replace $dL/d\Omega$ in (2.7) by $k_{\max}$.

If (2.7) is integrated over angles, one obtains

$$k_{\max}\frac{dW}{dk_0}=\frac{\alpha W_{Kl}}{4\pi}\frac{1}{k_{\max}}[R_0+R_1+R_2],\tag{2.12}$$

where

$$R_0=\ln[m_K(m_K-2x)/m_l^2][4(m_K-1)/x-4+2x]-4(1-x)[x/(m_K-2x)+2/x],\tag{2.13}$$

$$R_1=\ln[m_K(m_K-2x)/m_l^2]\times[4m_l^2x\,\mathrm{Re}h_1/m_K^2+(-4x^2)\,\mathrm{Re}(h_1-h_2)/m_K]+[4x(1-x)/m_K(m_K-2x)]\times[2k_0\,\mathrm{Re}(h_1-h_2)+2(1-m_K+x)\,\mathrm{Re}h_1],\tag{2.14}$$

$$R_2=\frac{8x^3}{3m_l^2m_K}\frac{(1-x)^2}{(m_K-2x)^2}[3(m_K-2x)-2(1-x)]\times(|h_1|^2+|h_2|^2),\tag{2.15}$$

$x=k_0/k_{\max}$, and masses are expressed in units of $k_{\max}$.

The Appendix gives Eq. (2.7) integrated over all angles θ greater than an arbitrary angle θ_0.

R_0 and S_0 are inner bremsstrahlung terms, proportional to $|M_B|^2$; R_1 and S_1 arise from interference between M_B and M_S and therefore are linear in the h_i, while R_2 and S_2 are proportional to $|M_S|^2$ and are quadratic in the h_i.

For the electron decay mode the dominant term in Eq. (2.12) could well be R_2.[9] Inner bremsstrahlung corrections to that mode may be obtained from R_0:

$$k_{\max}\frac{dW_B}{dk_0}=\frac{\alpha W_{Ke}}{4\pi}\bigg\{[13.1+\ln(1-x)]\left[\frac{4}{x}-4+2x\right]+(-4)(1-x)\left[\frac{x}{2(1-x)}+\frac{2}{x}\right]\bigg\},\tag{2.16}$$

where $x=k_0/k_{\max}$. Interference terms due to R_1 are even smaller and may be neglected. As an example of the magnitude of the correction due to (2.16): For $k_0=0.4k_{\max}$, h_1 and h_2 real with $h_1=\frac{1}{2}$ ($M=7$ pion masses if there were no h_1 structure other than the boson) and, say, $h_1^2=h_2^2$, the correction is 5% (with $h_2^2=0$ the correction is 10%). Since inner bremsstrahlung varies as $1/x$, while the contribution to h_1^2 from the vector boson varies as $x^3(1-x)/M^4$, the correction for other values of k_0 and M can readily be estimated.

The formula for $k_{\max}dW/dk_0$ ($\theta\geq\pi/2$), (see reference 5 and the Appendix) i.e., the rate for events in which the electron falls in the backward cone, also needs a correction arising from inner bremsstrahlung:

$$k_{\max}\frac{dW_B}{dk_0}\left(\theta\geq\frac{\pi}{2}\right)=\frac{\alpha W_{Ke}}{4\pi}\bigg\{\left[\ln\left(\frac{2-x}{1-x}\right)\right]\left[\frac{4}{x}-4+2x\right]+\frac{4}{x}\frac{1}{x-2}\bigg\}.\tag{2.17}$$

III. MEASUREMENT OF THE FORM FACTORS

Using the rates of the preceding section, we give a procedure for measuring h_1 and h_2. In particular, the quantities $\mathrm{Im}|h_1\pm h_2|$ are of interest, since they should vanish if T invariance holds. It is clear that if the following quantities were known,

$$\begin{aligned}H_1&=\mathrm{Re}(h_1-h_2)\\&=|h_1|\cos\phi_1-|h_2|\cos\phi_2,\\H_2&=|h_1+h_2|^2\\&=|h_1|^2+|h_2|^2+2|h_1|\,|h_2|\cos(\phi_1-\phi_2),\\H_3&=|h_1|^2+|h_2|^2,\end{aligned}\tag{3.1}$$

then $\mathrm{Im}|h_1-h_2|$ would also be known since

$$2H_3-H_2-H_1^2=[\mathrm{Im}(h_1-h_2)]^2.\tag{3.2}$$

Further, if $\mathrm{Re}h_1$ were known, then $\mathrm{Im}|h_1+h_2|$ would follow, since

$$H_2-(2\,\mathrm{Re}h_1-H_1)^2=[\mathrm{Im}(h_1+h_2)]^2.\tag{3.3}$$

$\mathrm{Re}h_1$ and the quantities (3.1) occur as unknowns in rate expressions derived from Eqs. (2.7) and (2.12) and listed in Table I. The four expressions determine the four unknowns. It is necessary to specify for which particle, muon or electron, the rate is to be calculated, since only for the muon mode is the inner bremsstrahlung amplitude M_B expected to be sufficiently

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[9] That terms quadratic in h_1, h_2 are more likely to dominate in the electron mode can be seen by examining the dependence on lepton mass m_l of Eqs. (2.12) through (2.15). The nonradiative decay rate W_{Kl} varies roughly as m_l^2, so that, in going from muon to electron mode, R_0 and R_1 terms decrease as $m_l^2\ln(1/m_l^2)$. The quadratic terms, on the other hand, are almost the same for the two modes. [Electron inner bremsstrahlung can be large in angular distributions, however, near $\theta=0$, $1/(l\cdot k)\to O(1/m_e^2)$.]

TABLE I. Form factors occurring in the rate expressions. Column 3 gives the combinations of form factors which occur in the theoretical expressions for each of the quantities in column 1. The symbols e and μ of column 2 indicate for which mode, $K \to e\nu\gamma$ or $K \to \mu\nu\gamma$, the expression is calculated; in expressions intended for the e mode, some terms proportional to lepton mass can be neglected. ϕ is the angle between gamma and neutrino; Ω refers to the angle between gamma and e or μ; k_0 is the gamma energy.

Measurement	Mode	Form factors
$k_{max}d^2W/dk_0d\Omega$ $(\phi=0)$	e or μ	$\lvert h_1+h_2\rvert^2$
$k_{max}d^2W/dk_0d\Omega$ $(\phi=\pi)$[a]	μ	$\mathrm{Re}(h_1-h_2)$, $\lvert h_1-h_2\rvert^2$
$k_{max}dW/dk_0$[b]	e	$\lvert h_1\rvert^2+\lvert h_2\rvert^2$
$k_{max}dW/dk_0$[a]	μ	$\mathrm{Re}h_1$, $\mathrm{Re}(h_1-h_2)$, $\lvert h_1\rvert^2+\lvert h_2\rvert^2$

[a] Inner bremsstrahlung terms dominate.
[b] Inner bremsstrahlung terms are a small correction; $\mathrm{Re}h_1$ and $\mathrm{Re}(h_1-h_2)$ terms are negligible.

large that terms linear in the h_i, arising from the interference between M_B and M_S, must also be included.

The first two rates of the table apply to events in which the momenta of all three final particles lie in a straight line. There are two kinds of such events, depending on whether the angle ϕ between neutrino and gamma is 0 or π. The two types of events are distinguished experimentally by observation of the kinetic energy $L=l_0-m_l$ of the massive lepton. If gamma ray and neutrino emerge in the same direction ($\phi=0$), then from kinematics L must have the value

$$\begin{aligned} L&=L_{max} \\ &=(m_K-m_l)^2/2m_K \quad (\phi=0). \end{aligned} \tag{3.4}$$

The rate for such events is

$$k_{max}\frac{d^2W}{dk_0d\Omega}(k_0,\theta=\pi)\bigg|_{\phi=0} =\frac{\alpha W_{Kl}}{2\pi^2}\frac{k_{max}}{m_K^2m_l^2}k_0^3H_2 \tag{3.5}$$

$$=4.29\times10^{-4}W_{K\mu}x^3H_2 \text{ (muon)} \tag{3.6}$$

$$=4.72\times10^{-4}W_{K\mu}x^3H_2 \text{ (electron)}, \tag{3.7}$$

$x=k_0/k_{max}$. The rates are given in terms of the inverse lifetime $W_{K\mu}$ for the process $K\to\mu+\nu$; the appropriate k_{max} is to be used in each case.

If gamma and neutrino emerge in opposite directions ($\phi=\pi$), then L can have a spectrum of values

$$L=(m_K-m_l-2k_0)^2/2(m_K-2k_0) \quad (\phi=\pi). \tag{3.8}$$

The rate for these events is

$$k_{max}\frac{d^2W}{dk_0d\Omega}\bigg|_{\phi=\pi} =\frac{\alpha W_{Kl}}{\pi m_l^2}\frac{(k_{max}-k_0)}{k_{max}}\frac{dL}{d\Omega} \times\left[m_K-2k_0H_1+\frac{k_0^2}{m_K}|h_1-h_2|^2\right] \tag{3.9}$$

$$=1.07\times10^{-4}W_{K\mu}\left[\frac{(2.10-2x)^2-0.20}{(2.10-2x)}\right]^2 \times[4.40-4.20H_1x+x^2|h_1-h_2|^2]x \text{ (muon)}, \tag{3.10}$$

where

$$\frac{dL}{d\Omega}(\phi=\pi)=\frac{k_0}{2\pi}\left[\frac{(m_K-2k_0)^2-m_l^2}{2(m_K-2k_0)}\right]^2 \times\frac{1}{m_K(k_{max}-k_0)}. \tag{3.11}$$

For these events the angle θ between charged lepton and gamma will be π or 0, according as k_0 is greater or lesser than $(m_K-m_l)/2$ $(=0.82k_{max}$ for the muon mode, $\approx k_{max}$ for the electron mode). Because the events at $\phi=0$, Eq. (3.5), can have only the one value of L, Eq. (3.4), the measurement of L to distinguish $\phi=0$ from $\phi=\pi$ events, does not have to be accurate unless L is very large ($L_{max}=150$ Mev for muons, 247 Mev for electrons).

Measurement of (3.9) is not possible in the electron case. Except at the very highest gamma energies, the electron emerges at $\theta=0$; consequently the inner bremsstrahlung, which is generally not a problem for this mode, becomes large. The full effects of inner bremsstrahlung are not apparent from (3.9) because exactly at $\theta=0$ such effects are small: (3.9) is the constant term in an expansion of the rate in powers of θ. For $k_0<(m_K-m_l)/2$ $(\theta\approx0)$ the following term should be added to the bracket in (3.9):

$$\frac{2m_Kk_{max}}{m_l^2k_0^2}\frac{[(m_K-2k_0)^2-m_l^2]^2}{4(k_{max}-k_0)}\theta^2 \approx\frac{2m_Kk_{max}^2}{m_l^2}\frac{(1-x)^3}{x^2}\theta^2. \tag{3.12}$$

For the electron mode this term is very large. For the muon mode at $x=\frac{1}{2}$ and $\theta=10°$, the half-angle of a cone subtending 0.1 sr, this term is $\approx m_K/2\pi$.

Using the calculation of Eq. (2.5) with an intermediate vector boson of mass $M=$seven pion masses (this mass is taken to be typical of the masses of the particles involved in intermediate states) as a rough guide, one can estimate h_1 to be approximately $\frac{1}{4}$. Barring fortuitous cancellations, the H_1 and inner bremsstrahlung terms in (3.9) would then be in the ratio $x/4$ to 1.

The third and fourth entries of the table come from Eq. (2.12) specialized for the e and μ decay modes, respectively. As was mentioned previously, the R_2 terms are comparable in magnitude for μ and e modes, but the other two are much suppressed in the electron case[5,9] since M_B becomes small. Neglecting terms of higher order in m_e/m_K, one gets for the electron spectrum

$$k_{max}dW/dk_0=4.61\times10^{-3}W_{K\mu}x^2(1-x)H_3. \tag{3.13}$$

For (3.13) to be a valid approximation, h_1 must be of order $\frac{1}{4}$; then corrections due to R_0 will be small. Such corrections were given in Sec. II, Eq. (2.16).

In the muon case, on the other hand, all terms in (2.12) must be retained and inner bremsstrahlung radiation dominates. Again using the vector boson calculations as a guide, with $h_2=0$ and k_0 in the range $k_0 \geqslant k_{\max}/2$, R_1 plus R_2 terms may be expected to contribute from 10 to 15% to the total rate, with R_1 and R_2 terms contributing in the ratio 8–9 to 1. R_1 has a zero in this range of k_0 at approximately $\mathrm{Re}h_2=4\,\mathrm{Re}h_1$, but due to (3.5) and (3.9) such a zero cannot be taken to mean $h_1=h_2=0$.

IV. MUON POLARIZATION

We compute the differential transition rate $d^2W_\pm/dk_0d\Omega$ for production of muons with spin along $\pm\mathbf{n}=\pm\mathbf{k}\times\mathbf{l}/kl\sin\theta$ in terms of a quantity P:

$$d^2W_\pm/dk_0d\Omega=(d^2W/dk_0d\Omega)[\tfrac{1}{2}(1\pm P)]. \tag{4.1}$$

P is the net polarization along n. For negative muons P is given by

$$P\xi=\frac{4k_0l\sin\theta}{m_Km_\mu}\left\{\left[\frac{m_K}{k_0}-\frac{m_Kl_0}{l\cdot k}-1\right]\mathrm{Im}h_1+\frac{m_K\nu_0}{l\cdot k}\mathrm{Im}h_2\right\}, \tag{4.2}$$

$$\xi=S_0+S_1+S_2.$$

The detailed expression for ξ is given by Eqs. (2.8) through (2.10). For positive muons P has opposite sign.

V. INTERMEDIATE STATES

In this section we consider the intermediate states contributing to the weight functions of a dispersion relation for h_1 and h_2. By virtue of the locality hypothesis, the lepton momenta occur in expressions for structure terms only in the combination $(l+\nu)$. The reaction $K\to\gamma+l+\nu$ thus has the analytic properties of a three-particle vertex function, the third "particle" being $l+\nu$ with $(\text{mass})^2=(l+\nu)^2$. It is convenient to write the dispersion relations for the process $K+\gamma\to l+\nu$ which is physical for $s=(l+\nu)^2\geqslant m_K^2$, and then obtain the matrix element for the decay process by continuing analytically in s to $m_K^2\geqslant s\geqslant m_l^2$. We assume, as is reasonable from perturbation theory applied to the three-particle vertex function,[10] that the matrix element is analytic in s except perhaps for a pole at $s=m_K^2$ and a cut for $s\geqslant(m_K+m_\pi)^2$. In this region of s the weight functions can be obtained from unitarity.

$$\mathrm{Im}(\nu l|T|K\gamma)=\pi\sum\nolimits_n(\nu l|T|n)\delta(E_n-E)(n|T^\dagger|K\gamma). \tag{5.1}$$

$\mathrm{Im}(\nu l|T|K\gamma)$ contains more than is wanted, however, since it contains contributions from inner bremsstrahlung radiation, while what is desired is a dispersion relation for the structure terms only. For example, when $n=K$ the matrix element $(K|T^\dagger|K\gamma)$ represents absorption by the K meson of a photon of zero energy (in virtue of the conservation of energy). At zero photon energy the form of this matrix element is known exactly: It diverges as $1/(q\cdot k)$ and therefore cannot contribute to the structure matrix element. A similar consideration rules out contributions from $n=\nu l\gamma$. It follows that the separation of the amplitude into M_B and M_S corresponds to a separation of intermediate states into K or $\nu l\gamma$ and all others.

States n contributing to M_S contain no leptons, if terms of higher order in the weak and electromagnetic couplings are dropped. The factor $(\nu l|T|n)$ must represent the weak step, therefore, while $(n|T^\dagger|K\gamma)$ is electromagnetic and conserves parity and strangeness. The states n accordingly have the same strangeness and charge as the K meson and have baryon number zero: $n=K+\pi$, $K+2\pi$, $\cdots 2K+\bar{K}$, $\cdots$, $N\bar{Y}$ or $\bar{N}Y$ (where $Y=$hyperon), etc.

In the frame in which the center of mass of the two leptons is at rest, the photon wave functions obtained from the h_1 and h_2 amplitudes in Eq. (2.1) transform as $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}\times\mathbf{k}$, respectively, i.e., as $E1$ and $M1$ photons. Consequently, the states n have $J=1$; furthermore, states contributing to h_2 have the same parity as the K meson, while those contributing to h_1 have opposite parity. It now follows from parity conservation applied to $(n|T^\dagger|K\gamma)$ that the state $n=K\pi$ contributes only to h_2. Contributions to h_1 start from the cut at $s=(m_K+2m_\pi)^2$ [or from the pole at $s=M^2$, Eq. (2.5), if $M<m_K+2m_\pi$]. A $J=1$, $n=K\pi$ resonance of narrow width centered at $s=(m^*)^2$ would contribute to h_1 and h_2 an amount

$$h_1^*=0,$$
$$h_2^*=\frac{Gm_K^2}{(m^*)^2-(k-q)^2}\,\frac{M^2}{M^2-(k-q)^2}, \tag{5.2}$$

where G is a dimensionless coupling constant depending on the strengths of the couplings $K\gamma$ to K^* and K^* to the vector boson.

Because of final state and initial state strong interactions, the matrix elements $(\nu l|T|n)$ and $(n|T^\dagger|K\gamma)$ can exhibit an energy behavior which comes from a strong process such as $(n|T|n)$. As a consequence, the energy behavior of matrix elements such as $(K\pi|T|K\pi)$ is relevant to the present discussion, even though they are not explicitly present in the sum (5.1).

VI. THE RATIO h_1/h_2

In this section the test proposed in KST for the existence of the intermediate vector boson is discussed. We first summarize the proposal. Any such test must start from the one effect which the boson has on the matrix element for $K\to l\nu\gamma$: The boson requires h_1 and

[10] R. Oehme, Phys. Rev. 111, 1430 (1958).

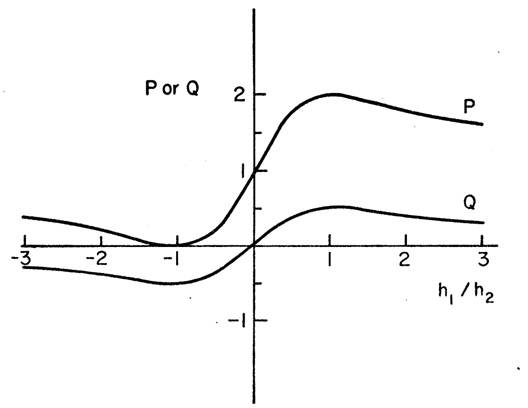

FIG. 2. The functions P and Q, Eqs. (6.3) and (6.4), plotted against h_1/h_2. They vary slowly near the points $h_1/h_2=\pm 1$.

h_2 to have the form

$$h_1=(m_K^2+M^2d_1)/[M^2-(k-q)^2],\qquad h_2=M^2d_2/[M^2-(k-q)^2], \tag{6.1}$$

where d_1 and d_2 represent the (incalculable) effects of strongly interacting intermediate particles. The test in KST, which applies only if d_1 and d_2 are constants or very nearly so, is to show, by a measurement of dW/dk_0 for $K\to e\nu\gamma$, that h_1 and h_2 have the energy dependence given by the denominator of (6.1).

Even in the limit $M^2\to\infty$, d_1 and d_2 could possess an energy dependence simulating a boson if they were not constants. The test for the boson therefore involves an auxiliary test of the behavior of d_1 and d_2. This test is in essence a measurement of the ratio

$$h_1/h_2=(m_K^2+M^2d_1)/M^2d_2. \tag{6.2}$$

The denominators of (6.1) conveniently cancel out of (6.2), and any residual energy dependence would be due to strong interactions only.

Actually, the suggestion in KST is to measure the energy dependence of either of the two ratios

$$P=(h_1+h_2)^2/(h_1^2+h_2^2)=(1+h_1/h_2)^2/[1+(h_1/h_2)^2], \tag{6.3}$$

or

$$Q=h_1h_2/(h_1^2+h_2^2)=(h_1/h_2)/[1+(h_1/h_2)^2], \tag{6.4}$$

whichever is experimentally the more convenient, rather than to measure the energy dependence of the ratio h_1/h_2 directly. P and Q, rather than h_1/h_2, are the quantities obtainable directly from experiment since the spectrum of $K\to e\nu\gamma$ is quadratic in the h_i.

It is argued that h_1 and h_2 are not simply related in any way (the states contributing to d_1 and d_2 have opposite parities), so that if the ratio (6.2) is a constant, then d_1 and d_2 separately must be constants.

If the result $P\approx$constant, $Q\approx$constant were obtained experimentally, some care would be needed in its interpretation. The functions P and Q are plotted against h_1/h_2 in Fig. 2. P and Q are seen to have zero derivative at $h_1/h_2=\pm 1$. The functions P and Q therefore would be nearly constant in the region $h_1/h_2\approx 1$, even though h_1/h_2 itself were not.

An implicit assumption of the argument in KST is that, barring resonances, the probability that d_1 and d_2 be rapidly varying is small. Then the probability of rapid variation in d_1 and d_2 simultaneously would be even smaller, and the most probable interpretation of the result, $h_1/h_2\approx$constant, would be d_1 and d_2 individually constants. The alternative interpretation, that no weak boson exists and both d_1 and d_2 vary rapidly but in the same way, would be much less probable.

However, the alternative should be considered as a serious one, because it is possible to give h_1 and h_2 rapid energy behavior without invoking resonances or improbable mechanisms. In fact, if the weight functions in the dispersion treatment of the h_i simply changed sign, the resulting energy dependence could be quite rapid.

To understand how this dependence would arise, one can study dispersion relations which the h_i satisfy:

$$d_i=\frac{1}{\pi}\int_{s_{Ti}}^{\infty}\frac{\rho_i(s')ds'}{s'-s},\quad (i=1,2), \tag{6.5}$$

where s_{T1} and s_{T2} are $(m_K+m_\pi)^2$ and $(m_K+2\pi)^2$; $s=(k-q)^2$; and ρ_1 and ρ_2 are given, apart from kinematical factors, by the quantity (5.1) evaluated for transitions of $E1$ and $M1$ photons, respectively. d_i, rather than h_i, is written in Eq. (6.5); it is taken to apply in the limit $M^2\to\infty$ [cf. Eq. (6.1)]. We investigate what types of energy dependence are allowed for the d_i by the relations (6.5) when $M^2\to\infty$ and compare the results to the energy dependence expected for the case M^2 finite and d_i constant.

To facilitate the comparison, one can expand the denominator of Eq. (6.5) in powers of s/s':

$$d_i=\frac{1}{\pi}\int_{s_{Ti}}^{\infty}\rho_i(s')ds'\left(\frac{1}{s'}+\frac{s}{(s')^2}+\frac{s^2}{(s')^3}+\cdots\right) \tag{6.6}$$

$$\equiv A_0\left[1+\frac{A_1}{A_0}\left(\frac{s}{m_K^2}\right)+\frac{A_2}{A_0}\left(\frac{s}{m_K^2}\right)^2+\cdots\right]. \tag{6.7}$$

The corresponding expansion in the case M^2 finite and d_i constant yields

$$h_i=(\text{const})\left[1+\frac{m_K^2}{M^2}\left(\frac{s}{m_K^2}\right)+\left(\frac{m_K^2}{M^2}\right)^2\left(\frac{s}{m_K^2}\right)^2+\cdots\right]. \tag{6.8}$$

Large A_n, $n>0$, in (6.7) therefore correspond to a boson of low mass. These expansions can be understood as expansions in the photon energy, $x=k_0/k_{\max}$, about

its maximum value, $x=1$, since $s/m_K^2 \approx (1-x)$ (lepton mass neglected).

One now studies the magnitudes of the quantities A_n/A_0. The case $n=1$ will be discussed in detail; the discussion for $n>1$ is similar. The quantities A_0 and A_1 are expressible in terms of the weight function ρ,

$$A_0 = \frac{1}{\pi}\int_{s_T}^{\infty} \frac{\rho(s')ds'}{s'}, \tag{6.9}$$

$$A_1 = \frac{m_K^2}{\pi}\int_{s_T}^{\infty} \frac{\rho(s')ds'}{(s')^2}. \tag{6.10}$$

First suppose $\rho(s')$ did not change sign. Then A_1/A_0 is a weighted average of the quantity m_K^2/s' with normalized weight function $\rho(s')/s'$. A_1/A_0 would be greater than m_K^2/s_T, and its exact magnitude would depend on how rapidly $\rho(s')/s'$ fell off to zero. If $\rho(s')$ fell off slowly, A_1/A_0 would be very small, and expression (6.7) would simulate (6.8) only if M^2 were taken very large.

Now suppose $\rho(s')$ changed sign. Then as one integrated away from threshold, $|A_0|$ and $|A_1|$ would first increase, then decrease, but $|A_1|$ less so than $|A_0|$ because larger values of (s') contribute less to $|A_1|$ than to $|A_0|$. The ratio A_1/A_0 would therefore increase, and the equivalent M^2 in expression (6.8) could even be below s_T.

The ratios A_n/A_0 with $n>1$ would be enhanced also, although in general each succeeding A_n/A_0 would not be exactly m_K^2/M^2 smaller than the preceding A_{n-1}/A_0, as is the case for the coefficients in (6.8). In principle, therefore, a measurement of the higher coefficients would distinguish between the series (6.7) and (6.8); in practice, however, it would be surprising to find that the measured values of the higher coefficients agreed with the ideal values given in Eq. (6.8), even for the case of finite M^2: Small deviations are to be expected, due to departures of d_1 and d_2 from rigorous constancy.

VII. PION MODES

In this section the magnitude of the various terms in the $\pi \to e\nu\gamma$ and $\pi \to \mu\nu\gamma$ rates will be estimated using the vector boson calculation as a guide.

The general formulas of the preceding sections and the Appendix also apply to the pion case, if the K-meson lifetimes and mass are replaced by the corresponding pion quantities. As in the K-meson case, the general conclusion is that terms quadratic in h_1, h_2 predominate in the electron mode, while inner bremsstrahlung and terms linear in h_1, h_2 predominate in the muon mode.[1,9]

In Eq. (2.5) with m_K replaced by the pion mass m_π, we take $M=3m_\pi$, a mass not allowed for the boson but consistent with the order of magnitude of masses of intermediate states allowed by symmetry principles. With this mass the inner bremsstrahlung still dominates in the muon mode. At $k_0=\frac{2}{3}k_{\max}$, the R_1 terms of Eq. (2.12) are a 2% correction (taking $\mathrm{Re}h_2=0$). To indicate how the rates (3.5) and (3.9) are affected in magnitude, we quote the pion analog of Eq. (3.6):

$$k_{\max}\frac{d^2W}{dk_0 d\Omega}\bigg|_{\phi=0} = 1.3\times10^{-6} W_{\pi\mu} x^3 H_2. \tag{7.1}$$

For the electron mode, with $M=4m_\pi$ (approximately the lowest mass allowed for the intermediate vector boson), $h_2^2=0$,[11] and $k_0=0.9k_{\max}$, the ratio of h_1^2 terms to inner bremsstrahlung terms in $k_{\max}dW/dk_0$ is 2.4 to 1, while the absolute rate for the inner bremsstrahlung is $k_{\max}dW_B/dk_0=2.23\times10^{-7}W_{\pi\mu}$.

VIII. CONCLUSION

For the decay $K \to \mu\nu\gamma$, inner bremsstrahlung predominates over structure radiation. With strong inner bremsstrahlung, however, the interference terms between the inner bremsstrahlung amplitude and the structure-dependent amplitudes become quite measurable. With the aid of additional information from the spectrum of $K \to e\nu\gamma$, it is then possible to determine $\mathrm{Re}h_1$ and $\mathrm{Re}h_2$ completely, while $\mathrm{Im}(h_1 \pm h_2)$ can be determined up to sign.

Two measurements, one a measurement of the form factors of the processes $K \to \mu\nu\gamma$ and $K \to e\nu\gamma$, and the other a measurement of the transverse polarization of the muons from $K \to \mu\nu\gamma$, have been proposed as a test of the validity of time-reversal invariance.

From measurements on the total number of gamma rays emitted at a given energy, together with the number of such events in which the three final particles emerge collinearly, it is possible to determine $\mathrm{Im}|h_1-h_2|$ and $\mathrm{Im}|h_1+h_2|$. These quantities will not vanish if T invariance is violated. Data from both modes $K \to e\nu\gamma$ and $K \to \mu\nu\gamma$ is to be combined, under the assumption that, if the muon is a "heavy electron," the two sets of form factors $h_i(\mu)$ and $h_i(e)$ will be identical. Since the quantity $|h_1+h_2|^2$ can be measured equally well using either mode [see Eqs. (3.6) and (3.7)], this assumption of equivalence of e and μ can be checked.

The muon polarization normal to the production plane has been calculated on the assumption that the leptons couple by the vector variants only. The polarization test is not sensitive to this assumption, however. Tensor or scalar[12] variants containing phase factors forbidden by time-reversal invariance would also give rise to normal polarization.

The spark chamber would be the best device for carry-

[11] h_2^2 can be calculated[1] in terms of the π^0 lifetime τ_0 using the conserved vector current hypothesis. In the notation of the present paper, h_2^2 is given by $h_2^2=1/(\pi m_\pi \alpha^2 \tau_0)=3\times10^{-4}$ for $\tau_0=1\times10^{-16}$ sec.

[12] The scalar term violates the assumption of local coupling of the leptons, since this principle requires that in structure terms the lepton momenta occur only in the combination $(l+\nu)_\mu$.

ing out the second test (as well as the first). If the chamber were lined with lead to convert the gamma, then two particles from the K decay, as well as the electron from the muon decay, would be visible. The chamber can distinguish between electron and muon mode in (1.1) and can separate both these modes from 2γ events:

$$\begin{aligned} K^{\pm} &\rightarrow \mu^{\pm}+\nu+\pi^0(2\gamma),\\ K^{\pm} &\rightarrow e^{\pm}+\nu+\pi^0(2\gamma). \end{aligned} \tag{8.1}$$

It should be noted that the muon mode of the decay (8.1) can also be used as a test of T invariance; the literature[13] is in error regarding this point. It is not true that the transverse polarization of the muons from (8.1) vanishes independently of whether h_1 and h_2 are real, simply because the coupling is assumed to be V and A only. On the contrary, a test utilizing polarization measurements is possible for the muon mode of (8.1) just as for the muon mode of (1.1). A correct calculation of the polarization to be expected in the decays (8.1) for V and A as well as other couplings is given by Ivanter.[14]

The two tests suggested in KST for monitoring the strong-interaction background in connection with the search for the intermediate vector boson must be applied with care, as they are insensitive when h_1 and h_2 are similar in magnitude.

It is impossible to predict exactly what energy dependence will be imparted to h_1 and h_2 by the strongly interacting intermediate states; but, in view of its relevance to the proposed search for the intermediate vector boson, a less ambitious question has been discussed: What energy dependence could conceivably be imparted? Of course, it is conceivable that there be a resonance in a strongly interacting intermediate state; this resonance would impart rapid energy variation to the appropriate h_i. The point to be stressed, however, is that even in the absence of resonances, rapid energy variation is *a priori* as likely as slow variation, since zeros as well as peaks in the weight functions can give rise to rapid energy behavior. Such zeros could arise if contributions from successive cuts had opposite sign; there is no need to invoke any behavior more unusual than this. Accordingly, the proposed test for the weak boson is ambiguous.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Sherwood Parker and Dr. Sheldon Penman for their questions and discussions which led to the writing of this paper, and Dr. S. Courtenay Wright for clarification of experimental points. The author also thanks Dr. R. H. Dalitz and Mr. Herman Chew for helpful discussions on dispersion relations and inner bremsstrahlung radiation.

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[14] I. G. Ivanter, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 111 (1958) [translation: Soviet Phys.—JETP **8**, 79 (1959)].

APPENDIX

In this Appendix we calculate the probability $k_{\max}dW/dk_0\ (\theta_0,k_0)$ that the lepton e or μ will make an angle θ_0 or greater with a photon of energy k_0; that is, we integrate $k_{\max}d^2W/dk_0d\Omega$ over a cone of half-angle $(\pi-\theta_0)$.

The integrations are simpler if done with $L=l_0-m_l=L(\theta,k_0)$ rather than θ as variable of integration, and the results will be given in terms of $k_{\max}dW/dk_0\ (L_0,k_0)$, the probability that the lepton e or μ will have kinetic energy $L\geqslant L_0$ when the photon has energy k_0:

$$k_{\max}\frac{dW}{dk_0}(L_0,k_0)=\frac{\alpha W_{Kl}}{4\pi}(T_0+T_1+T_2), \tag{A1}$$

where

$$T_0=\ln\left(\frac{x}{x-\Delta}\right)\left[\frac{4(m_K-1)}{x}-4+2x\right]+\frac{2\Delta}{x^2(x-\Delta)}\times[m_K\Delta-2x(L_{\max}+m_l+\Delta)+2x^2]-2\Delta, \tag{A2}$$

$$T_1=\left(\frac{4x\,\mathrm{Re}h_1}{m_K}\right)\left\{\frac{m_l^2}{m_K}\ln\left(\frac{x}{x-\Delta}\right)-\frac{\Delta}{x^2}\left[\frac{\Delta m_K}{2}-x(\Delta+2-m_K)\right]\right\}+\left[\frac{4x}{m_K}\mathrm{Re}(h_1-h_2)\right]\left[\Delta-x\ln\left(\frac{x}{x-\Delta}\right)\right], \tag{A3}$$

$$T_2=\frac{4H_2}{m_l^2}\left[-\frac{x\Delta^2}{2}+\frac{x^2\Delta(1+\Delta)}{m_K}-\frac{x^3\Delta}{m_K}\right]+\frac{4H_3}{m_l^2}\left\{\frac{\Delta^3}{3}+x\left[\frac{\Delta^2}{2}-\frac{\Delta^2(1+2\Delta/3)}{m_K}\right]\right\}. \tag{A4}$$

$L_{\max}=(m_K-m_l)^2/2m_K$; $x=k_0/k_{\max}$ and all masses are in units of $k_{\max}$; $\Delta=L_{\max}-L_0$.

To calculate $dW/dx(\theta_0,x)$ one determines what range of L corresponds to the range $\theta\geqslant\theta_0$ and then applies Eq. (A1). The analytic relationship between L and θ is obtained from a quadratic equation

$$\Delta=L_{\max}-L=[-c_2\pm(c_2^2-4c_1c_3)^{\frac{1}{2}}]/2c_1, \tag{A5}$$

where

$$\begin{aligned} c_1&=x^2\sin^2\theta+m_K(m_K-2x),\\ -c_2&=2(L_{\max}+m_l)x^2\sin^2\theta+2xm_K(1-x),\\ c_3&=x^2\sin^2\theta. \end{aligned} \tag{A6}$$

The relationship between L and θ can also be seen graphically. If L is plotted against θ in polar coordinates, the result is a lobe symmetric about the horizontal axis $(\theta=0,\pi)$. The position of the lobe depends on k_0; for

$$x<m_K/(m_K+m_l)\equiv x'. \tag{A7}$$

The lobe encloses the origin, while for $x>x'$ the lobe lies in the backward cone

$$\theta \geqslant \theta' \equiv \arcsin[m_K(1-x)/m_l x] \geqslant \pi/2. \qquad (A8)$$

As x approaches unity, the cone gradually narrows down about $\theta=\pi$. For $x>x'$, a ray from the origin at an angle $\theta>\theta'$ intersects the lobe twice: Two values of L, corresponding to the two roots of (A5), are allowed at every value of $\theta>\theta'$. If $x>x'$, the largest and smallest values of L are both at $\theta=\pi$; and if in addition $\theta_0>\theta'$, some intermediate values of L lie outside the cone $\theta>\theta_0$.

From the foregoing discussion it is clear that the procedure for obtaining $dW/dx(\theta_0,k_0)$ from dW/dx (L_0,k_0) depends on k_0 and θ_0. Accordingly, the rates will be given subscripts a, b, or c to indicate in which region they apply:

$$\begin{aligned}
&\text{(a)}\quad x \leqslant m_K/(m_K+m_l) \text{ and all } \theta_0,\\
&\text{(b)}\quad m_K/(m_K+m_l) \leqslant x \leqslant m_K/(m_K+m_l \sin\theta_0) \text{ and } \theta_0>\pi/2,\\
&\text{(c)}\quad m_K/(m_K+m_l \sin\theta_0) \leqslant x \text{ and } \theta_0>\pi/2;\\
&\text{or}\\
&\qquad m_K/(m_K+m_l) \leqslant x \text{ and } \theta_0 \leqslant \pi/2.
\end{aligned} \qquad (A9)$$

For $K\rightarrow e\nu\gamma$, the rates (a) apply at all but the highest energies; for $K\rightarrow \mu\nu\gamma$ the rates (b) or (c) apply when $0.82<x$.

We list the results in one place and then summarize the procedure used for each case.

The results are

$$\frac{dW_a}{dx}(\theta_0,k_0)=\frac{dW}{dx}[L_0(\theta_0,k_0),k_0], \qquad (A10)$$

$$\frac{dW_b}{dx}(\theta_0,k_0)=\frac{dW}{dx}-\left\{\frac{dW}{dx}[L_{0+}(\theta_0,k_0)] - \frac{dW}{dx}[L_{0-}(\theta_0,k_0)]\right\}, \qquad (A11)$$

$$\frac{dW_c}{dx}(\theta_0,k_0)=\frac{dW}{dx}. \qquad (A12)$$

In case (a) the range of integration is $L_0(\theta_0,k_0) \leqslant L \leqslant L_{\max}$; the L_0 to be used in (A10) comes from the positive (negative) root of (A5) when θ_0 is less than (greater than) $\pi/2$.

Case (b) corresponds to $\theta_0>\theta'$; i.e., the cone $\theta\geqslant\theta_0$ cuts the lobe. Some intermediate values of L fall outside the cone $\theta\geqslant\theta_0$; accordingly, the bracket in (A11) subtracts the events in the range $L_{0+}\leqslant L\leqslant L_{0-}$ from the total number of events, dW/dx, Eq. (2.12). $L_{0\pm}$ comes from the $\pm$ root of (A5).

Case (c) corresponds to $\theta_0<\theta'$, i.e., k_0 so large that all leptons are constrained to fall within θ_0.

The case $\theta_0=\pi/2$ is particularly simple: in region (a) the discriminant in (A5) vanishes giving $\Delta(\pi/2,x) = x/(m_K-x)$, while region (b) does not exist for $\theta=\pi/2$ (or in fact for any $\theta\leqslant\pi/2$).