Cross Sections at High Energies*

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If the difference $\sigma_+(E)-\sigma_-(E)$ of particle and antiparticle total cross sections changes sign at most at a finite number of energies, then the odd part $f_+(E)-f_-(E)$ of the forward scattering amplitude has a very useful representation, as a rational times a "Herglotz" function. The representation implies a correlation between the high-energy asymptotic behavior of $\sigma_+(E)-\sigma_-(E)$ and $f_+(E)-f_-(E)$. For example, if $\{f_+(E)-f_-(E)\}/E\ln^m E$ is bounded, and $m\leq\frac{1}{2}$ then we have the Pomeranchuk result that $\sigma_+(E)-\sigma_-(E)\rightarrow 0$. Even if $m>\frac{1}{2}$ it seems likely that although the difference of $\sigma_+(E)$ and $\sigma_-(E)$ may not tend to zero, their ratio does tend to one.

POMERANCHUK[1] has remarked that the total cross sections $\sigma_\pm(E)$ for a particle and its antiparticle incident on any target should approach each other as $E\rightarrow\infty$. In order to use his method of proof, it is necessary to assume that

(A) the difference $\sigma_+(E)-\sigma_-(E)$ approaches a constant $\Delta\sigma$ very rapidly as $E\rightarrow\infty$;

(B) the odd part of the forward scattering amplitude

$$g(E^2)\equiv\frac{f_+(E)-f_-(E)}{2E}=\frac{f_+(E)-f_+(-E)}{2E} \tag{1}$$

satisfies a once-subtracted dispersion relation, where $f_\pm(E)$ are the particle and antiparticle forward scattering amplitudes for lab energy E; and

(C) as $E\rightarrow\infty$, $g(E^2)$ remains bounded. Since the spectral function of $g(E^2)$ is

$$\rho(E^2)\equiv\frac{1}{\pi}\operatorname{Im}g(E^2)=\frac{(E^2-m^2)^{\frac{1}{2}}}{4\pi^2E}[\sigma_+(E)-\sigma_-(E)] \tag{2}$$

(except perhaps for a finite range of E), assumptions (A) and (B) would lead to $g(E^2)\sim(\Delta\sigma)\ln E$, violating (C) unless $\Delta\sigma=0$.

One naturally wonders what would happen if $\sigma_+(E)-\sigma_-(E)$ were to oscillate or grow as $E\rightarrow\infty$, or if $g(E^2)$ were not bounded. Actually, it is hard to justify (A) and (C). If partial waves higher than $l=kR_\pm$ may be neglected for large E, then $|g(E^2)|\leq\frac{1}{2}(R_+^2+R_-^2)$. If absorption is a maximum for the other partial waves, then $\sigma_\pm(E)\rightarrow 2\pi R_\pm^2$. With constant $R_\pm$, assumptions (A), (B), and (C) would be justified, although one could still question whether $\sigma_\pm(E)$ approach constants fast enough to justify the method of proof.

However, Froissart's recent work[2] suggests strongly that $R_\pm$ may be functions of E, growing perhaps as fast as $\ln E$. Whether or not this is the case, it seems worthwhile to spell out precisely what assumptions are needed to prove the asymptotic equality of $\sigma_\pm(E)$.

We shall present here a rigorous generalization of Pomeranchuk's theorem that has the advantages of virtually doing away with assumptions (A) and (B), and of giving information on the behavior of $\sigma_+(E)-\sigma_-(E)$ even if assumption (C) is relaxed. It is at least part of our motivation to demonstrate here some mathematical methods which may prove generally useful in dispersion theory.

The assumption we make in place of (A) is that the difference $\sigma_+(E)-\sigma_-(E)$ does not change sign an infinite number of times. This certainly follows from (A) (unless $\Delta\sigma=0$) and hence is a far weaker starting point for proving $\Delta\sigma=0$.

Our work is based on the fact that if a function $g(z)=g^*(z^*)$ is analytic except for a cut on the real axis, with spectral function $\pi\rho(\omega)=\operatorname{Im}g(\omega+i\epsilon)$, and if $\rho(\omega)$ changes sign at most a finite number of times, then $g(z)$ may be written

$$g(z)=R(z)H(z), \tag{3}$$

where $R(z)$ is rational, and $H(z)$ is a "Herglotz function," i.e., $\operatorname{Im}H(z)\geq 0$ for $\operatorname{Im}z\geq 0$. [This follows immediately from a recent theorem of Symanzik,[3] who showed that if a function $g_1(z)$ satisfies the above postulates, but also has spectral function $\rho_1(\omega)\geq 0$, then $g_1(z)=P(z)H(z)$, where $P(z)$ is a polynomial (containing all "non-CDD zeroes"). If $\rho(\omega)$ changes sign a finite number of times we can find a polynomial $Q(z)$ such that $\rho_1(\omega)\equiv Q(\omega)\rho(\omega)\geq 0$, and hence $g_1(z)\equiv Q(z)g(z)=P(z)H(z)$, so $g(z)=P(z)H(z)/Q(z)$.]

The usefulness of this way of writing $g(z)$ stems from the fact that any such $H(z)$ has the representation[4]

$$H(z)=H(0)+z\left\{A+\int_{-\infty}^{\infty}\frac{p(\omega)}{\omega(\omega-z)}d\omega\right\}, \tag{4}$$

where $H(0)$, A, $p(\omega)$ are real with $A\geq 0$, $p(\omega)\geq 0$.

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[1] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 725 (1958) [Soviet Phys.—JETP **34**(7), 499 (1958)].

[2] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[3] K. Symanzik, J. Math. Phys. **1**, 249 (1960), Appendix B and private communication. The theorem presented here was originally proven without using Symanzik's results, but the proof while more straightforward was much longer and inelegant.

[4] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, 1943), p. 23. A little trivial rewriting is needed to put their Eq. (2.3) into the form of Eq. (3).

Furthermore, since $\hat{H}\equiv -1/H$ is also a Herglotz function, it has the corresponding representation

$$\hat{H}(z)=\hat{H}(0)+z\left\{\hat{A}+\int_{-\infty}^{\infty}\frac{\hat{p}(\omega)}{\omega(\omega-z)}d\omega\right\}. \quad (5)$$

These formulas provide a direct and rigorous way of relating the asymptotic behavior of $g(z)$ as $z\to\infty$, the number of subtractions needed in $g(z)$, and the asymptotic behavior of $\rho(\omega)$ and $\hat{\rho}(\omega)$.

Clearly $p(\omega)$ and $\hat{p}(\omega)$ are nice enough at infinity to make both $\int(p/\omega^2)d\omega$ and $\int(\hat{p}/\omega^2)d\omega$ convergent. What we will be making use of here is the less obvious property that $\int(p/\omega)d\omega$ and $\int(\hat{p}/\omega)d\omega$ cannot both diverge, assuming $\rho=0$ for $\omega<\omega_0$. [For if $x>0$

$$-H(-x)\geqq -H(0)+\tfrac{1}{2}\int_{\omega_0}^{x}p(\omega)\omega^{-1}d\omega, \quad (6)$$

and so if $\int(p/\omega)d\omega$ diverges then $-H(-x)\to\infty$ as $x\to\infty$. Likewise, if $\int(\hat{p}/\omega)d\omega$ diverges then $-\hat{H}(-x)\to\infty$. But if $H\to-\infty$ then $\hat{H}\to 0$. A similar result holds even if p is not restricted to vanish below some ω_0.]

We are now in a position to prove our main theorem: One of the two integrals

$$J\equiv\int^{\infty}\frac{\rho(\omega)}{|g(\omega+i\epsilon)|^2}\omega^{-1}d\omega, \quad (7)$$

$$I\equiv\int^{\infty}\rho(\omega)\omega^{-1}d\omega \quad (8)$$

must converge.

[Note that

$$p(\omega)=\rho(\omega)/R(\omega), \quad (9)$$

$$\hat{p}(\omega)=p(\omega)/|H(\omega+i\epsilon)|^2=\rho(\omega)R(\omega)/|g(\omega+i\epsilon)|^2. \quad (10)$$

Since R is rational, $R(\omega)\to C\omega^n$ as $\omega\to\infty$, where $C\neq 0$ and n is an integer. If $n\leqq -1$ then the convergence of $\int(p/\omega^2)d\omega$ implies that I converges. If $n\geqq 1$, then the convergence of $\int(\hat{p}/\omega^2)d\omega$ implies that I converges. If $n=0$ then the convergence of $\int(p/\omega)d\omega$ or $\int(\hat{p}/\omega)d\omega$ implies, respectively, that I or J converges.]

If we now make Pomeranchuk's assumption that $|g(E)|$ is bounded (say, by M), then $I\leqq JM^2$, *so I must converge.* Using (2), this shows that $[\sigma_+(E)-\sigma_-(E)]\times\ln E\to 0$ in the mean as $E\to\infty$. This is the form of Pomeranchuk's theorem proposed by Amati, Fierz, and Glaser[5] [and needed if $g(E^2)$ is to be written without subtractions, as is suggested[6] by the low energy $\pi^{\pm}p$ data]. However, their method of proof has been questioned,[7] and at any rate their assumptions were certainly as strong as Pomeranchuk's. This result was also obtained by Sugawara and Kanazawa,[8] who assumed that the amplitudes $f_{\pm}(E)/E$ approach constants at infinity. We have not assumed that anything approaches a constant at infinity.

[5] D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters **4**, 89 (1960).

[6] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

If $|g|\to\infty$ as $E\to\infty$, then our theorem yields the weaker result that J is convergent, and thus correlates the asymptotic behavior of $\sigma_+(E)-\sigma_-(E)$ with that of $|g(E^2)|$. Suppose for example that $|f_{\pm}(E)|\sim E\ln^m E$ as $E\to\infty$. Then $|g(E^2)|=O(\ln^m E)$, so that

$$\int^{\infty}\frac{[\sigma_+(E)-\sigma_-(E)]}{E\ln^{2m}E}dE<\infty, \quad (11)$$

and hence for $m\leqq\frac{1}{2}$ we still have $\sigma_+-\sigma_-\to 0$.

The optical theorem would allow $\sigma_{\pm}(E)$ in this example to grow as fast as $\ln^m E$, and with maximum absorption in partial waves with $l\leqq kR_{\pm}(\omega)$, $\sigma_{\pm}$ would actually be given by $C_{\pm}\ln^m E$ as $E\to\infty$. Then Eq. (11) implies that $C_+=C_-$ for $m\leqq 1$. Hence, though $\sigma_+-\sigma_-$ may not tend to zero, the ratio σ_+/σ_- does tend to one.

This result may be much more general than indicated by the restriction $m\leqq 1$. For if we assume that $\sigma_{\pm}(E)$ actually become equal to $C_{\pm}\ln^m E$ for E large enough, then direct calculation of the once-subtracted dispersion integral for $g(E^2)$ gives $g\to(C_+-C_-)\ln^{m+1}E$, so $C_+=C_-$, and hence $\sigma_+/\sigma_-\to 1$.

Pomeranchuk has also given a different sort of justification for the equality of $\sigma(\infty)$ for π^+p and π^-p, which is capable of a very broad extension. A generalized version of his remark is as follows: If two sets A_m and B_n of particles (e.g., $\pi^{\pm}$, π^0, and p,n) form two irreducible representations of *any* symmetry group (e.g., isospin), and if all individual inelastic processes $A_m+B_n\to A_{m'}+B_{n'}$ (for $m\neq m'$ or $n\neq n'$) have vanishing cross sections as $E\to\infty$, then the symmetry implies the equality of all total cross sections for any of the A_m striking any of the B_n. Of course, there is no reason to suppose that global symmetry or the "eight-fold way" should become exact for high energy total cross sections,[8] so the most we can hope for in this direction is that some approximate equalities may become apparent in the high-energy πN, KN or NN, YN total cross sections.

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[7] M. Sugawara and A. Kanazawa (to be published).

[8] M. Gell-Mann and F. Zachariasen (to be published).