Analytical Solutions for Velocity-Dependent Nuclear Potentials*

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The two-nucleon potential, with the necessary invariance requirements, is assumed to be a quadratic function of momentum: $v=-V_0J_1(r)-(\lambda/M)\ \mathbf{p}\cdot J_2(r)\mathbf{p}$, where $J_1(r)$ and $J_2(r)$ are two short-range functions. For simplicity $-J_2(r)$ is assumed to be a square well of unit depth. The Schrödinger equation is solved (neglecting Coulomb forces) for three different choices of $J_1(r)$. Numerical results for the phase shifts are given for these three potentials (v_1, v_2, and v_3) for the singlet S, D, and G states. Reasonably good fits are obtained.

I. INTRODUCTION

THE interaction between two nucleons at small distances has been the subject for discussion for many years, but there is still no definite and convincing description. Meson theory fails to provide a satisfactory, complete answer, and there is little experimental evidence in favor of any one out of the many possible phenomenological approaches.

One successful fit to the observed singlet-even phase shifts is the Gammel-Thaler potential. Though for the two-body problem it represents a simple static potential, it is mathematically difficult to use for calculating the properties of nuclei, since special techniques must be used for such a potential. As an alternative to the hard core of the Gammel-Thaler potential we discuss here a velocity-dependent potential with the property that ordinary methods of calculation, such as perturbation theory, can be applied without too much difficulty.[1]

Of course it is known that there are certain uniqueness theorems, in that a complete knowledge of one set of phase shifts at all energies would determine the singlet-even potential.[2] On the other hand, this potential is only determined if it is assumed to be static. It is possible to find velocity-dependent potentials which also fit the singlet S-phase shifts.[3] If the phase shifts were known accurately for a large range of energies for many different angular momentum states, this would serve to determine the nature of the potential. However, the singlet-even phase shifts are known only up to an energy of about 340 Mev for the lowest three angular momentum states. (Above this energy, pion production becomes significant and the concept of a nucleon-nucleon potential loses its original meaning.) It therefore remains an open question to what extent this limited amount of phase shift data can be fitted with reasonable accuracy to a static potential on the one hand and to a velocity-dependent potential on the other.

In this paper, we take the simple, though physically unrealistic, shape of a square well with some modifications.

In Sec. II we assume the same range for the velocity-dependent and static potentials.[4] In Sec. III we modify the static square well by adding a "Jost" potential and a delta-function potential. With this potential we can still find an analytic form for the S-wave function.

It may be of interest to mention that Feshbach *et al.*[5] assume an energy-independent boundary condition for the logarithmic derivative of the wave function, but that we find a boundary condition which is somewhat energy-dependent [Eq. (4)].

In Sec. IV we use square wells with different ranges together with a delta-function added to the static potential. To make such a potential more realistic we add a one-pion exchange tail to it.

II. SQUARE WELL-SQUARE WELL (Equal Ranges)

The central potential is of the form

$$v_1(r,\mathbf{p})=-V_0J_1(r)-(\lambda/M)\mathbf{p}\cdot J_2(r)\mathbf{p}, \tag{1}$$

where

$$J_1(r)=J_2(r)=1-U(r-b),$$

and

$$U(x)=0,\quad x<0;\quad =\tfrac{1}{2},\quad x=0;\quad =1,\quad x>0.$$

$\mathbf{p}$ is the relative momentum operator, and M is the nucleon mass.

The Schrödinger equation for the lth partial wave (neglecting Coulomb forces) reduces to

$$\begin{aligned}&[1-\lambda+\lambda U(r-b)]\left(R''+\frac{2}{r}R'\right)\\&\quad+\left[k^2+\frac{MV_0}{\hbar^2}[1-U(r-b)]\right.\\&\quad\left.-[1-\lambda+\lambda U(r-b)]\frac{l(l+1)}{r^2}\right]R=-\lambda R'\delta(r-b).\end{aligned} \tag{2}$$

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[1] R. E. Peierls, *Proceedings of the International Conference on Nuclear Structure*, Kingston (University of Toronto Press, Toronto, 1960), p. 7.
[2] I. M. Gel'fand and B. M. Levitan, Doklady Akad. Nauk, S.S.S.R. **77**, 557 (1951).
[3] M. A. B. Bég, Ann. Phys. **13**, 110 (1961).
[4] M. Razavy, O. Rojo, and J. S. Levinger, *Proceedings of the International Conference on Nuclear Structure*, Kingston (University of Toronto Press, Toronto, 1960), p. 128. There is an error in Eq. (3) of this reference. α is defined there as $\lambda/2(1-\lambda)$, while it should be $\lambda/(2-\lambda)$.
[5] H. Feshbach, E. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961).

Primes mean derivative with respect to r, the relative coordinate, and $k^2=EM/\hbar^2$ (where E is the energy in the center-of-mass system). The solution of (2) is

$$\begin{aligned} R_{\mathrm{I}} &= B_l j_l(k'r), & r<b, \\ R_{\mathrm{II}} &= A_l[j_l(kr)-\tan\eta_l n_l(kr)], & r>b, \end{aligned} \tag{3}$$

where $k'^2=k^2/(1-\lambda)+MV_0/[\hbar^2(1-\lambda)]$. The boundary conditions at the edge of the square well, $(r=b)$, are

$$R_{\mathrm{I}}(b)=R_{\mathrm{II}}(b), \tag{4a}$$

$$(1-\lambda)R_{\mathrm{I}}'(b)=R_{\mathrm{II}}'(b). \tag{4b}$$

Boundary condition (4b) is derived by multiplying (2) by r^2dr, integrating from $(b-\epsilon)$ to $(b+\epsilon)$, and taking the limit as ϵ goes to zero. We keep in mind that R_{I} and R_{II} are continuous functions of r in their respective regions:

$$\begin{aligned} &(1-\lambda)\int_{b-\epsilon}^{b+\epsilon}\left(R''+\frac{2}{r}R'\right)r^2dr \\ &+\lambda\int_{b-\epsilon}^{b+\epsilon}U(r-b)\left[R''+\frac{2}{r}R'\right]r^2dr \\ &+\int_{b-\epsilon}^{b+\epsilon}\left\{k^2+\frac{MV_0}{\hbar^2}[1-U(r-b)]\right. \\ &\left.-[1-\lambda+\lambda U(r-b)]\frac{l(l+1)}{r^2}\right\}Rr^2dr \\ &= -\lambda\int_{b-\epsilon}^{b+\epsilon}R'\delta(r-b)r^2dr. \end{aligned} \tag{5}$$

Integrating the first integral by parts, we have

$$\begin{aligned} &(1-\lambda)\int_{b-\epsilon}^{b+\epsilon}(R''r^2+2rR')dr \\ &= (1-\lambda)[(b+\epsilon)^2R'(b+\epsilon)-(b-\epsilon)^2R'(b-\epsilon)]. \end{aligned}$$

The second term on the left in (5) becomes

$$\begin{aligned} &\lambda\int_{b-\epsilon}^{b+\epsilon}U(r-b)\left(R''+\frac{2}{r}R'\right)r^2dr \\ &= \lambda(b+\epsilon)^2R'(b+\epsilon)-\lambda\int_{b-\epsilon}^{b+\epsilon}R'\delta(r-b)r^2dr. \end{aligned}$$

The third term vanishes, since

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{b-\epsilon}^{b+\epsilon}\Big[&k^2+\frac{MV_0}{\hbar^2}[1-U(r-b)] \\ &-[1-\lambda+\lambda U(r-b)]\frac{l(l+1)}{r^2}\Big]Rr^2dr=0. \end{aligned}$$

Substituting the above in (5) and taking the limit as $\epsilon\to 0$

$$\lim_{\epsilon\to 0}R'(b+\epsilon)=R_{\mathrm{II}}'(b), \quad \lim_{\epsilon\to 0}R'(b-\epsilon)=R_{\mathrm{I}}'(b),$$

we find Eq. (4b). Substituting R_{I} and R_{II} in the boundary conditions, we find the phase shift η_l:

$$\tan\eta_l=\frac{j_l'(kb)-(1-\lambda)\gamma_l(k'b)j_l(kb)}{n_l'(kb)-(1-\lambda)\gamma_l(k'b)n_l(kb)}, \tag{6}$$

where $\gamma_l(k'b)=j_l'(k'b)/j_l(k'b)$. For $l=0$, the expression (6) can be written as

$$k\cot(\eta_0+kb)=(1-\lambda)k'\cot k'b+\frac{\lambda}{b}. \tag{7}$$

Note added in proof. This result for η_0 was also obtained by R. L. Carovillano (Ph.D. Thesis, Indiana University, Bloomington, Indiana, July, 1959, unpublished.)

The effective range r_0 and the scattering length a can be found by calculating $k\cot\eta_0$ directly from (7) and then expanding it in powers of k. They have the following forms:

$$\begin{aligned} a&=\frac{-b(1-\lambda)(1-\zeta\cot\zeta)}{(1-\lambda)\zeta\cot\zeta+\lambda}, \\ \tfrac{1}{2}r_0&=\frac{D+b(1-\lambda)(1-\zeta\cot\zeta)+(b/3)[(1-\lambda)\zeta\cot\zeta+\lambda]^2}{(1-\lambda)^2(1-\zeta\cot\zeta)^2}, \end{aligned} \tag{8}$$

where $\zeta=[MV_0/\hbar^2(1-\lambda)]^{\frac{1}{2}}b$ and

$$D=(b/2\zeta)(\cot\zeta-\zeta/\sin^2\zeta).$$

III. SQUARE WELL-"JOST" POTENTIAL

A different form of potential for which there is an analytic solution for the S wave function is the static "Jost" potential[6] outside the range of the velocity-dependent part. In order to get a satisfactory ^1S phase shift we have to add a δ-function term to the static part, and we thus choose the following potential:

$$\begin{aligned} v_2(r,\mathbf{p})=&V_J(c)J_1(r)-\frac{\lambda}{M}\hbar^2\frac{\delta(r-c)}{r} \\ &+V_J(r)U(r-c)-(\lambda/M)\mathbf{p}\cdot J_2(r)\mathbf{p}, \end{aligned} \tag{9}$$

where $J_1(r)=J_2(r)=1-U(r-c)$ and V_J is the "Jost" potential defined by

$$V_J(r)=(2\nu\hbar^2\mu^2e^{-\mu r})M^{-1}(1-\nu e^{-\mu r})^{-2}.$$

The Schrödinger equation for this potential can be solved for the ^1S state in the same way as was done for v_1; only, this time, new boundary conditions are found for the derivatives of the wave function.

At $r=c$, $u_{\mathrm{I}}(c)=u_{\mathrm{II}}(c)$ and $u_{\mathrm{II}}'(c)=(1-\lambda)u_{\mathrm{I}}'(c)$ where

[6] R. Jost, Helv. Phys. Acta **20**, 256 (1947).

$u=rR$. The phase shift η_0 is given by

$$\tan(\eta_0+kb)=\frac{K_3-FK_2}{FK_1-K_4}, \tag{10}$$

where

$$F=(1-\lambda)k'\cot k'c,\quad k'^2=\frac{\mu y'(c)+k^2}{1-\lambda},$$

$$K_1=1+4K^2+y(c),\qquad K_2=2Ky(c),$$

$$K_3=k\left[1+4K^2+y(c)+\frac{2}{\mu}y'(c)\right],\quad K_4=y'(c)-2Kky(c),$$

with $y(r)=2\nu e^{-\mu r}(1-\nu e^{-\mu r})^{-1}$, $y'(c)=(dy/dr)_{r=c}$, and $K=k/\mu$.

We use Born approximation for the ^1D phase shift. For the potential of the form

$$v=V(r)-\frac{\lambda}{M}\frac{\hbar^2\delta(r-c)}{r}-\frac{\lambda}{M}\mathbf{p}\cdot[1-U(r-c)]\mathbf{p},$$

the radial part of the Schrödinger equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(k^2-\frac{l(l+1)}{r^2}\right)R=\hat{W}(r,k)R, \tag{11}$$

where

$$\hat{W}(r,k)=\frac{1}{1-\lambda+\lambda U(r-c)}\left\{\frac{M}{\hbar^2}V(r)-\frac{\lambda\delta(r-c)}{r}-\lambda k^2[1-U(r-c)]-\lambda\delta(r-c)\frac{d}{dr}\right\}.$$

The Born approximation for the phase shift is given by

$$\tan\eta_l=-k\int_0^\infty j_l(kr)[\hat{W}(r,k)j_l(kr)]r^2dr.$$

For $V(r)=V_J(c)[1-U(r-c)]+V_J(r)U(r-c)$, we have

$$\tan\eta_l=\frac{(kc)^3}{2(1-\lambda)}\left(\frac{V_0}{E}+\lambda\right)[j_l^2(kc)-j_{l-1}(kc)j_{l+1}(kc)]-[\ln(1-\lambda)]kcj_l(kc)[j_l(kc)+cj_l'(kc)]-k\int_c^\infty\frac{2\nu\mu^2e^{-\mu r}}{(1-\nu e^{-\mu r})^2}r^2j_l^2(kr)dr. \tag{12}$$

IV. SQUARE WELLS OF UNEQUAL RANGE

Meson theory predicts that when the distance between the two interacting nucleons is large enough, ($r>1.6$ f), the potential is static, well behaved and is of the Yukawa form (OPEP). Therefore we cannot expect that the potential that we have discussed in Sec. II should be satisfactory. Indeed, the ^1D phase shifts for the potential v_1 turn out to be very small for all energies. To improve the fit for ^1D and ^1G phase shifts we take a short-range velocity-dependent part with a static part of longer range, together with a Yukawa tail. Here too, we add the additional term $-(\hbar^2/M)\lambda\delta(r-c)/r$ to the potential:

$$v_3(r,\mathbf{p})=-V_0[1-U(r-b)]-\frac{\hbar^2\lambda}{M}\frac{\delta(r-c)}{r}+V^{\mathrm{OP}}U(r-b)-\frac{\lambda}{M}\mathbf{p}\cdot[1-U(r-c)]\mathbf{p}, \tag{13}$$

where for the even, spin-singlet potential, $V^{\mathrm{OP}}=-(10.83e^{-0.708r}/0.708r)$ Mev. For $r<b$ the Schrödinger equation can be solved exactly for all angular momentum states, as was done in Sec. II.

The "phase shift" δ_0 for ^1S states is given at $r=b$ by

$$\tan(\delta_0+kb)=\frac{k}{q}\tan\{q(b-c)+\tan^{-1}k[(1-\lambda)k'\cot k'c]^{-1}\}, \tag{14}$$

where $q^2=(MV_0/\hbar^2)+k^2$. (See the appendix for δ_l.) To calculate the effect of OPEP, we use Born approximation. If δ_l represents the phase shift at $r=b$, and δ_l^{OP} the contribution of the Yukawa potential to the phase shift, then the over-all phase shift η_l is given by

$$\tan\eta_l=\tan\delta_l+\tan\delta_l^{\mathrm{OP}}.$$

We use

$$\tan\delta_l^{\mathrm{OP}}=-k\int_0^\infty j_l(kr)\left(\frac{M}{\hbar^2}V^{\mathrm{OP}}\right)\times[j_l(kr)-\tan\delta_l(kr)]r^2dr. \tag{15}$$

The ratio of V^{OP}/E for $r\geq b\geq 1.6$ f shows that Born approximation is accurate enough even at 20 Mev. The second term in the bracket of the expression (15) can be neglected when $\tan\delta_l$ is very small.

V. NUMERICAL RESULTS

We have tried to fit the ^1S phase shifts to Breit's values (YLAM),[7] and then used the same parameters to calculate the ^1D and ^1G phase shifts. The set of numbers given below do not necessarily represent the best results. It is possible that further adjustments of the parameters would give better fits. In all our calculation we have neglected the Coulomb force. Table I gives the

TABLE I. Parameters for different potentials.[a]

	b (f)	c (f)	V_0 (Mev)	λ	μ (f^{-1})
v_1 (square wells)	2.4	···	16.9	−0.21	···
v_2 (square-Jost)	···	0.7	75.5	−1.43	2
v_3 (square wells with Yukawa)	1.6	0.5	51	−1.64	···

[a] The parameters in the table are to be used as follows: for potential v_1 use Eq. (7), for v_2 use Eq. (10), for v_3 use Eq. (14).

[7] Breit, Hull, Lassila, and Pyatt, Phys. Rev. 120, 2227 (1960).

TABLE II. The ^1S phase shifts.[a]

E (Mev)	20	100	180	260	340
$\eta_0(v_1)$	0.871	0.217	−0.082	−0.187	−0.197
$\eta_0(v_2)$	0.852	0.341	0.103	−0.079	−0.192
$\eta_0(v_3)$	0.846	0.448	0.169	−0.032	−0.190
Breit's K_0 (YLAM)	0.856	0.380	0.136	−0.033	−0.195
Gammel-Thaler	0.859	0.379	0.120	−0.069	···

[a] Phase shifts in radians. The effective range r_0 and the scattering length a for v_1 and v_2 are as follows: $a(v_1)=a(\text{G-T})=-23.6$ f, $r_0(v_1)=r_0(\text{G-T})=2.65$ f, $a(v_2)=\infty$, $r_0(v_2)=2.71$ f. $\eta_0(v_{1,2,3})$ are the phase shifts calculated for potentials $v_{1,2,3}$. G-T refers to Gammel and Thaler.

TABLE III. ^1D phase shifts.[a]

E (Mev)	100	180	260	340
$\eta_2(v_2)$	0.114	0.200	0.246	0.259
$\eta_2(v_3)$	0.052	0.126	0.189	0.270
Breit's K_2 (YLAM)	0.072	0.120	0.160	0.184
Gammel-Thaler	0.096	0.181	0.239	···

[a] Phase shifts in radians. $\eta_2(v_{2,3})$ are the phase shifts calculated for potentials $v_{2,3}$.

TABLE IV. ^1G phase shifts.[a]

E (Mev)	100	180	260	340
$\eta_4(v_3)$	0.006	0.012	0.017	0.023
Breit's K_4 (YLAM)	0.007	0.012	0.016	0.017

[a] Phase shifts in radians. $\eta_4(v_3)$ is the phase shift calculated for potential v_3.

values of the parameters used with the three different forms of potential. In Tables II–IV, the values of ^1S, ^1D, and ^1G phase shifts for different shapes are compared with Breit's phase shifts. The scattering lengths and the effective ranges for v_1 and v_2 are given below Table II, together with the values given by Gammel-Thaler. These results suggest that one can explain all the scattering data using a velocity-dependent potential, without introducing a great number of new parameters. In fact, for the singlet potential we have three (for v_1), and four (for v_2 or v_3) adjustable parameters, compared to the three parameters used by Gammel-Thaler.[8] Once more we want to emphasize that the above potentials are chosen to illustrate how one can explain the experimental results of nucleon-nucleon scattering without using a hard core. Two of our potentials (v_1 and v_3) have been used for a perturbation calculation of the energy of a neutron gas.[9] However, we have so far no physical evidence in favor of such a velocity-dependent interaction.[10]

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APPENDIX

For the angular momentum state l, the "phase shift" [corresponding to δ_0 of Eq. (14)] is given by

$$\tan\delta_l=\frac{j_l'(kb)-\mathcal{G}_l j_l(kb)}{n_l'(kb)-\mathcal{G}_l n_l(kb)},$$

where

$$\mathcal{G}_l=\frac{j_l'(qb)+\Lambda_l n_l'(qb)}{j_l(qb)+\Lambda_l n_l(qb)},$$

$$\Lambda_l=\frac{j_l'(qc)-F_l(k'c)j_l(qc)}{n_l(qc)F_l(k'c)-n_l'(qc)},\quad F_l(k'c)=(1-\lambda)\frac{j_l'(k'c)}{j_l(k'c)}-\frac{\lambda}{c},$$

and

$$j_l'(qb)=\frac{d}{dr}j_l(qr)\bigg|_{r=b}.$$

[8] J. L. Gammel and R. M. Thaler, *Progress in Elementary Particle and Cosmic-Ray Physics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. V, p. 99.

[9] J. S. Levinger and L. M. Simmons, Phys. Rev. **124**, 916 (1961).

[10] O. Rojo and J. S. Levinger, following paper [Phys. Rev. **125**, 273 (1962)].